

COMPOSITIO MATHEMATICA

RAM SINGH

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Compositio Mathematica, tome 19, n° 1 (1968), p. 78-82

<http://www.numdam.org/item?id=CM_1968__19_1_78_0>

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On a class of star-like functions¹

by

Ram Singh

1. Introduction

A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular and univalent in $|z| < 1$, is said to be star-like if $\operatorname{Re}\{zf'(z)/f(z)\} \geq 0$ in $|z| < 1$. Various sub classes of the class of star-like functions have been considered by different authors. The class S_α of star-like functions of order α ($0 \leq \alpha \leq 1$) i.e. the class of star-like functions $f(z)$ for which $\operatorname{Re}\{zf'(z)/f(z)\} \geq \alpha$, was first introduced by Robertson [1] and has been studied among others by Merkes [5] and Schild [4]. The class $S_{\frac{1}{2}}$ is particularly important. Marx [2] and Strohäcker [6] proved that if $f(z)$ maps $|z| < 1$ on to a convex domain, then $f(z) \in S_{\frac{1}{2}}$. Somewhat later Gabriel [3] showed that the functions of the class $S_{\frac{1}{2}}$ played an important role in the solution of certain differential equations. In the present paper we investigate the class \bar{S} of star-like functions $f(z)$ such that $\operatorname{Re}\{zf'(z)/f(z)\}$ lies in the circle with centre at $(1, 0)$ and radius unity i.e. the class of functions $f(z)$ such that $|zf'(z)/f(z) - 1| \leq 1$ in $|z| < 1$.

2. A representation formula

THEOREM 1: $f(z) \in \bar{S}$ if and only if

$$(2.1) \quad f(z) = z \exp \left\{ \int_0^z \phi(t) dt \right\},$$

where $\phi(z)$ is regular and $|\phi(z)| \leq 1$ in $|z| < 1$.

PROOF. Suppose that $f(z) \in \bar{S}$. Then $|zf'(z)/f(z) - 1| \leq 1$. Also $\{zf'(z)/f(z) - 1\} = 0$ when z is 0. Therefore, by Schwarz' lemma it follows that

$$(2.2) \quad \frac{zf'(z)}{f(z)} - 1 = z\phi(z),$$

where $\phi(z)$ is regular and $|\phi(z)| \leq 1$ in $|z| \leq 1$.

¹ The author wishes to express his thanks to Prof. V. Singh for helpful guidance.

From (2.2), by integration, it easily follows that $f(z)$ has the representation (2.1).

Conversely, if $f(z)$ has the representation (2.1), then by differentiation etc., we find that $|zf'(z)/f(z)-1| \leq 1$ and therefore $f(z) \in \bar{S}$.

3. Distortion theorems

THEOREM 2. *For all $f(z) \in \bar{S}$, we have*

$$(3.1) \quad |z| e^{-|z|} \leq |f(z)| \leq |z| e^{|z|},$$

and

$$(3.2) \quad (1-|z|) e^{-|z|} \leq |f'(z)| \leq (1+|z|) e^{|z|}.$$

PROOF: Since any function $f(z) \in \bar{S}$ has the representation (2.1), where $\phi(z)$ is regular and $|\phi(z)| \leq 1$ in $|z| < 1$, (3.1) follows immediately.

Inequalities (3.1) are sharp. Equality being attained for the function $f(z) = ze^z$.

Again differentiating (2.1) we get

$$(3.3) \quad f'(z) = [1+z\phi(z)] \exp \left\{ \int_0^z \phi(t) dt \right\}.$$

From the above representation of $f'(z)$, (3.2) at once follows.

The function $f(z) = ze^z$ shows that (3.2) is sharp.

From (3.2) we have the following:

COROLLARY 1: The conformal map of $|z| < 1$ as yielded by functions of class \bar{S} contains all the points of circle $|w| < 1/e$.

4. Coefficient regions

THEOREM 3: *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \bar{S}$, then*

$$(4.1) \quad |a_n| \leq \frac{1}{(n-1)}.$$

PROOF. The proof is based on a method of Clunie. If $f(z) \in \bar{S}$, then we know that

$$(4.2) \quad z f'(z) - f(z) = f(z) \cdot z\phi(z).$$

If we put $\psi(z) = z\phi(z) = \sum_{n=1}^{\infty} b_n z^n$, then $|\psi(z)| < 1$ in $|z| < 1$, and (4.2) may be written as

$$(4.3) \quad \sum_{n=2}^{\infty} (n-1)a_n z^n = [z + \sum_{n=2}^{\infty} a_n z^n] \sum_{n=1}^{\infty} b_n z^n.$$

Comparing the coefficients of z^n on both the sides of (4.3) we get

$$(n-1)a_n = b_1 a_{n-1} + b_2 a_{n-2} + b_3 a_{n-3} + \dots + b_{n-1}, \quad n \geq 2.$$

This shows that the coefficient a_n on the left side of (4.3) depends only on the coefficients a_2, a_3, \dots, a_{n-1} on the right hand side of (4.3). Hence for $n \geq 2$, we may write

$$(4.4) \quad \sum_{k=2}^n (k-1)a_k z^k + \sum_{k=n+1}^{\infty} c_k z^k = [z + \sum_{k=2}^{n-1} a_k z^k] \psi(z)$$

say. Squaring the moduli of both sides of (4.4) and integrating round $|z| = r < 1$, we get, using the fact that $|\psi(z)| < 1$ in $|z| < 1$,

$$\sum_{k=2}^n (k-1)^2 |a_k|^2 r^{2k} + \sum |c_k|^2 r^{2k} < 1 + \sum_{k=2}^{n-1} |a_k|^2$$

making $r \rightarrow 1$, we find that

$$\sum_2^n (k-1)^2 |a_k|^2 \leq 1 + \sum_2^{n-1} |a_k|^2$$

or

$$(n-1)^2 |a_n|^2 \leq 1 - \sum_2^{n-1} (k-2)k |a_k|^2.$$

It, therefore follows that for all $n \geq 2$

$$|a_n| \leq \frac{1}{(n-1)}.$$

Equality is attained for $f(z) = ze^{z^{n-1}/n-1}$, a function of \bar{S} for which $a_2 = a_3 = \dots = a_{n-1} = 0$.

5. Radius of convexity

THEOREM 4. Each function $f(z) \in \bar{S}$ maps

$$(5.1) \quad |z| \leq \frac{\sqrt{13}-3}{2}$$

onto a convex domain.

PROOF. We know that if $f(z) \in \bar{S}$, then

$$\frac{zf'(z)}{f(z)} = 1 + z\phi(z).$$

Logarithmic differentiation of the above equation yields

$$\begin{aligned} \frac{zf''(z)}{f'(z)} &= \frac{zf'(z)}{f(z)} - 1 + \frac{z\phi(z) + z^2\phi'(z)}{1 + z\phi(z)} \\ &= z\phi(z) + \frac{z\phi(z) + z^2\phi'(z)}{1 + z\phi(z)}. \end{aligned}$$

Now making use of the fact that $|\phi'(z)| \leq 1 - |\phi(z)|^2/1 - |z|^2$, we get

$$(5.2) \quad \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{|z|[2|\phi| - 2|\phi||z|^2 + |z| - |z| \cdot |z\phi|^2]}{(1 - |z\phi|)(1 - |z|^2)}.$$

$f(z)$ will map $|z| \leq r$ onto a convex domain if the right hand side of (5.2) is less than or equal to 1 for $|z| \leq r$, i.e. if

$$(5.3) \quad (1 - |z\phi|)(1 - |z|^2) \geq |z|[2|\phi| - 2|\phi| \cdot |z|^2 + |z| - |z| \cdot |z\phi|^2].$$

Putting $|z| = p$, $|\phi| = q$ and $pq = x$, where $0 \leq p \leq 1$, $0 \leq q \leq 1$, $0 \leq x \leq p$, (5.3) may be written as

$$(5.4) \quad p^2x^2 - 3(1 - p^2)x + (1 - 2p^2) \geq 0.$$

We consider $f(x) = p^2x^2 - 3(1 - p^2)x + (1 - 2p^2)$. Clearly $f(x)$ decreases for $0 \leq x \leq p$, if

$$2p^2x - 3(1 - p^2) < 0$$

or

$$2p^2 \cdot p - 3(1 - p^2) < 0$$

or

$$(5.5) \quad p < \frac{1}{2}[(5 - 2\sqrt{6})^{\frac{1}{2}} + (5 + 2\sqrt{6})^{\frac{1}{2}} - 1]$$

The value of p as given by (5.5) is easily seen to be less than $\frac{1}{16}$.

The least value of $f(x)$ is $f(p)$, when p is given by (5.5). Therefore, (5.4) will hold if

$$f(p) \geq 0$$

or

$$p^4 + 3p^3 - 2p^2 - 3p + 1 \geq 0$$

or

$$(1 - p^2)(p^2 + 3p - 1) \leq 0$$

or

$$p^2 + 3p - 1 \leq 0$$

or

$$(5.6) \quad p \leq \frac{\sqrt{13} - 3}{2}.$$

Clearly the value of p as given by (5.6) is less than that given by (5.5). Hence every function $f(z) \in \bar{S}$ maps $|z| \leq \sqrt{13} - 3/2$ onto a convex domain.

The function $f_0(z) = ze^z$ shows that this bound for the radius of convexity cannot be improved.

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