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SATYA NARAIN SRIVASTAVA

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**On the mean values of integral functions  
and their derivatives represented  
by Dirichlet series \***

by

Satya Narain Srivastava

**1**

Consider the Dirichlet series

$$(1.1) \quad f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \quad (s = \sigma + it, \lambda_1 \geq 0, \lambda_n < \lambda_{n+1} \rightarrow \infty \text{ with } n).$$

Let  $\sigma_c$  and  $\sigma_a$  be the abscissa of convergence and the abscissa of absolute convergence, respectively, of  $f(s)$ . If  $\sigma_c = \sigma_a = \infty$ ,  $f(s)$  represents an integral function.

Let the maximum modulus over a vertical line be as

$$M(\sigma) = \text{l.u.b.}_{-\infty < t < \infty} |f(\sigma + it)|$$

and the maximum term as

$$\mu(\sigma) = \max_{n \geq 0} |a_n e^{\lambda_n(\sigma + it)}|.$$

If

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = D < \infty,$$

we know ([1], p. 68)

$$(1.2) \quad M(\sigma) < \mu(\sigma + D + \varepsilon), \quad (\varepsilon > 0; \sigma > \sigma(\varepsilon)).$$

The mean values of  $f(s)$  are defined as

$$(1.3) \quad I_2(\sigma) = I_2(\sigma, f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^2 dt$$

$$(1.4) \quad m_{2,k}(\sigma) = m_{2,k}(\sigma, f) = \lim_{T \rightarrow \infty} \frac{1}{T e^{k\sigma}} \int_0^\sigma \int_{-T}^T |f(x + it)|^2 e^{kx} dx dt,$$

where  $k$  is a positive number.

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In this paper we obtain a lower bound of  $m_{2,k}(\sigma, f^{(1)})$  in terms of  $m_{2,k}(\sigma)$  and  $\sigma$ , where  $m_{2,k}(\sigma, f^{(1)})$  is the mean value of  $f^{(1)}(s)$ , the first derivative of  $f(s)$ , that is

$$(1.5) \quad m_{2,k}(\sigma, f^{(1)}) = \lim_{T \rightarrow \infty} \frac{1}{T e^{k\sigma}} \int_0^\sigma \int_{-T}^T |f^{(1)}(x+it)|^2 e^{kx} dx dt.$$

We also study some properties of  $m_{2,k}(\sigma)$ .

We first prove the following lemmas.

**LEMMA 1.**  $m_{2,k}(\sigma)$  is a steadily increasing function of  $\sigma$ .

**PROOF:** We have

$$|f(s)|^2 = \sum_{n=1}^\infty |a_n|^2 e^{2\sigma\lambda_n} + \sum_{m \neq n} a_m \bar{a}_n e^{\sigma(\lambda_m + \lambda_n) + it(\lambda_m - \lambda_n)},$$

the series on the right being absolutely and uniformly convergent in any finite  $t$ -range. Hence integrating term by term we obtain

$$\frac{1}{2T} \int_{-T}^T |f(s)|^2 dt = \sum_{n=1}^\infty |a_n|^2 e^{2\sigma\lambda_n} + \sum_{m \neq n} a_m \bar{a}_n \frac{\sin T(\lambda_m - \lambda_n)}{T(\lambda_m - \lambda_n)}.$$

The term involving  $T$  is bounded for all  $T, m$  and  $n$  so that the double series converges uniformly with respect to  $T$  and each term tends to zero as  $T \rightarrow \infty$ . Thus we get

$$(1.6) \quad I_2(\sigma) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(s)|^2 dt = \sum_{n=1}^\infty |a_n|^2 e^{2\sigma\lambda_n}.$$

The series in (1.6) is absolutely and uniformly convergent and so again integrating term by term we obtain

$$(1.7) \quad \begin{aligned} m_{2,k}(\sigma) &= \lim_{T \rightarrow \infty} \frac{1}{T e^{k\sigma}} \int_0^\sigma \int_{-T}^T |f(x+it)|^2 e^{kx} dx dt \\ &= 2 \sum_{n=1}^\infty |a_n|^2 \frac{(e^{2\lambda_n\sigma} - e^{-k\sigma})}{2\lambda_n + k}. \end{aligned}$$

$m_{2,k}(\sigma)$  is steadily increasing follows from (1.7).

**LEMMA 2.**  $\log m_{2,k}(\sigma)$  is a convex function of  $\sigma$ .

**PROOF.** From (1.3) and (1.4), we have

$$m_{2,k}(\sigma) = \frac{2}{e^{k\sigma}} \int_0^\sigma I_2(x) e^{kx} dx.$$

Therefore,

$$\begin{aligned} \frac{d(\log m_{2,k}(\sigma))}{d(\sigma)} &= \left\{ \frac{2I_2(\sigma) - km_{2,k}(\sigma)}{m_{2,k}(\sigma)} \right\} \\ &= \left\{ \frac{2I_2(\sigma)}{m_{2,k}(\sigma)} - k \right\}, \end{aligned}$$

which increases with  $\sigma > \sigma_1$ , since  $e^{k\sigma}I_2(\sigma)$  is a convex function of  $e^{k\sigma}m_{2,k}(\sigma)$  ([2], p. 135).

Hence,

$$\frac{d^2(\log m_{2,k}(\sigma))}{d\sigma^2} > 0 \quad \text{for } \sigma > \sigma_1.$$

LEMMA 3. *If  $f(s)$  is an integral function of Ritt-order  $\rho$  and lower order  $\lambda$  and  $D < \infty$ , then*

$$\lim_{\sigma \rightarrow \infty} \frac{\sup \log \log m_{2,k}(\sigma)}{\inf \sigma} = \frac{\rho}{\lambda}.$$

PROOF. We have from (1.6)

$$(1.8) \quad \{\mu(\sigma)\}^2 \leq I_2(\sigma) \leq \{M(\sigma)\}^2.$$

If  $D < \infty$ , we get from (1.2) and (1.8)

$$\mu(\sigma) \leq \{I_2(\sigma)\}^{\frac{1}{2}} \leq M(\sigma) < \mu(\sigma + D + \varepsilon) \quad \varepsilon > 0; \sigma > \sigma(\varepsilon).$$

Therefore,

$$(1.9) \quad \begin{aligned} \lim_{\sigma \rightarrow \infty} \frac{\sup \log \log I_2(\sigma)}{\inf \sigma} &= \lim_{\sigma \rightarrow \infty} \frac{\sup \log \log \mu(\sigma)}{\inf \sigma} \\ &= \lim_{\sigma \rightarrow \infty} \frac{\sup \log \log M(\sigma)}{\inf \sigma} = \frac{\rho}{\lambda}. \end{aligned}$$

Also, since  $I_2(x)$  is an increasing function of  $x$ ,

$$\begin{aligned} m_{2,k}(\sigma) &= \frac{2}{e^{k\sigma}} \int_0^\sigma I_2(x) e^{kx} dx \\ &\leq \frac{2I_2(\sigma)}{e^{k\sigma}} \int_0^\sigma e^{kx} dx \\ &= \frac{2}{k} I_2(\sigma) (1 - e^{-k\sigma}) \end{aligned}$$

and we get, on using (1.9),

$$\lim_{\sigma \rightarrow \infty} \frac{\sup \log \log m_{2,k}(\sigma)}{\inf \sigma} \leq \lim_{\sigma \rightarrow \infty} \frac{\sup \log \log I_2(\sigma)}{\inf \sigma} = \frac{\rho}{\lambda}.$$

Further for  $h > 0$

$$\begin{aligned} m_{2,k}(\sigma+h) &= \frac{2}{e^{k(\sigma+h)}} \int_0^{\sigma+h} I_2(x)e^{kx} dx \\ &\geq \frac{2}{e^{k(\sigma+h)}} \int_\sigma^{\sigma+h} I_2(x)e^{kx} dx \\ &\geq \frac{2I_2(\sigma)}{k} (1-e^{-kh}) \end{aligned}$$

and we again get, on using (1.9),

$$\lim_{\sigma \rightarrow \infty} \sup \inf \frac{\log \log m_{2,k}(\sigma)}{\sigma} \geq \lim_{\sigma \rightarrow \infty} \sup \inf \frac{\log \log I_2(\sigma)}{\sigma} = \frac{\rho}{\lambda},$$

and thus Lemma 3 follows.

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**THEOREM 1.** *If  $m_{2,k}(\sigma, f^{(1)})$  is the mean value of  $f^{(1)}(s)$  the first derivative of an integral function  $f(s)$  other than an exponential polynomial, then*

$$(2.1) \quad m_{2,k}(\sigma, f^{(1)}) \geq \frac{1}{2^2} \left\{ \frac{\log m_{2,k}(\sigma) - 1}{\sigma} \right\}^2 m_{2,k}(\sigma)$$

for  $\sigma > \sigma_0$ , where  $\sigma_0$  is a number depending on the function  $f$ .

**PROOF.** We have

$$\begin{aligned} m_{2,k}(\sigma, f^{(1)}) &= \lim_{T \rightarrow \infty} \frac{1}{Te^{k\sigma}} \int_0^\sigma \int_{-T}^T |f^{(1)}(x+it)|^2 e^{kx} dx dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{Te^{k\sigma}} \int_0^\sigma \int_{-T}^T \left| \lim_{\varepsilon \rightarrow 0} \frac{f(x+it) - f(x(1-\varepsilon)+it)}{\varepsilon x} \right|^2 e^{kx} dx dt \\ &\geq \lim_{\varepsilon \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{\varepsilon^2 \sigma^2 Te^{k\sigma}} \int_0^\sigma \int_{-T}^T \{|f(x+it)| - |f(x(1-\varepsilon)+it)|\}^2 e^{kx} dx dt. \end{aligned}$$

By Minkowski's inequality ([3], p. 384)

$$\begin{aligned} \left[ \int_{-T}^T \{|f(x+it)| - |f(x(1-\varepsilon)+it)|\}^2 dt \right]^{\frac{1}{2}} &\geq \left\{ \int_{-T}^T |f(x+it)|^2 dt \right\}^{\frac{1}{2}} \\ &\quad - \left\{ \int_{-T}^T |f(x(1-\varepsilon)+it)|^2 dt \right\}^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned}
 m_{2,k}(\sigma, f^{(1)}) &\geq \lim_{\varepsilon \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{\varepsilon^2 \sigma^2 T e^{k\sigma}} \int_0^\sigma \left[ \left\{ \int_{-T}^T |f(x+it)|^2 dt \right\}^{\frac{1}{2}} \right. \\
 &\quad \left. - \left\{ \int_{-T}^T |f(x(1-\varepsilon)+it)|^2 dt \right\}^{\frac{1}{2}} \right]^2 e^{kx} dx \\
 &= \lim_{\varepsilon \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{\varepsilon^2 \sigma^2 T e^{k\sigma}} \int_0^\sigma \left[ \left\{ e^{kx} \int_{-T}^T |f(x+it)|^2 dt \right\}^{\frac{1}{2}} \right. \\
 &\quad \left. - \left\{ e^{kx} \int_{-T}^T |f(x(1-\varepsilon)+it)|^2 dt \right\}^{\frac{1}{2}} \right]^2 dx.
 \end{aligned}$$

Again, using Minkowski's inequality, we obtain

$$\begin{aligned}
 m_{2,k}(\sigma, f^{(1)}) &\geq \lim_{\varepsilon \rightarrow 0} \lim_{T \rightarrow \infty} \frac{1}{\varepsilon^2 \sigma^2 T e^{k\sigma}} \left[ \left\{ \int_0^\sigma e^{kx} \int_{-T}^T |f(x+it)|^2 dx dt \right\}^{\frac{1}{2}} \right. \\
 &\quad \left. - \left\{ \int_0^\sigma e^{kx} \int_{-T}^T |f(x(1-\varepsilon)+it)|^2 dx dt \right\}^{\frac{1}{2}} \right]^2 \\
 &\geq \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 \sigma^2} \left[ \{m_{2,k}(\sigma)\}^{\frac{1}{2}} - \left\{ \frac{e^{-\varepsilon k\sigma}}{(1-\varepsilon)} e^{k((\sigma-\sigma\varepsilon)\varepsilon/(1-\varepsilon))} m_{2,k}(\sigma-\sigma\varepsilon) \right\}^{\frac{1}{2}} \right]^2 \\
 &= \lim_{\varepsilon \rightarrow 0} \left[ \frac{\{m_{2,k}(\sigma)\}^{\frac{1}{2}} - \{(1-\varepsilon)^{-1} m_{2,k}(\sigma-\sigma\varepsilon)\}^{\frac{1}{2}}}{\varepsilon\sigma} \right]^2.
 \end{aligned}$$

Now take  $g(\sigma) = \log m_{2,k}(\sigma)/\sigma$ ;  $g(\sigma)$  is a positive indefinitely increasing function of  $\sigma$  for  $\sigma > \sigma_0 = \sigma_0(f)$ , in fact  $\log m_{2,k}(\sigma)$  is a convex function of  $\sigma$ , and so we have

$$\begin{aligned}
 m_{2,k}(\sigma, f^{(1)}) &\geq \lim_{\varepsilon \rightarrow 0} \left\{ \frac{e^{\sigma g(\sigma)/2} - (1-\varepsilon)^{-\frac{1}{2}} e^{(\sigma-\sigma\varepsilon)g(\sigma-\sigma\varepsilon)/2}}{\varepsilon\sigma} \right\}^2 \\
 &\geq \lim_{\varepsilon \rightarrow 0} \left\{ \frac{e^{\sigma g(\sigma)/2} - (1+\varepsilon/2 + \frac{3}{8}\varepsilon^2 + \dots) e^{(\sigma-\sigma\varepsilon)g(\sigma)/2}}{\varepsilon\sigma} \right\}^2 \\
 &= \left\{ \frac{g(\sigma)}{2} e^{\sigma g(\sigma)/2} - \frac{1}{2\sigma} e^{\sigma g(\sigma)/2} \right\}^2 \\
 &= \frac{1}{2^2} \left\{ \frac{\log m_{2,k}(\sigma) - 1}{\sigma} \right\}^2 m_{2,k}(\sigma).
 \end{aligned}$$

**COROLLARY 1.** If  $m_{2,k}(\sigma, f^{(1)})$  is the mean value of  $f^{(1)}(s)$ , the first derivative of an integral function  $f(s)$  other than an exponential polynomial, then

$$\lim_{\sigma \rightarrow \infty} \sup \inf \frac{\log \left\{ \frac{m_{2,k}(\sigma, f^{(1)})}{m_{2,k}(\sigma)} \right\}^{\frac{1}{2}}}{\sigma} \geq \frac{\rho}{\lambda},$$

where  $\rho$  and  $\lambda$  are the Ritt-order and lower order of  $f(s)$  respectively, and  $D < \infty$ .

This follows from Theorem 1 and Lemma 3.

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**THEOREM 2.** *Let  $m_{2,k}(\sigma, f^{(r)})$ , ( $r = 1, 2, \dots, p$ ), be the mean value of  $f^{(r)}(s)$ , the  $r$ -th derivative of an integral function  $f(s)$  other than an exponential polynomial. If  $\lambda \geq \delta > 0$  and  $D < \infty$ , then*

$$m_{2,k}(\sigma) < m_{2,k}(\sigma, f^{(1)}) < \dots < m_{2,k}(\sigma, f^{(p)})$$

for  $\sigma > \sigma_0 = \max(\sigma_1, \sigma_2, \dots, \sigma_p)$ .

**PROOF:** Writing the above corollary for the  $r$ -th derivative, we have

$$\lim_{\sigma \rightarrow \infty} \sup \frac{\log \left\{ \frac{m_{2,k}(\sigma, f^{(r)})}{m_{2,k}(\sigma, f^{(r-1)})} \right\}^{\frac{1}{2}}}{\sigma} \geq \frac{\rho}{\lambda}.$$

Therefore,

$$m_{2,k}(\sigma, f^{(r)}) > e^{2\sigma(\lambda-\epsilon)} m_{2,k}(\sigma, f^{(r-1)})$$

for  $\sigma > \sigma_r$ .

If  $\lambda \geq \delta > 0$

$$m_{2,k}(\sigma, f^{(r)}) > m_{2,k}(\sigma, f^{(r-1)})$$

for  $\sigma > \sigma_r$ .

Giving  $r$  the values  $r = 1, 2, \dots, p$ , we get

$$m_{2,k}(\sigma) < m_{2,k}(\sigma, f^{(1)}) < \dots < m_{2,k}(\sigma, f^{(p)})$$

for  $\sigma > \sigma_0 = \max(\sigma_1, \sigma_2, \dots, \sigma_p)$ .

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**THEOREM 3.** *Let  $m_{2,k}(\sigma, f^{(p)})$  be the mean value of  $f^{(p)}(s)$  the  $p$ -th derivative of an integral function  $f(s)$  other than an exponential polynomial. If  $\lambda \geq \delta > 0$  and  $D < \infty$ , then*

$$(4.1) \quad m_{2,k}(\sigma, f^{(p)}) > \frac{1}{2^{2p}} \left\{ \frac{\log m_{2,k}(\sigma) - 1}{\sigma} \right\}^{2p} m_{2,k}(\sigma)$$

for  $\sigma > \sigma_0 = \max(\sigma_1, \sigma_2, \dots, \sigma_{p-1}, \sigma_1^1, \sigma_2^1, \dots, \sigma_p^1)$ .

**PROOF:** Writing (2.1) for the  $r$ -th derivative, we have

$$\frac{m_{2,k}(\sigma, f^{(r)})}{m_{2,k}(\sigma, f^{(r-1)})} \geq \frac{1}{2^2} \left\{ \frac{\log m_{2,k}(\sigma, f^{(r-1)}) - 1}{\sigma} \right\}^2$$

for  $\sigma > \sigma_r^1$ .

Giving  $r$  the values  $r = 1, 2, \dots, p$  and multiplying them, we get

$$\frac{m_{2,k}(\sigma, f^{(p)})}{m_{2,k}(\sigma)} \geq \frac{1}{2^{2p}} \left\{ \frac{\log m_{2,k}(\sigma, f^{(p-1)}) - 1}{\sigma} \right\}^2 \left\{ \frac{\log m_{2,k}(\sigma, f^{(p-2)}) - 1}{\sigma} \right\}^2 \dots \left\{ \frac{\log m_{2,k}(\sigma) - 1}{\sigma} \right\}^2$$

for  $\sigma > \sigma_0^1 = \max(\sigma_1^1, \sigma_2^1, \dots, \sigma_p^1)$ .

Using Theorem 2, we get

$$\frac{m_{2,k}(\sigma, f^{(p)})}{m_{2,k}(\sigma)} > \frac{1}{2^{2p}} \left\{ \frac{\log m_{2,k}(\sigma) - 1}{\sigma} \right\}^{2p}$$

for  $\sigma > \sigma_0 = \max(\sigma_1, \sigma_2, \dots, \sigma_{p-1}, \sigma_1^1, \sigma_2^1, \dots, \sigma_p^1)$ .

**COROLLARY 1.** If  $m_{2,k}(\sigma, f^{(p)})$  is the mean value of  $f^{(p)}(s)$ , the  $p$ -th the derivative of an integral function  $f(s)$  other than an exponential polynomial, then

$$\lim_{\sigma \rightarrow \infty} \sup \frac{\log \left\{ \frac{m_{2,k}(\sigma, f^{(p)})}{m_{2,k}(\sigma)} \right\}^{1/2p}}{\inf \sigma} \geq \frac{\rho}{\lambda},$$

where  $\rho$  and  $\lambda$  are the Ritt-order and lower order of  $f(s)$  respectively, and  $D < \infty$ .

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Department of Mathematics  
and Astronomy,  
Lucknow University,  
Lucknow (India)