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Some formulae by means of fractional derivatives

by

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1

The object of this paper is to obtain some formulae, believed to be new, by using the conception of fractional derivatives as defined by

$$(1) \quad D_w^\lambda [w^{\mu-1}] = \frac{d^\lambda w^{\mu-1}}{dw^\lambda} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)} w^{\mu-\lambda-1}.$$

2

To start with, let us consider the elementary identity

$$[1-u(x+y-xy)]^{-a}(1-uz)^{-\lambda} = (1-u\overline{x+y+xy})^{-a}(1-uz)^{-\lambda}$$

or

$$\begin{aligned} \sum_{m,n=0}^{\infty} \frac{(a)_m(\lambda)_n}{m!n!} (x+y-xy)^m z^n u^{m+n} \\ = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^r (a)_{r+m}(\lambda)_n}{r!m!n!} (xy)^r (x+y)^m z^n u^{r+m+n} \end{aligned}$$

Now we multiply both the sides by u^{b-1} and then apply the operator, D_u^{b-c} , so that

$$\begin{aligned} \sum_{m,n=0}^{\infty} \frac{(a)_m(\lambda)_n}{m!n!} (x+y-xy)^m z^n D_u^{b-c} [u^{b+m+n-1}] \\ = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^r (a)_{r+m}(\lambda)_n}{r!m!n!} (xy)^r (x+y)^m z^n D_u^{b-c} [u^{b+r+m+n-1}]. \end{aligned}$$

Using the definition (1) it can be easily seen that

$$D_u^{b-c} [u^{b+m+n-1}] = \frac{\Gamma(b+m+n)}{\Gamma(c+m+n)} u^{c+m+n-1}$$

and

$$D_u^{b-c} [u^{b+r+m+n-1}] = \frac{\Gamma(b+r+m+n)}{\Gamma(c+r+m+n)} u^{c+r+m+n-1}.$$

Thus we arrive at

$$F_1[b; a, \lambda; c; u(x+y-xy), uz] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} (-uxy)^r \\ \times F_1[b+r; a+r, \lambda; c+r; u(x+y), uz]$$

or, putting $u = 1$,

$$(2) \quad F_1(b; a, \lambda; c; x+y-xy, z] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} (-xy)^r \\ \times F_1(b+r; a+r, \lambda; c+r; x+y, z).$$

In case we put $z = x+y-xy$ and use the formula [2, p. 239]

$$(3) \quad F_1(\alpha; \beta, \beta'; \gamma; x, x) = {}_2F_1(\alpha, \beta + \beta'; \gamma; x),$$

(2) reduces to

$$(4) \quad {}_2F_1(a+\lambda, b; c; x+y-xy) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} (-xy)^r \\ \times F_1(b+r; a+r, \lambda; c+r; x+y, x+y-xy).$$

On the other hand, if we put $z = x+y$, it follows from (2) that

$$(5) \quad F_1(b; a, \lambda; c; x+y-xy, x+y) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} (-xy)^r \\ \times {}_2F_1(a+\lambda+r, b+r; c+r; x+y).$$

For $\lambda = 0$, (5) reduces to the known result due to Burchall and Chaundy [1].

In order to find an inverse of (2), we consider

$$[1-u(x+y)]^{-a} (1-uz)^{-\lambda} = [1-u(x+y-xy)-uxy]^{-a} (1-uz)^{-\lambda}$$

or

$$\sum_{m, n=0}^{\infty} \frac{(a)_m (\lambda)_n}{m! n!} (x+y)^m z^n u^{m+n} \\ = \sum_{r, m, n=0}^{\infty} \frac{(a)_{r+m} (\lambda)_n}{r! m! n!} (xy)^r (x+y-xy)^m z^n u^{r+m+n}.$$

We multiply both the sides by u^{b-1} and apply the operator D_u^{b-c} , thus having

$$\sum_{m, n=0}^{\infty} \frac{(a)_m (\lambda)_n}{m! n!} (x+y)^m z^n D_u^{b-c} [u^{b+m+n-1}] \\ = \sum_{r, m, n=0}^{\infty} \frac{(a)_{r+m} (\lambda)_n}{r! m! n!} (xy)^r (x+y-xy)^m z^n D_u^{b-c} [u^{b+r+m+n-1}],$$

which yields, on putting $u = 1$,

$$(6) \quad {}_1F_1(b; a, \lambda; c; x+y, z) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} (xy)^r \\ \times {}_1F_1(b+r; a+r, \lambda; c+r; x+y-xy, z).$$

Putting $z = x+y$ and using (3), (6) becomes

$$(7) \quad {}_2F_1(a+\lambda, b; c; x+y) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} (xy)^r \\ \times {}_1F_1(b+r; a+r, \lambda; c+r; x+y-xy, x+y),$$

while for $z = x+y-xy$, we have

$$(8) \quad {}_1F_1(b; a, \lambda; c; x+y, x+y-xy) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} (xy)^r \\ \times {}_2F_1(a+\lambda+r, b+r; c+r; x+y-xy).$$

Again, for $\lambda = 0$, (8) yields the known formula due to Burchnall and Chaundy [1].

Similarly, if we consider the identity

$$[1-u(x+y-xy)]^{-b} = [(1-ux)(1-uy)+uxy(1-u)]^{-b}$$

or

$$\sum_{n=0}^{\infty} \frac{(b)_n}{n!} (x+y-xy)^n u^{a+n-1} = \sum_{r=0}^{\infty} \frac{(-1)^r (b)_r}{r!} (xy)^r \\ \times \sum_{m, n, p=0}^{\infty} \frac{(b+r)_m (b+r)_n (-r)_p}{m! n! p!} x^m y^n u^{a+r+m+n+p-1}$$

and apply the operator D_u^{a-c} , we get

$$(9) \quad {}_2F_1(a, b; c; u(x+y-xy)) = \sum_{r=0}^{\infty} \frac{(-1)^r (a)_r (b)_r}{r! (c)_r} (uxy)^r \\ \times F_D(a+r; b+r, b+r, -r; c+r; ux, uy, u).$$

For $u = 1$, we get the formula due to Burchnall and Chaundy [1].

3

We take up another elementary result

$$(2-2x-2y)^{-\alpha} = [(1-2x)+(1-2y)]^{-\alpha}$$

or

$$\sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n}}{m! n!} x^m y^n = 2^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{n!} (1-2x)^{-\alpha-n} (1-2y)^n.$$

Now we multiply both the sides by $x^{\beta-1}y^{\beta'-1}$, so that

$$\sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n}}{m!n!} x^{\beta+m-1} y^{\beta'+n-1} = 2^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{n!} \\ \times \sum_{r=0}^{\infty} \frac{(\alpha+n)_r}{r!} 2^r x^{\beta+r-1} \sum_{k=0}^n \frac{(-n)_k}{k!} 2^k y^{\beta'+k-1}$$

Applying the operators $D_x^{\beta-\gamma}$ and $D_y^{\beta'-\gamma'}$, we obtain

$$(10) \quad F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = 2^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{n!} \\ \times {}_2F_1(\alpha+n, \beta; \gamma; 2x) {}_2F_1(-n, \beta'; \gamma'; 2y).$$

Using Euler's formula

$${}_2F_1(a, b; c; x) = (1-x)^{-a} {}_2F_1\left(a, c-b; c; -\frac{x}{1-x}\right),$$

we rewrite (10) as

$$F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = 2^\alpha (1-2x)^{-\beta}, \\ \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{n!} {}_2F_1\left(\gamma-\alpha-n, \beta; \gamma; -\frac{2x}{1-2x}\right) {}_2F_1(-n, \beta'; \gamma'; 2y),$$

which on putting $\gamma = \gamma' = \alpha$, $\beta' = \beta$, $2x/(2x-1) = y$ and using the relation [2, p. 238]

$$F_2(\alpha; \beta, \beta'; \alpha, \alpha; x, y) = (1-x)^{-\beta} (1-y)^{-\beta'} {}_2F_1\left[\beta, \beta'; \alpha; \frac{xy}{(1-x)(1-y)}\right],$$

becomes

$$(11) \quad {}_2F_1\left[\beta, \beta; \alpha; -\frac{y^2}{(1-y)^2}\right] \\ = 2^\alpha (1-y)^{2\beta} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{n!} [{}_2F_1(-n, \beta; \alpha; 2y)]^2$$

Dividing y by β and taking $\beta \rightarrow \infty$, we get from (11) that

$$(12) \quad {}_0F_1(-; 1+\alpha; -y^2) = 2^{1+\alpha} e^{-2y} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(1+\alpha)_n} [L_n^{(\alpha)}(2y)]^2$$

or, if we prefer,

$$(13) \quad J_\alpha(y) = \frac{2y^\alpha}{\Gamma(1+\alpha)} e^{-y} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(1+\alpha)_n} [L_n^\alpha(y)]^2.$$

Next, we consider

$$[(1+x)(1-y) + (1-x)(1+y)]^n = 2^n(1-xy)^n$$

or

$$\sum_{r=0}^n \binom{n}{r} (1+x)^{n-r} (1-x)^r (1-y)^{n-r} (1+y)^r = 2^n \sum_{r=0}^n \frac{(-n)_r}{r!} x^r y^r.$$

Multiplying both the sides by $x^{\alpha-1} y^{\gamma-1}$ and applying the operators $D_x^{\alpha-\beta}$ and $D_y^{\gamma-\delta}$, we get

$$(14) \quad \sum_{r=0}^n \binom{n}{r} F_1(\alpha; -n+r, -r; \beta; -x, x) F_1(\gamma; -n+r, -r; \delta; y, -y) = 2^n {}_3F_2(-n, \alpha, \gamma; \beta, \delta; xy).$$

Lastly, we deal with the identity

$$[(1-y) - (1-x)]^n = (x-y)^n$$

or

$$\sum_{r=0}^n \frac{(-n)_r}{r!} (1-x)^r (1-y)^{n-r} = \sum_{r=0}^n \frac{(-n)_r}{r!} x^{n-r} y^r$$

or

$$\begin{aligned} \sum_{r=0}^n \frac{(-n)_r}{r!} \sum_{k=0}^r \frac{(-r)_k}{k!} x^{\alpha_1+k-1} \sum_{i=0}^{n-r} \frac{(-n+r)_i}{i!} y^{\alpha_1+i-1} \\ = \sum_{r=0}^n \frac{(-n)_r}{r!} x^{\alpha_1+n-r-1} y^{\alpha_1+r-1}. \end{aligned}$$

Applying the operators $D_x^{\alpha_1-b_1}$ and $D_y^{\alpha_1-\beta_1}$, it yields

$$(15) \quad \begin{aligned} \sum_{r=0}^n \frac{(-n)_r}{r!} {}_2F_1(-r, a_1; b_1; x) {}_2F_1(-n+r, \alpha_1; \beta_1; y) \\ = \frac{(a_1)_n x^n}{(b_1)_n} {}_3F_2\left(-n, 1-b_1-n, \alpha_1; 1-a_1-n, \beta_1; \frac{y}{x}\right). \end{aligned}$$

If we continue this process of performing operators p times with respect to x and l times with respect to y and then suppress some parameters, we arrive at

$$(16) \quad \begin{aligned} \sum_{r=0}^n \frac{(-n)_r}{r!} {}_{1+p}F_p \left[\begin{matrix} -r, a_1, a_2, \dots, a_p; \\ b_1, \dots, b_p; \end{matrix} x \right] {}_{1+l}F_l \left[\begin{matrix} -n+r, \alpha_1, \dots, \alpha_l; \\ \beta_1, \dots, \beta_m; \end{matrix} y \right] \\ = \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} x^n {}_{1+a+l}F_{p+m} \\ \left[\begin{matrix} -n, 1-b_1-n, \dots, 1-b_q-n, \alpha_1, \dots, \alpha_l; \\ 1-a_1-n, \dots, 1-a_p-n, \beta_1, \dots, \beta_m \end{matrix} (-1)^{p+l} \frac{y}{x} \right]. \end{aligned}$$

REFERENCES

BURCHNALL, J. L. and CHAUNDY, T. W.

- [1] "Expansions of Appell's double hypergeometric functions" *Quart. J. Maths.* (Oxford), vol. 11, No. 44, pp. 249—270 (1940).

ERDELYI, A.

- [2] *Higher transcendental functions*. Vol. 1 (1953).

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