COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 19, n° 1 (1968), p. 1-15
<http://www.numdam.org/item?id=CM_1968__19_1_1_0>

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I

by

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Introduction

This paper is the first of a series to be published under the same title and numbered I, II, ...

In this work we study a non-standard type of pseudo-normed linear spaces, herein called V-spaces.

V-spaces depart from the classical normed linear spaces ([7], [36]) in that the usual requirements on the norm function

(0.1) $\quad ||\alpha x|| = |\alpha| \, ||x||$ for all x and for all scalars α,

(0.2) $\quad ||x+y|| \leqq ||x||+||y||$ for all x, y,

are replaced by

(0.3) $\quad ||\alpha x|| = ||x||$ for all x and all scalars $\alpha \neq 0$,

(0.4) $$||x+y|| \begin{cases} \leqq \text{Max } \{||x||, ||y||\} & \text{for all } x, y, \\ = \text{Max } \{||x||, ||y||\} & \text{if } ||x|| \neq ||y||, \end{cases}$$

and, also, by the additional condition that the norm of an element is either 0 or is equal to ρ^n for a fixed real ρ, $1 < \rho < \infty$, and some integer n. A V-space is assumed to be complete with respect to its norm and the field of scalars to have characteristic 0 ([10]). Thus, in the usual terminology, a V-space is a complete strongly non-Archimedean pseudo-normed linear space over a field of scalars with characteristic 0 and a trivial valuation.

The author's attention was directed to this abstract structure by the following example. A classical method to obtain information about the asymptotic behaviour of a real valued function is to compare it with the elements of an "asymptotic sequence" of functions (see Erdelyi [9], van der Corput [38], [39]). C. A. Swanson and M. Schulzer [32], [33], have extended this method of comparison to functions defined on some neighbourhood of a non-isolated point of a Hausdorff space and with ranges in an arbitrary Banach space. It will be shown, in Part III, that when applied to the elements of a linear space of functions, the results of this method can be expressed by assigning to each function a norm under which the space is a V-space.

Linear spaces satisfying the defining properties of a V-space, except for the retention of (0.1) in place of (0.3), have been systematically investigated by A. F. Monna [24], [25]. Most of the results of Monna are valid under the additional conditions that the space be separable or locally compact. Except in trivial cases, V-spaces are neither locally compact nor separable.

In Parts I and II we investigate the basic topological and algebraic properties of V-spaces. A notion of utmost importance in this work is that of "distinguishability". "Distinguished sets" and "distinguished bases" are defined in Section 5, Part I. The concept of distinguishability has been introduced by Monna [24, V], [25, I] under a different name and through another formal definition. Monna has shown that in non-Archimedean normed linear spaces over a field with a non-trivial valuation, distinguished bases exist only under restrictive conditions. However, by use of a modified form (Theorem I-6.1) of the classical Riesz's Lemma ([7], [36]), it is proved in Part II that a V-space admits a distinguished basis. It follows (Theorem I-5.6) that an element belongs to the space if and only if it is a sum of a formal series in terms of the elements of a distinguished basis. Thus, the rôle of a distinguished basis in a V-space is similar to the rôle of a complete orthogonal basis in a Hilbert space.

We also consider V-algebras and give theorems on the existence of inverses and on the spectra of elements of a V-algebra. Most of these theorems are simple modifications of the classical theorems of the theory of normed rings ([7], [26]).

Examples of V-spaces and V-algebras will be displayed in Part III. "Asymptotic spaces" are constructed by widening the scope of the method of C. A. Swanson and M. Schulzer [32], [33], referred to above. We also define "moment spaces" in which, for example, one can interpret the methods of Lanczos [21] or Clenshaw [2] for the approximation of the solutions of certain differential equations.

Later Parts will be devoted to the study of linear and non-linear operators on V-spaces. By setting a proper norm on these operators, the set of bounded operators forms a V-space of which the set of bounded linear operators is a subspace.

Elementary theorems of the theory of bounded linear operators on Banach spaces still apply in V-spaces. However, important differences will be exemplified: a continuous linear operator is not necessarily bounded; the uniform boundedness theorem does not hold.

We give a simple characterization of bounded linear operators. As applications of this important theorem we derive a result of H. F. Davis [4] and indicate how asymptotic expansions of the Laplace transforms of certain functions of two variables can be obtained (see V. A. Ditkin and A. P. Prudnikov [6]).

Theorem IV 6.5 allows the comparison of the spectra of two bounded linear operators when the norm of their difference is less than 1. The result is obtained by showing that an inequality proved by C. A. Swanson [34], [35] for linear transformations with eigenvalues on a Hilbert space can be modified into an equality in V-spaces.

The problem considered by C. A. Swanson and M. Schulzer in [32] and [33] is that of the existence and approximation of "asymptotic solutions" of certain equations in Banach spaces. In Part V, we extend the results of Swanson and Schulzer to arbitrary V-spaces and V-algebras (Theorems 2.2, 3.2, 4.3). Our methods of proof are different than those of [32] and [33]. Our hypotheses are weaker and consequently our proofs are more involved. Possible simplifications of the hypotheses are mentioned.

In Part VI we consider continuous linear functionals. It is known that continuous linear functionals on a V-space are bounded (Monna [24], III) and that the Hahn-Banach Theorem is valid (Monna [24], III, Cohen [3], Ingleton [17]); we give a new proof of the latter using distinguished bases.

The main result of this chapter is a representation theorem for linear functionals on certain bounded V-spaces. The representation theorem is a generalization of a theorem of H. F. Davis [4] which asserts that the space of continuous linear functionals on the space of asymptotically convergent power series in a real variable is isomorphic to the space of polynomials in that variable.

It is shown (Section VI-2) that a new norm, called "*norm", can be defined on the set of finite linear combinations of the elements of a distinguished basis of a V-space X, and that, under this norm, this set is a V-space isomorphic to a subspace of the dual of X. This isomorphism is isometric and is obtained by use of a particular type of inner product.

In Parts IV, V and VI, applications of the theorems are shown using some of the examples of asymptotic spaces described in Part III.

The author wishes to express his gratitude to Professor Charles A. Swanson of the University of British Columbia (Vancouver, Canada), whose never failing interest and advice has been invaluable to him in the preparation of this work.

1. Definitions and notations

DEFINITION 1.1. A *pseudo-valued space* is a non-Archimedean pseudo-normed linear space over a trivially valued field with characteristic 0.

Thus, if X is a pseudo-valued space, F its field of scalars, θ and 0 are the additive identities of X and F, respectively, then

(I.1) $|\theta| = 0$,

(I.2) $|\alpha x| = |x|$ for all $x \in X$ and all $\alpha \in F$, $\alpha \neq 0$,

(I.3) $|x+y| \leqq \text{Max}\,\{|x|, |y|\}$ for all $x, y \in X$.

It follows easily that for all $x, y \in X$, $|x| \neq |y|$ implies:

(I.4) $$|x+y| = \text{Max}\,\{|x|, |y|\}.$$

DEFINITION 1.2. A pseudo-valued space X will be called a *valued space* if

(I.5) $$|x| = 0 \quad \text{implies } x = \theta.$$

2. Topological properties

The topology considered on the (pseudo)-valued space X is the topology induced by the metric d:

$$d(x, y) = |x-y|.$$

The open ball $S(x, r)$, the closed ball $S'(x, r)$ and the sphere $B(x, r)$, with center x and radius r, are defined by

(I.6a) $S(x, r) = \{y \in X : d(x, y) < r\}, \quad r > 0,$

(I.6b) $S'(x, r) = \{y \in X : d(x, y) \leqq r\}, \quad r \geqq 0,$

(I.6c) $B(x, r) = \{y \in X : d(x, y) = r\}, \quad r \geqq 0.$

THEOREM 2.1. If X is a (pseudo-) valued space, then

(i) for any $x, y \in X$ and r,

$$S(x, r) = S(y, r) \text{ or } S(x, r) \cap S(y, r) = \emptyset$$
$$S'(x, r) = S'(y, r) \text{ or } S'(x, r) \cap S'(y, r) = \emptyset$$

(where $\emptyset$ denotes the empty set).

(ii) For any $x \in X$ and $r > 0$, $S(x, r)$, $S'(x, r)$, $B(x, r)$ are all closed and open.

THEOREM 2.2. If X is a (pseudo-) valued space and $r > 0$, then

(i) $S(\theta, r)$ and $S'(\theta, r)$ are subspaces of X;

(ii) The quotient topologies on the quotient spaces $X/S(\theta, r)$ and $X/S'(\theta, r)$ are both discrete.

PROOF: (i) is easily verified.

(ii) The natural mapping from a topological group to one of its quotient groups is a continuous open mapping. The points in the quotient groups $X/S(\theta, r)$ and $X/S'(\theta, r)$ are translates of the balls $S(\theta, r)$ and $S'(\theta, r)$ respectively. The balls $S(\theta, r)$ and $S'(\theta, r)$ are both closed and open (Theorem 2.1. (ii)), hence the points in the quotient groups are both closed and open ([28], p. 59).

3. Sequences and series

THEOREM 3.1. If X is a (pseudo-) valued space:

(i) A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} |x_n - x_{n+1}| = 0.$$

(ii) A series $\sum_n x_n$ in X is a Cauchy series if and only if

$$\lim_{n\to\infty} d(x_n, \theta) = \lim_{n\to\infty} |x_n| = 0.$$

The proof of this theorem is omitted. It is a mere modification of the proof of a similar theorem for fields with a non-Archimedean valuation. See [31], p. 28 or [40], p. 240.

The proof of the following theorem is also omitted (cf. Lemma 5.5 below). Part (i) is quoted, without proof, in [29], p. 139. Part (ii) follows from Theorem 3.1 (ii) and inequality (I.3).

THEOREM 3.2. If X is a (pseudo-) valued space:

(i) A convergent series is unconditionally convergent, i.e. any reordering of its terms converges to the same sum(s).

(ii) If $\sum_n x_n$ is convergent and has sum x, then

$$|x| \leqq \sup_n |x_n|. \tag{1.7}$$

4. Compactness

In this section we give a characterization of the compact subsets of a (pseudo-) valued space X.

If the topology of X is discrete, a subset of X is compact if and only if it is finite. Thus X itself is not compact. Since each point forms a neighbourhood of itself, X is locally compact.

If the topology of X is not discrete, then X is neither compact nor locally compact. Indeed every neighbourhood V of θ contains a ball $S(\theta, r)$ for some r. This ball contains a point x such that $|x| \neq 0$. Thus V contains the discrete subspace generated by x.

We shall use the following definition:

Definition 4.1. Let $A \subset X$. The set $\Omega(A)$ defined by

$$\Omega(A) = \{r : |x| = r \text{ for some } x \in A\}$$

will be called the *norm range* of A.

Theorem 4.2. Let X be a (pseudo-) valued space, and A be a subset of X.

(i) A is compact if and only if for each $r > 0$ it is a finite union of disjoint compact subsets $K_1, K_2, \ldots, K_{n(r)}$, such that $x \in K_i$ and $y \in K_j$ implies

$$\text{(I.8)} \qquad \begin{aligned} |x-y| &< r \qquad \text{for } i = j, \\ |x-y| &\geqq r \qquad \text{for } i \neq j. \end{aligned}$$

(ii) If A is compact and does not contain θ, except possibly as an isolated point, then its norm range $\Omega(A)$ is finite.

Proof: (i) A set is certainly compact if it is a finite union of compact sets.

For the converse, let A be compact and $r > 0$ be arbitrary. The family

$$\mathscr{S} = \{S(x, r) : x \in A\}$$

is an open cover of A. We can extract from $\mathscr{S}$ a finite subcover $\{S(x_1, r), S(x_2, r), \ldots, S(x_{n(r)}, r)\}$ such that the $S(x_i, r)$ are disjoint. (See th. 2.1. (i)). Then (I.8) is satisfied with K_i replaced by $S(x_i, r)$. Take $K_i = A \cap S(x_i, r)$. K_i is compact since it is the intersection of a compact set and a closed set (Th. 2.1. (ii)). Then (I.8) holds.

(ii) Since θ is at most an isolated point of A, there exists $r > 0$ such that

$$|x| \geqq r \qquad \text{for all } x \in A, \ x \neq \theta.$$

Consider, for this particular value of r, the sets K_i, $i = 1, 2, \ldots, N_{(r)}$ of (i). Then, $\theta \notin K_i$, $x \in K_i$ and $y \in K_i$ imply

$$|x| \geqq r, \ |y| \geqq r \text{ and } |x-y| < r.$$

By (I.4), $|x| = |y|$ and the conclusion follows.

REMARKS: (i) The fact that the valuation on the field F is the trivial valuation is responsible for a high discretization in a (pseudo-) valued space. As a result, we may say, loosely speaking, that compactness is a very restrictive property and that it is rather difficult for a subset of a (pseudo-) valued space to be compact.

No convex set is compact unless it is reduced to a single point (or to a subset of $S(\theta, 0)$). No set with a non-empty interior is compact unless the space is discrete and the set is finite.

One can expect that compactness will not play an important rôle in this theory.

(ii) The results of Theorem 4.2 may be compared with Property 4, in Theorem 2 of Monna, [24], Part I, page 1048.

Monna has shown that if a non-Archimedean normed linear space over a field of scalars with the trivial valuation is locally compact, then the field of scalars is finite. ([24], Part II, p. 1061.)

5. Distinguished bases

Let A be a subset of a topological linear space X, over a field F. The subspace (A) generated by A is the set of all the finite linear combinations of elements of A. The topological closure of (A) will be denoted by $[A]$ and be called the closed subspace generated by A.

We introduce the following definition:

DEFINITION 5.1. Let A be a subset of a topological linear space X.

(i) A is said to be a *completely independent set* if

$$x \notin [A \backslash \{x\}] \qquad \text{for each } x \in A.$$

(ii) A is called a *complete basis* if it is a completely independent set and $[A] = X$.

Clearly, in an arbitrary topological linear space, a completely independent set is also linearly independent. The converse is not true as is shown by the following example. Let $C[0, 1]$ be the space of all the real valued continuous functions f on $[0, 1]$, with the uniform norm:

$$||f|| = \sup_{0 \leqq \lambda \leqq 1} |f(\lambda)|;$$

let X be the subspace of $C[0, 1]$ generated by $\Phi_0 \cup E_0$, where

$$\begin{aligned} \Phi_0 &= \{\varphi_n : n = 0, 1, 2, \ldots\}, & \varphi_n(\lambda) &= \lambda^n, \\ E_0 &= \{e_{-r} : r > 0\}, & e_{-r}(\lambda) &= e^{-r\lambda}. \end{aligned}$$

It is known (Weierstrass' Theorem) that the set Φ_0 is a completely independent set which is a complete basis but not a Hamel basis of X. Thus $\Phi_0 \cup E_0$ is a linearly independent but not completely independent set.

THEOREM 5.2. A complete basis A of a topological linear space X is also a Hamel basis of X if and only if (A) contains an open set.

PROOF: If A is a Hamel basis $(A) \supset X$. Conversely, suppose that (A) contains an open set. Then (A) contains an interior point and (A) is a subgroup of X. It is known ([18], p. 106) that any subgroup of a topological group which contains an interior point is closed (and open). Thus, A is a linearly independent set and $(A) = [A] = X$.

Returning to the theory of valued spaces, we introduce the notion of distinguishability in the following way.

DEFINITION 5.3. Let A be a non-empty subset of a (pseudo-) valued space X.

(i) A is said to be a *distinguished set* if no element of A has norm equal to 0, and, if for any finite subset of distinct points $x_1, x_2, \ldots, x_n$ of A,

$$|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n| = \operatorname*{Max}_i |x_i|,$$

$\alpha_i \in F$, whenever $\alpha_i \neq 0$ for $i = 1, 2, \ldots, n$.

(ii) A is called a *distinguished basis of* X if A is a distinguished set and a complete basis of A.

In a later paper of this series, we show that there exists a norm on the space X of the example above, under which X is a pseudo-valued space. The Hamel basis $\Phi_0 \cup E_0$ will be shown to be neither a complete nor a distinguished basis.

The essential feature of a distinguished set A in a (pseudo-) valued space is the following: if $x, y \in A$, $x \neq y$ and $|x| = |y| = r$, then $|\alpha x + \beta y| = r$, except when $\alpha = \beta = 0$.

The author has not been able to show the existence of distinguished bases in arbitrary (pseudo-) valued spaces. Nevertheless, under an important additional assumption on the norm range of the space, we shall prove, in Part II, that a (pseudo-) valued space has a distinguished basis. This assumption will be satisfied in all the examples and applications to follow.

In the case of an arbitrary (pseudo-) valued space, we have:

THEOREM 5.4. A (pseudo-) valued space admits a Hamel basis which is a distinguished set.

The proof is identical to the proof of the existence of a Hamel basis in a linear space ([36], p. 45). A distinguished Hamel basis is a maximal distinguished set.

Definition 5.3 (ii) is slightly redundant. Indeed, we shall prove in Theorem 5.6(i) below that a distinguished set is completely independent. Thus a distinguished basis A in a (pseudo-) valued space X is a distinguished subset such that $[A] = X$. To prove Theorem 5.6 we shall need the following lemma, which is an improvement over Theorem 3.2 (ii).

LEMMA 5.5. Let A be a distinguished subset of a (pseudo-) valued space X. Let $\{x_n\}$ be an at most countable subset of A. If $\alpha_n \in F$, $\alpha_n \neq 0$ for each n, and $x = \sum_n \alpha_n x_n$, then

$$|x| = \sup_n |x_n|.$$

PROOF: By Theorem 3.1 (ii), given $r < |x_1|$, there exists N such that for all $n > N$, $|x_n| < r$. By Theorem 3.2 (ii)

$$|\sum_{n \geqq N+1} \alpha_n x_n| \leqq r,$$

and, since A is distinguished,

$$|\sum_{n=1}^{N} \alpha_n x_n| = \max_{1 \leqq n \leqq N} |x_n| \geqq |x_1| > r.$$

Thus,

$$|x| = |\sum_{n=1}^{N} \alpha_n x_n + \sum_{n>N} \alpha_n x_n| = \max_{1 \leqq n \leqq N} |x_n| = \sup_n |x_n|.$$

Note that this supremum is attained.

THEOREM 5.6. Let X be a (pseudo-) valued space.

(i) A distinguished subset A of X is completely independent.

(ii) If A is a distinguished basis of X, then every $x \in X$ can be represented uniquely (except for order) by a series $\sum_{n=1}^{\infty} \alpha_n x_n$, with $x_n \in A$, $\alpha_n \in F$, $n = 1, 2, \ldots$.

PROOF: (i) Let x_0 be an arbitrary point of A. Suppose that $x_0 \in [B]$, where $B = A \backslash \{x_0\}$. Then there exists a sequence $\{y_n\}$

in (B) which converges to x_0. It follows that there exists an element y of (B)

$$y = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m, \quad \alpha_i \neq 0, \; x_i \in B, \quad i = 1, 2, \ldots, m,$$

such that $|y - x_0| < |x_0|$, i.e.

$$|\sum_{i=0}^{m} \alpha_i x_i| < |x_0| \leqq \operatorname*{Max}_{0 \leqq i \leqq m} |x_i|, \quad \alpha_0 = -1.$$

This contradicts the distinguishability of A.

(ii) Since $x \in [A]$, there exists a sequence $\{y_n : n = 1, 2, \ldots\}$ in (A) which converges to x. Let $z_1 = y_1$ and $z_n = y_n - y_{n-1} \in (A)$ for $n = 2, 3, \ldots$. Then the series $\sum_{n=1}^{\infty} z_n$ converges to x.

Let

$$z_n = \alpha_{n1} x_{n1} + \alpha_{n2} x_{n2} + \ldots + \alpha_{n,p(n)} x_{n,p(n)}, \quad \alpha_{nj} \neq 0, \quad x_{nj} \in A.$$

The set $\{x_{nj} : n = 1, 2, \ldots, j = 1, 2, \ldots, p(n)\}$ is a countable set. Let its elements be ordered into a sequence $\{x_m : m = 1, 2, \ldots\}$ such that $|x_m| \geqq |x_{m+1}|$ for all m.

For each integer $m \geqq 1$, there exists an integer $N(m)$ such that

$$|z_n| < |x_m| \text{ for all } n \geqq N(m).$$

Therefore, for each m, the number of integers n such that $x_m = x_{nj}$ for some j, $1 \leqq j \leqq p(n)$, is finite. The series $\sum_{n=1}^{\infty} z_n$ can thus be reordered by grouping the terms in x_m, for each integer m.

The uniqueness (except for order) follows from Lemma 5.5.

Consequences of the above theorem are:

(i) If A is a distinguished subset of X, then

$$\Omega(A) = \Omega((A)) \backslash \{0\} = \Omega([A]) \backslash \{0\}.$$

(ii) If A is a distinguished basis of X, then $\inf \Omega(A) = 0$ when X is not discrete, and in any case $\Omega(A) = \Omega(X) \backslash \{0\}$, i.e. for every $r \in \Omega(X)$, $r \neq 0$, there exists $x \in A$ such that $|x| = r$.

If A is a distinguished basis of a (pseudo) valued space X, the unique series

$$\text{(I.9)} \qquad \sum_{n=1}^{\infty} \alpha_n x_n, \quad x_n \in A, \quad \alpha_n \in F, \quad \alpha_n \neq 0,$$

which converges to a given point $x \in X$, will be called the expansion of x in terms of A. According to Theorem 3.2 (i), the terms of such an expansion can be reordered to give a non-increasing series, i.e. a series such that $|x_n| \geqq |x_{n+1}|$ for all $n \geqq 1$.

NOTATION: If $x \in X$ and $y \in A$, we shall denote by $(x, y)_A$ the coefficient of y in the expansion of x in terms of A. With this notation $(I.9)$ becomes

$$x = \sum_{n=1}^{\infty} (x, x_n)_A x_n. \tag{I.10}$$

Assuming that a (pseudo-) valued space X admits a distinguished basis, we can state

THEOREM 5.7. All distinguished bases of X have the same cardinality.

This theorem justifies:

DEFINITION 5.8. If a (pseudo-) valued space X admits a distinguished basis, the cardinality of this basis will be called the (algebraic) *dimension* of the space.

The proof of Theorem 5.7 is omitted; it is similar to the proof given by Dunford and Schwartz ([7], p. 253) for the invariance of the cardinality of complete orthonormal bases of a Hilbert space [1].

Theorems 5.6 and 5.7 indicate that, to some extent, the rôles of distinguished sets and distinguished bases in a (pseudo-) valued space are similar to the rôles of orthogonal sets and orthonormal bases in a Hilbert space ([7], pp. 252—253).

6. Modification of Riesz's Lemma

We conclude Part I of this work by proving a modified version of Riesz's Lemma. This theorem will be used in Part II to prove the existence of distinguished bases in certain types of pseudo-valued spaces.

In the case of a normed linear space X over the real or complex field with the usual valuations, Riesz's Lemma can be stated as follows ([36], p. 96; also [7], p. 578):

"Let Y be a closed, proper subspace of X. Then for each α such that $0 < \alpha < 1$, there exists a point $x_\alpha \in X$ such that $||x_\alpha|| = 1$ and $||y - x_\alpha|| > \alpha$ for all $y \in Y$."

If X is a (pseudo-) valued space, the above statement must be modified. The reason for the alteration is the impossibility of

[1] If A and B are two distinguished bases of X, the only modification to [7], p. 253, is the replacement of the words "$\cdots$ the inner product of a and b is non-zero $\cdots$", or of the symbol "$\cdots (a, b) \neq 0 \cdots$" by "$\cdots (a, b)_B \neq 0$ and $(b, a)_A \neq 0 \cdots$".

finding, for each x such that $|x| \neq 0$, a scalar α such that $|\alpha x| = 1$ (unless, of course, $|x| = 1$ for all $x \in X$ such that $|x| \neq 0$).

THEOREM 6.1. (Modified Riesz's Lemma)

Let Y be a closed, proper subspace of a (pseudo-) valued space X. For each α such that $0 < \alpha < 1$, there exists a point $x_\alpha \in X$ such that

$$|y-x_\alpha| > \alpha |x_\alpha| \text{ for all } y \in Y.$$

PROOF: (i) If there exists $z \in X$, $|z| \neq 0$, such that

$$|y-z| \geqq |z| \text{ for all } y \in Y,$$

take $x_\alpha = z$ for all α, $0 < \alpha < 1$.

(ii) If (i) fails, let $x_0 \in X\backslash Y$ and choose $y_0 \in Y$ such that

$$|y_0-x_0| < |x_0| \text{ (so that } |y_0| = |x_0|).$$

Define

$$\delta(y) = \frac{|y-x_0|}{|y|}, \quad y \in Y, \quad |y| \neq 0,$$
$$\delta = \inf_{y \in Y} \delta(y).$$

Then, $\delta \leqq \delta(y_0) < 1$. Moreover $\delta(y) < 1$ implies $|y-x_0| < |y|$ and hence $|y| = |x_0|$; therefore, since Y is closed,

$$\delta(y) = \frac{|y-x_0|}{|x_0|}$$

is bounded away from zero for $y \in Y$. Thus

$$0 < \delta < 1.$$

Let α be given, $0 < \alpha < 1$, and let

$$\delta' = \min \{\alpha^{-1}\delta, \tfrac{1}{2}(1+\delta)\}.$$

$\delta < \delta' < 1$. There exists $y_1 \in Y$ such that

$$\delta(y_1) = \frac{|y_1-x_0|}{|x_0|} < \delta'.$$

Let $x_\alpha = x_0-y_1$. Then

$$|x_\alpha| < \delta' |x_0| = \delta' |y_1|. \tag{I.11}$$

Now let $y \in Y$. If $|y-x_\alpha| \geqq |x_\alpha|$ the proof is finished, so we may assume

$$|y-x_\alpha| < |x_\alpha|.$$

Then $|y| = |x_\alpha| < |y_1|$ (by (I.11)) so that $|y+y_1| = |y_1|$. Hence since $y+y_1 \in Y$, we have by the definition of δ:

$$|y-x_\alpha| = |y+y_1-x_0| \geqq \delta|y+y_1| = \delta|y_1|,$$

and by (I.11)

$$|y-x_\alpha| > \frac{\delta}{\alpha\delta'}\,\alpha|x_\alpha| \geqq \alpha|x_\alpha|.$$

This completes the proof.

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(Oblatum 18–3–66)

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