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KULDIP KUMAR

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On the geometric means of an integral function

by

Kuldip Kumar

1.

Let $f(z)$ be an integral function of order ρ and lower order λ and

$$(1.1) \quad \lim_{r \rightarrow \infty} \frac{\sup \log n(r)}{\inf \log r} = \frac{\rho_1}{\lambda_1},$$

$n(r)$ being the number of zeros of $f(z)$ in $|z| \leq r$. Let $G(r)$ and $g_\delta(r)$ denote the geometric means of $|f(z)|$, defined as

$$(1.2) \quad G(r) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \right\}$$

and

$$(1.3) \quad g_\delta(r) = \exp \left\{ \frac{(\delta+1)}{2\pi r^{\delta+1}} \int_0^r \int_0^{2\pi} \log |f(xe^{i\theta})| x^\delta dx d\theta \right\}$$

In this paper we have obtained some of the properties of $G(r)$ and $g_\delta(r)$. Let

$$(1.4) \quad N(r) = \int_0^r \frac{n(x)}{x} dx$$

and

$$(1.5) \quad \lim_{r \rightarrow \infty} \frac{\sup n(r)}{\inf r^\rho} = \frac{c}{d}.$$

Using Jensen's formula in (1.2), we have

$$(1.6) \quad \log G(r) = \log |f(0)| + \int_0^r \frac{n(x)}{x} dx.$$

From (1.1), we have, for any $\varepsilon > 0$ and $r > r_0 = r_0(\varepsilon)$

$$r^{\lambda_1 - \varepsilon} < n(r) < r^{\rho_1 + \varepsilon}.$$

Using this in (1.6), we have, for almost all values of $r > r_0$

$$r^{\lambda_1-1-\varepsilon}G(r) < G'(r) < r^{\rho_1-1+\varepsilon}G(r).$$

Again from (1.5), we have for any $\varepsilon > 0$ and $r > r_0$

$$(d-\varepsilon)r^\rho < n(r) < (c+\varepsilon)r^\rho.$$

Substituting for $n(r)$ from (1.6), we have, for almost all $r > r_0$

$$(d-\varepsilon)r^{\rho-1}G(r) < G'(r) < (c+\varepsilon)r^{\rho-1}G(r).$$

2.

We shall now obtain some of the properties of $g_\delta(r)$. We may write (1.3) as

$$\log g_\delta(r) = \frac{(\delta+1)}{r^{\delta+1}} \int_0^r \log G(x)x^\delta dx.$$

Now, using (1.4) and (1.6), we get

$$(2.1) \quad \log g_\delta(r) = 0(1) + \frac{(\delta+1)}{r^{\delta+1}} \int_{r_0}^r N(x)x^\delta dx.$$

THEOREM 1. *Let $f(z)$ be an integral function of order ρ ($0 < \rho < \infty$) and let $f(0) \neq 0$. Further,*

(i) *if*

$$\liminf_{r \rightarrow \infty} \frac{N(r)}{r^\rho} = \beta,$$

then

$$\liminf_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} \geq \frac{\beta(\delta+1)}{(\rho+\delta+1)};$$

(ii) *if*

$$\limsup_{r \rightarrow \infty} \frac{N(r)}{r^\rho} = \alpha,$$

then

$$\limsup_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} \leq \frac{\alpha(\delta+1)}{(\rho+\delta+1)}.$$

PROOF. (i) Since

$$\liminf_{r \rightarrow \infty} \frac{N(r)}{r^\rho} = \beta,$$

therefore, for any $\varepsilon > 0$ and $r > r_1 = r_1(\varepsilon)$, we have

$$N(r) > (\beta - \varepsilon)r^\rho,$$

and so from (2.1)

$$\log g_\delta(r) > o(1) + \frac{(\beta - \varepsilon)(\delta + 1)}{(\rho + \delta + 1)} (r^{\rho + \delta + 1} - r_0^{\rho + \delta + 1})r^{-\delta - 1}.$$

Taking limit on both the sides leads to

$$\liminf_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} \geq \frac{\beta(\delta + 1)}{(\rho + \delta + 1)}.$$

(ii) If

$$\limsup_{r \rightarrow \infty} \frac{N(r)}{r^\rho} = \alpha,$$

we have, for any $\varepsilon > 0$ and $r > r_1 = r_1(\varepsilon)$, $N(r) < (\alpha + \varepsilon)r^\rho$. Substituting this in (2.1), integrating and proceeding to limits, the result follows.

THEOREM 2. *Let $f(z)$ be an integral function of finite non-integral order ρ , and let*

$$\lim_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} = \nu \quad \text{and} \quad \lim_{r \rightarrow \infty} \frac{\log G(r)}{r^\rho} = \mu,$$

then

$$\nu(\rho + \delta + 1) = \mu(\delta + 1).$$

PROOF. Since

$$\lim_{r \rightarrow \infty} \frac{\log g_\delta(r)}{r^\rho} = \nu,$$

therefore,

$$r^\rho(\nu - \varepsilon) < \log g_\delta(r) < r^\rho(\nu + \varepsilon), \quad \text{for } r > r_0(\varepsilon).$$

Hence,

$$\begin{aligned} \frac{(\delta + 1)}{r^{\delta + 1}} \int_{(1-\eta)r}^r \log G(x)x^\delta dx &= \frac{(\delta + 1)}{r^{\delta + 1}} \int_{r_0}^r \log G(x)x^\delta dx \\ &\quad - \frac{(\delta + 1)}{r^{\delta + 1}} \int_{r_0}^{(1-\eta)r} \log G(x)x^\delta dx, \quad \left(0 < \eta < \frac{1}{\delta + 1}\right) \\ &= o(1) + \log g_\delta(r) - (1-\eta)^{\delta + 1} \log g_\delta\{(1-\eta)r\} \\ &> o(1) + \nu\{(\rho + \delta + 1)\eta - \dots\}r^\rho - \varepsilon\{2 - (\rho + \delta + 1)\eta + \dots\}r^\rho. \end{aligned}$$

But,

$$\begin{aligned} \frac{(\delta+1)}{r^{\delta+1}} \int_{(1-\eta)r}^r \log G(x)x^\delta dx &\leq \frac{(\delta+1) \log G(r)}{r^{\delta+1}} \int_{(1-\eta)r}^r x^\delta dx \\ &< \frac{(\delta+1)\eta \log G(r)}{1-(\delta+1)\eta}, \quad \text{for } \rightarrow (\delta+1)\eta < 1. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\log G(r)}{r^\rho} &> o(1) + \frac{\nu\{(\rho+\delta+1)\eta - \dots\}\{1-(\delta+1)\eta\}}{(\delta+1)\eta} \\ &\quad - \frac{\varepsilon\{2-(\rho+\delta+1)\eta + \dots\}\{1-(\delta+1)\eta\}}{(\delta+1)\eta}. \end{aligned}$$

Since η is arbitrary, we get

$$(2.2) \quad \liminf_{r \rightarrow \infty} \frac{\log G(r)}{r^\rho} \geq \frac{\nu(\rho+\delta+1)}{(\delta+1)}.$$

Further,

$$\begin{aligned} \frac{(\delta+1)}{r^{\delta+1}} \int_r^{(1+\eta)r} \log G(x)x^\delta dx &= \frac{(\delta+1)}{r^{\delta+1}} \int_{r_0}^{(1+\eta)r} \log G(x)x^\delta dx \\ &\quad - \frac{(\delta+1)}{r^{\delta+1}} \int_{r_0}^r \log G(x)x^\delta dx < o(1) + \nu\{(\rho+\delta+1)\eta + \dots\} \\ &\quad \times r^\rho + \varepsilon\{2+(\rho+\delta+1)\eta + \dots\}r^\rho, \quad \text{for } \eta < 1, \end{aligned}$$

but,

$$\frac{(\delta+1)}{r^{\delta+1}} \int_r^{(1+\eta)r} \log G(x)x^\delta dx > (\delta+1)\eta \log G(r),$$

hence,

$$\frac{\log G(r)}{r^\rho} < o(1) + \frac{\nu\{(\rho+\delta+1)\eta + \dots\}}{(\delta+1)\eta} + \frac{\varepsilon\{2+(\rho+\delta+1)\eta + \dots\}}{(\delta+1)\eta}.$$

Therefore we get

$$(2.3) \quad \limsup_{r \rightarrow \infty} \frac{\log G(r)}{r^\rho} \leq \frac{\nu(\rho+\delta+1)}{(\delta+1)}.$$

Combining (2.2) and (2.3), we get the result.

3.

Combining (1.2) and (1.3), we obtain

$$\frac{g_\delta(r)}{G(r)} = \exp \left\{ \frac{-1}{r^{\delta+1}} \int_0^r x^{\delta+1} \frac{d}{dx} (\log G(x)) dx \right\}.$$

Using (1.6) in this, we get

$$(3.1) \quad \left\{ \frac{g_\delta(r)}{G(r)} \right\}^{1/N(r)} = \exp \left\{ \frac{-1}{r^{\delta+1}N(r)} \int_0^r n(x)x^\delta dx \right\}.$$

Let us set

$$\limsup_{r \rightarrow \infty} \left[\frac{\{g_\delta(r)\}^{1/N(r)}}{G(r)} \right] = \frac{P}{p}.$$

We now prove the following:

THEOREM 3. *If $f(z)$ is an integral function of order ρ ($0 < \rho < \infty$) and $f(0) \neq 0$, such that $n(r) \sim \phi(r)r^{\rho_1}$, where $\phi(r)$ is a positive continuous and indefinitely increasing function of r and $\phi(cr) \sim \phi(r)$ as $r \rightarrow \infty$ for every constant $c > 0$, then*

$$P = p = \exp \left\{ \frac{-\rho_1}{\rho_1 + \delta + 1} \right\}.$$

PROOF. Since $n(r) \sim \phi(r)r^{\rho_1}$ we have for any $\varepsilon > 0$ and $r \geq r_0(\varepsilon)$

$$(3.2) \quad (1 - \varepsilon)\phi(r)r^{\rho_1} < n(r) < (1 + \varepsilon)\phi(r)r^{\rho_1}$$

or

$$(1 - \varepsilon) \int_{r_0}^r \phi(x)x^{\rho_1 + \delta} dx < \int_{r_0}^r n(x)x^\delta dx < (1 + \varepsilon) \int_{r_0}^r \phi(x)x^{\rho_1 + \delta} dx.$$

Now, by Lemma V ([1], p. 54),

$$\int_{r_0}^r \phi(u)u^{\alpha-1} du \sim \frac{\phi(r)r^\alpha}{\alpha},$$

for every positive α , and so we get

$$(3.3) \quad \frac{(1 - \varepsilon)}{\rho_1 + \delta + 1} \phi(r)r^{\rho_1 + \delta + 1} + o(1) < \int_0^r n(x)x^\delta dx < \frac{(1 + \varepsilon)}{\rho_1 + \delta + 1} \phi(r)r^{\rho_1 + \delta + 1} + o(1).$$

Again, from (3.2), we have

$$(1 - \varepsilon) \int_{r_0}^r \phi(x)x^{\rho_1 - 1} dx < \int_{r_0}^r \frac{n(x)}{x} dx < (1 + \varepsilon) \int_{r_1}^r \phi(x)x^{\rho_1 - 1} dx,$$

giving,

$$(3.4) \quad \frac{(1 - \varepsilon)}{\rho_1} \phi(r)r^{\rho_1} + o(1) < N(r) < \frac{(1 + \varepsilon)}{\rho_1} \phi(r)r^{\rho_1} + o(1).$$

Combining (3.3) and (3.4) leads to

$$\frac{-\rho_1}{(\rho_1 + \delta + 1)} \left[\frac{(1 - \varepsilon)\phi(r)r^{\rho_1} + o(1)}{(1 + \varepsilon)\phi(r)r^{\rho_1}} \right] > \frac{-1}{r^{\delta+1}N(r)} \int_0^r n(x)x^\delta dx$$

$$> \frac{-\rho_1}{(\rho_1 + \delta + 1)} \left[\frac{(1 + \varepsilon)\phi(r)r^{\rho_1} + o(1)}{(1 - \varepsilon)\phi(r)r^{\rho_1}} \right].$$

Taking exponentials and proceeding to limits, we have, since ε is arbitrary and $n(r) \sim \phi(r)r^{\rho_1}$,

$$\lim_{r \rightarrow \infty} \exp \left\{ \frac{-1}{r^{\delta+1}N(r)} \int_0^r n(x)x^\delta dx \right\} = \exp \left\{ \frac{-\rho_1}{\rho_1 + \delta + 1} \right\}.$$

THEOREM 4. *If $f(z)$ has at least one zero and $f(0) \neq 0$, then*

- (i) $e^{-1} \leq p \leq P \leq 1;$
- (ii) $P \geq \exp \left(\frac{-\lambda_1}{\delta + 1} \right).$

PROOF. (i) Integrating by parts the integral in (3.1), we get

$$\left\{ \frac{g_\delta(r)}{G(r)} \right\}^{1/N(r)} = \exp \left\{ -1 + \frac{(\delta + 1)}{r^{\delta+1}N(r)} \int_0^r N(x)x^\delta dx \right\}.$$

Since $N(r)$ is non-decreasing function of r , we get

$$p \geq e^{-1} \quad \text{and} \quad P \leq 1.$$

Since $n(r)$ is non-decreasing function of r , (3.1) gives

$$\left\{ \frac{g_\delta(r)}{G(r)} \right\}^{1/N(r)} > \exp \left\{ \frac{-n(r)}{(\delta + 1)N(r)} \right\}.$$

But we know ([2], p. 17) that

$$\liminf_{r \rightarrow \infty} \frac{n(r)}{N(r)} \leq \lambda_1,$$

and so

$$\limsup_{r \rightarrow \infty} \left[\left\{ \frac{g_\delta(r)}{G(r)} \right\}^{1/N(r)} \right] \geq \exp \left(\frac{-\lambda_1}{\delta + 1} \right).$$

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Department of Mathematics
and Astronomy,
Lucknow University,
Lucknow (India)