DANA SCOTT

Extending the topological interpretation to intuitionistic analysis

Compositio Mathematica, tome 20 (1968), p. 194-210
<http://www.numdam.org/item?id=CM_1968__20__194_0>

Extending the topological interpretation to intuitionistic analysis

Dedicated to A. Heyting on the occasion of his 70th birthday

by

Dana Scott

The well-known Stone-Tarski interpretation of the intuitionistic propositional logic was extended by Mostowski to the quantifier logic in a natural way. For details and references the reader may consult the work Rasiowa-Sikorski [5], where intuitionistic theories are discussed in general, but where no particular theory is analysed from this point of view. The purpose of this paper is to present some classically interesting models for the intuitionistic theory of the continuum. These models will be applied to some simple independence questions. The idea of the model can also be used for models of second-order intuitionistic arithmetic (cf. the system of [6]), but lack of time and space force us to postpone this discussion to another paper. Also, the author has encountered some difficulty in verifying certain of the continuity assumptions (Axiom F4 of [6] for $\forall\alpha\exists\beta$ to be precise) and hopes to try to understand the motivation behind these principles better before presenting the details of the model. It is not impossible that there are several *distinct* intuitionistic notions of free-choice sequence (real number) with various continuity properties.

The paper has four sections In Section 1 the properties of order are discussed in a way that motivates the construction of the model in Section 2. In Section 3 the model is used to contrast classical and intuitionistic properties of order. In Section 4 the model is extended to include other relations, operations and higher-order notions, and the continuity question is touched on briefly.

The author hopes that the reader will be convinced that the model merits further study. The question is often raised whether the topological interpretation has any relevance at all to intuitionistic thought (e.g. [4], p. 189). Of course, the intended intuitionistic interpretation has to do with some abstract notion of "proof", but it seems fair to say that a formalization of this idea has *not* been carried far enough to treat analysis (again cf. [4] pp. 125ff.). However, the topological interpretation is not unrelated to this program, even though it is not the desired theory. One may view a neighborhood of a topological space as a kind of "proof": a proof that a point belongs to a more complicated set because the neighborhood of the point is included in the set. With this in mind, the usual topological facts follow the generally accepted intuitionistic prescriptions rather well, as follows:

Let X and Y be open subsets of a topological space T; let $t \in T$ be a point; and let $t \in U \subseteqq T$ be given so that U is an open neighborhood of t. Then U is a "proof" that $t \in (X \cap Y)$ if and only if $U \subseteqq X \cap Y$; which means that U is a "proof" that $t \in X$ *and* that $t \in Y$. Similarly U is a "proof" that $t \in (X \cup Y)$ if and only if U contains a neighborhood U' of t which is *either* a "proof" that $t \in X$ *or* a "proof" that $t \in Y$. Finally U is a "proof" that $t \in (X \Rightarrow Y) = \mathrm{In}\,((T \sim X) \cup Y)$ if and only if U is a "method" whereby, given a "proof" U' that $t \in X$ (i.e., $t \in U' \subseteqq X$), a "proof" that $t \in Y$ is produced (viz. $U \cap U'$).

Even if these remarks are not a philosophical justification of the topological interpretation (which indeed they are not), we at least see why it is that the "logic" of open sets proved to be formally an intuitionistic system. What may actually turn out to be the main interest of this study is the application of the reasonings of intuitionistic analysis to a very simple and well-known classical structure.

1. The properties of order in the continuum

The author must first admit to never having read a paper of Brouwer: he has taken nearly all of his information about the continuum from Heyting [2] and Kleene-Vesley [3]. Since the properties needed for this paper are quite elementary, these two references seem sufficient. In particular the relation $<$, the "measurable natural ordering" (written $< \circ$ in [3]), is very basic. It enjoys these two fundamental properties:

(1.1) $$\neg\,[x < y \wedge y < x];$$

(1.2) $$x < y \rightarrow x < z \vee z < y.$$

These are justified in [2, p. 25] and in [3, p. 143]. Note that transitivity follows at once from the above:

(1.3) $$x < y \wedge y < z \rightarrow x < z.$$

Next the "apartness relation", for which we shall write $\neq$, can be defined in terms of $<$:

$$x \neq y \leftrightarrow x < y \vee y < x. \tag{1.4}$$

Both in [2] and [3] the relation $\neq$ is written as $\#$. We have changed the notation for the sake of the model that will be presented below. In general, relations denoted by single symbols in the intuitionistic theory will have a very close connection with the corresponding classical relations. In order to keep these connections clearly in mind, it was found to be easier *not* to abbreviate an intuitionistically negated relation by drawing a stroke through the symbol for the relation to be negated. However, the negated relations are also of basic importance, and we use special symbols for them:

$$x \leqq y \leftrightarrow \neg\, y < x; \tag{1.5}$$

$$x = y \leftrightarrow \neg\, x \neq y. \tag{1.6}$$

We may also employ $>$ and $\geqq$ with the obvious meanings.

In both [2] and [3] the above use of $\leqq$ is avoided because it is not intuitionistically valid that:

$$x \leqq y \leftrightarrow x < y \vee x = y.$$

We shall see, nevertheless, that the model makes our usage natural even in the face of the above failure. Note, on the other hand, the validity of these two principles:

$$x < y \leftrightarrow x \leqq y \wedge x \neq y; \tag{1.7}$$

$$x = y \leftrightarrow x \leqq y \wedge y \leqq x. \tag{1.8}$$

The deductions of (1.7) and (1.8) from (1.1—(1.6) are quite straight forward, as are the proofs of the following:

$$x \leqq x; \tag{1.9}$$

$$x \leqq y \wedge y \leqq z \rightarrow x \leqq z; \tag{1.10}$$

$$x \leqq y \wedge y < z \rightarrow x < z; \tag{1.11}$$

$$x < y \wedge y \leqq z \rightarrow x < z. \tag{1.12}$$

From these it easily follows that $=$ is indeed an equality relation:

$$x = x; \tag{1.13}$$

$$x = y \wedge A(x) \rightarrow A(y), \tag{1.14}$$

where A is any first-order formula involving only $<$, $\neq$, $=$, and

$\leqq$. Note that the proofs of (1.10)—(1.12) make essential use of (1.3), which is the intuitionistic replacement for the *invalid* principle:

$$x < y \vee x = y \vee x > y.$$

The intuitionistic and classical theories are very close. One may show in general that an open sentence of the form

$$A_0 \wedge A_1 \wedge \cdots \wedge A_{n-1} \rightarrow B_0 \vee B_1 \vee \cdots \vee B_{m-1},$$

where the A_i and B_j are atomic formulae of the form $x < y$ or $x \neq y$ (maybe with other variables), is either provable from (1.1)—(1.4) (in intuitionistic logic!) or has a counterexample in the integers. The proof is effective. In case $x \leqq y$ and $x = y$ are allowed (especially in the conclusion), one must take $m = 1$. Surely it must be possible to give a simple decision method for all open consequences of (1.1)—(1.6). Maybe our model below will aid in seeing how the argument could be constructed. The quantified formulae should also be investigated, but the situation looks to be more involved.

We now extend our language by adding *variables* q, r, s (with or without various super- or subscripts) ranging over the *rational numbers*. We shall also use the usual rational *constants*, if necessary. In view of the constructive character of the rationals we can assume:

(1.15) $$q < r \vee \neg q < r.$$

The (open) theories of rationals in intuitionistic and classical logic coincide. (Instead of special variables we could, of course, have introduced a special predicate $x \in \boldsymbol{Q}$, but the formulas become too cumbersome.) Passing now for the first time to formulae involving existential quantifiers, we assume that the rationals are *dense* in the continuum in the following sense:

(1.16) $$\forall x \exists q, r[q < x \wedge x < r];$$

(1.17) $$\forall x, y[x < y \rightarrow \exists q[x < q \wedge q < y]].$$

These principles are justified in [3, p. 149, *R9.19].

Even though in the intuitionistic theory the order relations of an arbitrary real to the rationals are not necessarily decided, a real does in some sense determine a *cut* in the rationals. Indeed, we can prove at once from (1.17):

(1.18) *The following are equivalent*:

(i) $x < q$;

(ii) $\exists r < q[x < r]$;

(iii) $\exists r < q[x \leqq r]$.

A similar result (1.18′) holds with $<$ and $\leqq$ replaced by $>$ and $\geqq$, respectively. In these formulae, the bounded quantifiers should, of course, be interpreted as:

$$\exists r < q[\cdots] \leftrightarrow \exists r[r < q \wedge [\cdots]].$$

We also have:

(1.19) *The following are equivalent*:

(i) $x \leqq r$;

(ii) $\forall s > r[x < s]$;

(iii) $\forall s > r[x \leqq s]$.

By the way of proof, it is obvious that (i) $\to$ (ii) $\to$ (iii). Assume (iii) and recall that (i) means $\neg\, r < x$. So assume $r < x$. By (1.18′) we know there is an s with $r < s$ and $s < x$. By (iii), $x \leqq s$ follows, which gives a contradiction. We call (1.19′) the result of replacing all the relations by their converses.

2. The topological interpretation

The idea of this interpretation (which is carried out in classical mathematics) is to use the lattice of *open* subsets of a given topological space T as "truth values" for formulae. Thus to each formula A we associate an open subset $[\![A]\!]$ of the space T satisfying some simple rules:

(2.1) $$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!];$$

(2.2) $$[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!];$$

(2.3) $$[\![\neg A]\!] = \mathrm{In}(T \sim [\![A]\!]);$$

(2.4) $$[\![A \to B]\!] = \mathrm{In}((T \sim [\![A]\!]) \cup [\![B]\!]);$$

where In denotes the interior operator on subsets of T. In addition, the individual variables are interpreted as ranging over the elements ξ in some given domain $\mathscr{R}$ (we use "$\mathscr{R}$" here because we are thinking of models for the real numbers.) Thus for quantified formulae we have:

$$[\![\exists x \forall(x)]\!] = \bigcup_{\xi \in \mathscr{R}} [\![A(\xi)]\!]; \tag{2.5}$$

$$]\!]\forall x A(x)]\!] = \text{In} \bigcap_{\xi \in \mathscr{R}} [\![A(\xi)]\!]. \tag{2.6}$$

(Note that we really should write $A(\ulcorner\xi\urcorner)$ where $\ulcorner\xi\urcorner$ is the *name* of the object ξ in the formal language.)

As is proved in [5], all formulas A *provable* in Heyting's predicate logic are *valid* in this interpretation in the sense that $[\![A]\!] = T$. Let us assume further that all of (1.1)—(1.6) are valid in this particular interpretation as well as (1.16)—(1.17) where $\mathscr{R}$ is assumed to include the set $\boldsymbol{Q}$ of rational numbers. We wish to investigate more closely the behavior of the elements $\xi \in \mathscr{R}$.

To start out note that for each $t \in T$ an element $\xi \in \mathscr{R}$ determines a *cut* in $\boldsymbol{Q}$ where the upper part of the cut is the set

$$\{r \in \boldsymbol{Q} : t \in [\![\xi < r]\!]\}.$$

This suggests associating with $\xi \in \mathscr{R}$ a *function* $\xi : T \to \boldsymbol{R}$ (where $\boldsymbol{R}$ is the set of classical reals, and where we are using "ξ" again to denote the function) defined as:

$$\xi(t) = \inf \{r \in \boldsymbol{Q} : t \in [\![\xi < r]\!]\}.$$

In case $\xi = q \in \boldsymbol{Q}$, note that $\xi(t)$ is the *constant* function with value q. In general we have:

(2.7) *$\xi : T \to \boldsymbol{R}$ is continuous for all $\xi \in \mathscr{R}$.*

Proof: We note first that

$$\{t \in T : \xi(t) < q\} = [\![\xi < q]\!].$$

Because if $\xi(t) < q$, then by definition $t \in [\![\xi < r]\!]$ where $r < q$. But then obviously, $t \in [\![\xi < q]\!]$. Conversely, assume $t \in [\![\xi < q]\!]$. By (1.18) and (2.5), $t \in [\![\xi < r]\!]$ for some $r < q$. Thus $\xi(t) \leqq r < q$. Next we show that

$$\{t \in T : q < \xi(t)\} = [\![q < \xi]\!].$$

Assume first that $q < \xi(t)$. Then, by definition, there exists an $r > q$ such that $t \notin [\![\xi < s]\!]$ for all $s < r$. Take any s with $q < s < r$. By (1.3) we see

$$[\![q < \xi]\!] \cup [\![\xi < s]\!] = T;$$

hence, $t \in [\![q < \xi]\!]$. Assume for the converse that $t \in [\![q < \xi]\!]$. In view of (1.18′) we have $t \in [\![r < \xi]\!]$ for some $r > q$. Thus, for all $s < r$ we have $t \notin [\![\xi < s]\!]$; this means $r \leqq \xi(t)$ and so $q < \xi(t)$.

From these two equations we see that the inverse image under ξ of any open interval (q, q') is the open set $[\![q < \xi]\!] \cap [\![\xi < q']\!]$, which proves that the function $\xi : T \to \boldsymbol{R}$ is indeed continuous.

Now that we know that the functions are continuous, the open "truth" values of some other formulas are clear:

$$(2.8)\qquad [\![\xi < \eta]\!] = \{t \in T : \xi(t) < \eta(t)\}$$

$$(2.9)\qquad [\![\xi \neq \eta]\!] = \{t \in T : \xi(t) \neq \eta(t)\}$$

$$(2.10)\qquad [\![\xi \leqq \eta]\!] = \mathrm{In}\,\{t \in T : \xi(t) \leqq \eta(t)\}$$

$$(2.11)\qquad [\![\xi = \eta]\!] = \mathrm{In}\,\{t \in T : \xi(t) = \eta(t)\}$$

In fact (2.8) follows from (1.17) and the formulas used for the proof of (2.7) above; and the others follow by (1.4)—(1.6). Note that by (2.11) we have $\xi = \eta$ valid (that is, $[\![\xi = \eta]\!] = T$) if and only if $\xi(t) = \eta(t)$ for all $t \in T$. Hence it is safe simply to *identify* ξ with the continuous function it determines.

These simple facts suggest that the obvious classical model for the intuitionistic continuum is to take $\mathscr{R}$ to be the collection of *all* continuous functions $\xi : T \to \boldsymbol{R}$ and to use the rules (2.1)—(2.6), (2.8)—(2.11) for evaluating formulae. The reason for using all the continuous functions is that $\mathscr{R}$ should be *complete*. No doubt this statement could be justified more formally with reference to some suitable model of a higher-order intuitionistic theory of *species*. Since the author does not want to say at this moment what this theory is like, he must leave this matter somewhat vague.

3. Some independence results

Consider, for the sake of illustration, an open formula in two variables x and y. In view of (1.4)—(1.6), we need employ as atomic formulae only $x < y$ and $y < x$. Thus, corresponding to the given formula, there is a formula $A(P, Q)$ of propositional calculus such that the given formula is equivalent to

$$A(x < y, y < x).$$

We shall show that the universal statement

$$\forall x, y A(x < y, y < x)$$

is provable (intuitionistically) from (1.1)—(1.2) *if and only if* the propositional formula

$$[\neg\, [P \wedge Q] \to A(P, Q)]$$

is provable in Heyting's calculus. This gives a decision method for open formulae in two variables. No doubt the method can be extended to more variables, but the proper formulation of the result seems to be combinatorially rather complicated. We will discuss the case of three variables below.

Note first that the provability of the propositional formula is sufficient: because one can substitute $x < y$ for P and $y < x$ for Q and invoke (1.1). Suppose then that the formula is not provable. By the results stated in [5, pp. 385–396], there is a *metric* topological space T and there are assignments of open subsets $[\![P]\!]$ and $[\![Q]\!]$ to P and Q such that:

$$[\![\neg [P \wedge Q]]\!] \nsubseteq [\![A(P, Q)]\!].$$

Since we could relativize to a subspace, we can assume without loss of generality that

$$T = [\![\neg [P \wedge Q]]\!],$$

which means that $[\![P]\!]$ and $[\![Q]\!]$ are disjoint open sets.

We are now going to use our model for the intuitionistic continuum based on the metric space T. Let $\pi_P : T \to \boldsymbol{R}$ be defined by the formula

$$\pi_P(t) = \inf \{\delta(t, t') : t' \in T \sim [\![P]\!]\},$$

where δ is the metric in T; similarly for π_Q. These two *non-negative* functions are *continuous* and

$$[\![P]\!] = \{t \in T : \pi_P(t) > 0\} = [\![\pi_P > 0]\!];$$

similarly for π_Q. Define continuous functions $\xi, \eta : T \to \boldsymbol{R}$ by:

$$\xi(t) = 0;$$

$$\eta(t) = \pi_P(t) - \pi_Q(t).$$

Since $[\![P]\!]$ and $[\![Q]\!]$ are disjoint, we have

$$[\![\xi < \eta]\!] = [\![P]\!],$$

and

$$[\![\eta < \xi]\!] = [\![Q]\!];$$

which implies that

$$[\![A(\xi < \eta, \eta < \xi)]\!] = [\![A(P, Q)]\!] \neq T.$$

Thus the universal formula is *not* valid in the model.

There are several remarks to be made here. In the first place the choice of the model was made to depend on the formula. If we set $T = N^N$ (the *Baire* space, which is homeomorphic to the space *irrationals*, a zero-dimensional complete space), then by virtue of Theorem 4.1 in [5, pp. 130—131] this space can be used for *all* the counterexamples thus fixing the model.

Secondly, now that the space is fixed, we note that $\xi = 0$. If ξ' is any other continuous function, we replace ξ by ξ' and η by $\eta' = \xi' + \eta$. Obviously

$$[\![A(\xi' < \eta', \eta' < \xi')]\!] = [\![A(\xi < \eta, \eta < \xi)]\!];$$

and so

$$[\![\forall y A(\xi' < y, y < \xi')]\!] = [\![\forall y A(\xi < y, y < \xi)]\!].$$

But note that if $f : T \leftrightarrow T$ is an autohomeomorphism of T onto itself, then

$$[\![A(\xi < \eta \circ f, \eta \circ f < \xi)]\!] = f^{-1}([\![A(\xi < \eta, \eta < \xi)]\!]).$$

Inasmuch as the autohomeomorphism group is *transitive* on $T = N^N$, we conclude that

$$\bigcap_f f^{-1}([\![A(\xi < \eta, \eta < \xi)]\!]) = \emptyset.$$

This means that

$$[\![\forall y A(\xi < y, y < \xi)]\!] = \emptyset,$$

and in view of the remarks above

$$[\![\forall x \neg \forall y A(x < y, y < x)]\!] = T.$$

Among the more interesting of the formulae that fall within the scope of this discussion we may mention:

(3.1) $$\forall x \neg \forall y[\neg\neg\, x < y \rightarrow x < y]$$

(3.2) $$\forall x \neg \forall y[\neg\, x = y \rightarrow x \neq y]$$

(3.3) $$\forall x \neg \forall y[x < y \vee x = y \vee x > y];$$

(3.4) $$\forall x \neg \forall y[\neg\neg\, x < y \vee x = y \vee \neg\neg\, x > y];$$

(3.5) $$\forall x \neg \forall y[x = y \vee \neg\, x = y];$$

(3.6) $$\forall x \neg \forall y[x \leqq y \vee x \geqq y];$$

(3.7) $$\forall x \neg \forall y[x \leqq y \rightarrow x = y \vee x < y];$$

(3.8) $$\forall x \neg \forall y[\neg\, x = y \rightarrow x < y \vee x > y];$$

(3.9) $$\forall x \neg \forall y[\neg\, x = y \rightarrow \neg\neg\, x < y \vee \neg\neg\, x > y];$$

and, of course, many more, all of which are valid in the model.

The above formulae are closely related to those discussed in Chapter IV of [3]. Kleene shows that certain of them are *provable* in his system, which is a stronger theory than any we have considered so far in this paper. Now under suitable definitions our models *ought* to satisfy Kleene's axioms and more. Thus it would seem that we have somewhat stronger consistency results than those obtained by Kleene's method. However, judgement should be suspended until a better understanding of the model is obtained. We shall discuss the matter further in Section 4.

Passing now to three variables, the situation is more complicated. Suppose x_0, x_1, and x_2 are the variables. Let the propositional letter P_{ij} correspond to the atomic formula $x_i < x_j$. Let A be the given open formula, and let $A(P)$ be the propositional formula that results from replacing $x_i < x_j$ by P_{ij}. Let $B(P)$ be the conjunction of all formulae of the forms:

$$\neg\, [P_{ij} \wedge P_{ji}],$$

and

$$[P_{ij} \rightarrow P_{ik} \vee P_{kj}],$$

for $i, j, k < 3$. We wish to show that

$$\forall x_0, x_1, x_2 A$$

is provable if and only if the propositional formula

$$[B(P) \rightarrow A(P)]$$

is provable. In case of unprovability, we will find that

$$\forall x_0 \neg \forall x_1 x_2 A$$

is valid in the model.

As in the previous argument, sufficiency is obvious. Suppose then the propositional formula is unprovable. We evaluate the P_{ij} with open sets $[\![P_{ij}]\!]$ so that

$$[\![B(P)]\!] = T,$$

while

$$[\![A(P)]\!] \neq T.$$

We define *non-negative* continuous functions $\sigma_{ijkl} : T \rightarrow \boldsymbol{R}$ so that

$$[\![\sigma_{ijkl} > 0]\!] = [\![P_{ij}]\!] \cap [\![P_{kl}]\!].$$

We then introduce (by a formula which cost the author several hours to discover) functions $\pi_{ij} : T \rightarrow \boldsymbol{R}$ where

$$\pi_{ij} = \sigma_{ijik} + \sigma_{ijkj} - \sigma_{jijk} - \sigma_{jiki}.$$

This equation for the continuous function π_{ij} is, of course, understood as a functional equation. In view of the inclusion

$$[\![P_{ij}]\!] \subseteqq [\![P_{ik}]\!] \cup [\![P_{kj}]\!],$$

we see that

$$[\![\pi_{ij} > 0]\!] = [\![P_{ij}]\!].$$

The main reason for making the formula for π_{ij} so complicated was to assure:

$$\pi_{ij}+\pi_{jk}+\pi_{ki} = 0$$

for all $i, j, k < 3$. This means that we can solve the (overdetermined) system of equations

$$\xi_j - \xi_i = \pi_{ij}$$

for functions $\xi_0, \xi_1, \xi_2 : T \to \boldsymbol{R}$. Indeed let $\xi_0 = 0$ and $\xi_1 = \pi_{01}$ and $\xi_2 = \pi_{02}$. We then have

$$[\![\xi_i < \xi_j]\!] = [\![P_{ij}]\!]$$

so that

$$[\![A(\xi)]\!] = [\![A(P)]\!],$$

where $A(\xi)$ is the result of substituting the ξ's for the x's. Just as before we can show

$$[\![\forall x_1, x_2 A(\xi_0', x_1, x_2)]\!] = \emptyset$$

for all ξ_0', and the desired conclusion follows. The author was unable to come up with the proper π-formula for cases with more variables, though it would seem that the method should generalize.

4. Enlarging the model

Up to this point we have discussed the properties of only $<$, $\neq$, $\leqq$, and $=$. Suppose \$ is any relation between (classical) real numbers. We may extend \$ to the model $\mathscr{R}$ in the spirit of (2.8)—(2.11) by the formula:

$$[\![\xi \$ \eta]\!] = \mathrm{In}\,\{t \in T : \xi(t) \$\, \eta(t)\}. \tag{4.1}$$

In case \$ represents an *open* relation (open subset of $\boldsymbol{R}\times\boldsymbol{R}$), we can drop the In on the right-hand side. This we did in the cases of $<$ and $\neq$. In case \$ is a *closed* relation and \$$'$ is the complementary (open) relation, then

$$[\![\xi \$ \eta]\!] = [\![\neg\, \xi \$' \eta]\!].$$

This is what we did in the cases of $\leqq$ and $=$. The uniformity of these connections was the main motivation for the author's choice of notation. He feels that it makes comparison with the classical theory much easier and hopes others will agree with him. The author notes with satisfaction that Bishop in [1] (which is the deepest and most thorough mathematical development of intuitionistic analysis available) adopts the same notation for inequalities, of course based on an intuitive motivation.

By the way of example, let us consider the notions of closed and open intervals as defined in [3, *R8.1, p. 147 and *R13.1, p. 161]. After suitable logical simplifications, these two concepts can be defined as follows:

$$z \in [x, y] \leftrightarrow \neg\neg [x \leqq z \leqq y \vee x \geqq z \geqq y], \tag{4.2}$$

$$z \in (x, y) \leftrightarrow \neg\neg [x < z < y \vee x > z > y]. \tag{4.3}$$

In the above we have abbreviated a conjunction of successive relationships in the usual way. The exact reason why the doubly negated relationship is considered more basic than that without the $\neg\neg$ escapes the author.

In view of the topological equation

$$\mathrm{In}\,(X \cap Y) = \mathrm{In}\,(X) \cap \mathrm{In}\,(Y),$$

we see that if $A_0, A_1, \cdots, A_{n-1}, B_0$ are *atomic* formula of the kind just considered with the new relational symbols, then if

$$A_0 \wedge A_1 \wedge \cdots \wedge A_{n-1} \rightarrow B_0$$

is valid classically, it is also valid in our model. In case $B_0, B_1, \cdots, B_{n-1}$ all have symbols for *open* relations, then the same holds for

$$A_0 \wedge A_1 \wedge \cdots \wedge A_{n-1} \rightarrow B_0 \vee B_1 \vee \cdots \vee B_{m-1}.$$

By the way, it may turn out that the only interesting relations $\$$ that ought to be considered in the model are those that are either open or closed. But this remains to be seen.

Quantified formulae are as usual more difficult to understand. Suppose

$$\forall x \exists y [x \$ y]$$

is valid classically because there is a *continuous* function $f : \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that

$$\forall x [x \$ f(x)]$$

is true. Then obviously

$$\forall x \exists y [x \$ y]$$

will be valid in our model. The same remark can be made about formulae with implications of the kind considered above. Actually it is enough to have a *family* $\{f_i : i \in I\}$ of continuous functions such that

$$\forall x \exists i \in I [x \$ f_i(x)]$$

is classically valid for

$$\forall x \exists y [x \$ y]$$

to be valid in the model in case \$ happens to be *open*.

Besides relations, there are also some interesting functions on the elements of the model. For example, suppose $f : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ is *continuous*. We interpret the equation

$$\zeta = f(\xi, \eta),$$

for $\xi, \eta \in \mathscr{R}$, to mean that

$$\zeta(t) = f(\xi(t), \eta(t))$$

for all $t \in T$. Then ζ also belongs to $\mathscr{R}$. We can apply this method to the arithmetic operations $+$, $\cdot$, $-$ and to such functions as sin, cos, e^x, etc. Note that any (universal) (conditional) *equation* satisfied by functions on $\boldsymbol{R}$ is also valid on $\mathscr{R}$. Thus

(4.4) $$\forall x, y, z [x+y = x+z \to y = z]$$

is valid in the model; while

$$\forall x, y [x \cdot y = 0 \to x = 0 \vee y = 0]$$

is not. In fact,

(4.5) $$\neg \forall x, y [x \cdot y = 0 \to x = 0 \vee y = 0]$$

is valid.

This last remark points up the fact that if a sentence A has only constants (say, rational constants) and operations and relations like $+$, $=$, etc., then its value is invariant under all autohomeomorphisms of T. Now if $T = N^N$ (our favorite choice for T), then the only open sets so invariant are $\emptyset$ and T; thus if the sentence A is not valid, then $\neg A$ is valid. In particular, the law of excluded middle holds for such sentences in the model. Many people may regard this feature as undesirable; it is a consequence of our classical construction of the model. It does not seem too serious, however, when we realize that the most interesting sentences have parameters and then this conclusion does not hold.

We can weaken the unnegated (4.5) to make it valid; thus:

$$(4.6) \qquad \forall x, y[x \cdot y = 0 \wedge x \neq 0 \rightarrow y = 0].$$

One should not conclude, however, that a disjunctive conclusion with closed relations is always invalid. For example, we have:

$$(4.7) \qquad \forall x[x = x^2 \rightarrow x = 0 \vee x = 1].$$

This can easily be checked in the model, or we can prove it from other principles. Suppose $x = x^2$. By the obvious equations of algebra this can be written as

$$x \cdot (1-x) = 0.$$

Now by (1.2) we have $[0 < x \vee x < 1]$. In the first case we derive by (4.6) the equation $1-x = 0$, and so $x = 1$. In the second case $1-x \neq 0$ follows, so $x = 0$. We see that (1.2) in intuitionistic analysis replaces the invalid principle

$$[x > 0 \vee x = 0 \vee x < 0]$$

by the valid and almost as useful principle

$$[x > 0 \vee x < \varepsilon],$$

where ε can be any (very small) positive rational number.

The use of certain continuous functions and of some of their obvious properties sometimes gives us results about the pure theory of order which no doubt cannot be derived simply from (1.1)—(1.8). For example, the *average* $(x+y)/2$ clearly satisfies

$$(4.8) \qquad x < y \rightarrow x < (x+y)/2 < y.$$

From this and some other obvious arithmetic principles we derive easily:

$$(4.9) \quad \forall x, y \exists z[[x < y \vee x > y] \rightarrow [x < z < y \vee x > z > y]].$$

We can adjoin double negations and write the result as:

$$(4.10) \qquad \forall x, y \exists z[\neg\, x = y \rightarrow z \in (x, y)],$$

which seems to be slightly stronger (and more easily proved) than *R13.8 in [3, p. 161].

Other properties of order seem to require rather complicated proofs. In particular, the author could not see a more elementary proof of *R14.11 in [3, p. 168]. He can only remark that after the definitions are unwound the principle boils down to:

$$(4.11)\ \forall z[\neg\, [x < z < y] \to \neg\, x < z \vee \neg\, z < y] \to [\neg\neg\, x < y \to x < y].$$

That seems interesting, but the author finds the hypothesis very hard to work with. All in all, the quantified theory of order in the intuitionistic continuum may prove to be fairly difficult.

Partial functions are very annoying. In [2, p. 21] Heyting states that x^{-1} is defined only for $x \neq 0$. It seems rather difficult to make this procedure rigorous in a formal intuitionistic theory that allows the free formation of terms x^{-1} and not just the relation $y = x^{-1}$ (which is equivalent to $x \cdot y = 1$.) We can, however, show that

$$(4.12) \qquad \forall x[x \neq 0 \to \exists y[x \cdot y = 1]]$$

is valid in our model. For let $\xi \in \mathscr{R}$ and suppose $t \in [\![\xi \neq 0]\!]$. Then there is an integer $n \in N$, $n > 0$, such that $|\xi(t)| > 1/n$. Let f be any continuous function such that $f(u) = u^{-1}$ for $|u| \geqq 1/n$. We see that

$$[\![\xi > 1/n]\!] \subseteqq [\![\xi \cdot f(\xi) = 1]\!] \subseteqq [\![\exists y[\xi \cdot y = 1]]\!].$$

Hence,

$$[\![\xi \neq 0]\!] \subseteqq [\![\exists y[\xi \cdot y = 1]]\!],$$

which shows that (4.10) is valid. There ought to be a general principle to cover this situation, but the author is not quite sure what its proper formulation should be.

Instead of restricted variables such as the rational variables q, r, s, we could introduce predicates such as $\xi \in \boldsymbol{Q}$ into our formal language. By definition:

$$(4.13) \qquad [\![\xi \in \boldsymbol{Q}]\!] = \bigcup_{q \in \boldsymbol{Q}} [\![\xi = q]\!],$$

where on the right-hand side we have used q for the rational and for the *constant* function with value q. The reader should note that

$$[\![\xi \in \boldsymbol{Q}]\!] = T$$

if and only if the sets $[\![\xi = q]\!]$ are both open *and* closed in T. This means that T is partitional into such sets on each of which the function ξ has a constant value. Even though such functions are not strictly constant, we can show:

$$(4.14) \qquad [\![\exists x \in \boldsymbol{Q}[A(x)]]\!] = \bigcup_{q \in \boldsymbol{Q}} [\![A(q)]\!],$$

and

$$\llbracket \forall x \in \boldsymbol{Q}[A(x)] \rrbracket = \mathrm{In} \bigcap_{q \in \boldsymbol{Q}} \llbracket A(q) \rrbracket. \tag{4.15}$$

This means that the use of the predicates has the same effect as the use of restricted variables.

Another interesting predicate is defined by

$$\llbracket \xi \in \boldsymbol{D} \rrbracket = \bigcup_{a \in \boldsymbol{R}} \llbracket \xi = a \rrbracket, \tag{4.16}$$

where again a is used to denote a constant function. We can think of $\xi \in \boldsymbol{D}$ as meaning ξ is *definite.* It is just the point of the intuitionistic theory that it allows for "indefinite" numbers. That is why a statement

$$\forall x \exists y A(x, y)$$

is so strong because the y must exist not only for the definite but also for the indefinite numbers x.

It is seen that our model allows only for *continuous* functions on $\boldsymbol{R}$ to be automatically extended to $\mathscr{R}$. This limitation may indeed be essential. In view of Brouwer's Theorem on Continuity (cf. [2, p. 46] and [3, pp. 151ff.]) it may be reasonable to conjecture that if $\forall x \exists ! y A(x, y)$ is valid in the model, then there exists a continuous function $f : \boldsymbol{R} \to \boldsymbol{R}$ such that $\forall x A(x, f(x))$ is also valid (at least if $A(x, y)$ has no additional parameters). It is clear that we cannot weaken $\exists !$ to $\exists$ for this model, because

$$\forall x \exists y [y > 0 \leftrightarrow x \in \boldsymbol{Q}]$$

is valid, but there is no continuous function that can be used to obtain y even for all x where $x \in \boldsymbol{D}$ is valid. The above conjecture about $\exists !$ may be too optimistic: it will probably turn out that there are more functions $F : \mathscr{R} \to \mathscr{R}$ that are appropriate for the model than there are ordinary continuous functions $f : \boldsymbol{R} \to \boldsymbol{R}$, even though the F may satisfy the formal statement of continuity in the model.

The predicates introduced in (4.13) and (4.16) are what might be called "non-standard"-predicates to distinguish them from the kind of predicates \$ discussed at the beginning of this section. The non-standard notions are somehow peculiar to the model and have no counterpart in the classical theory. (This remark is somewhat misleading in connection with $\boldsymbol{Q}$; but note that the equation

$$\llbracket \xi \in \boldsymbol{Q} \rrbracket = \mathrm{In} \{ t \in T : \xi(t) \in \boldsymbol{Q} \}$$

is *not* in general correct for all $\xi \in \mathscr{R}$.) This suggests that we should have a *theory* of predicates (sets, species). A general predicate is represented by a function Ξ defined on $\mathscr{R}$ taking arbitrary open subsets of T as values. We write:

$$(4.17) \qquad [\![\xi \in \Xi]\!] = \Xi(\xi).$$

We may use variables X, Y, Z, etc., to range over these predicates. We note that the general *comprehension principle* is valid:

$$(4.18) \qquad \exists x \forall x [x \in X \leftrightarrow A(x)]$$

where $A(x)$ is an arbitrary formula. We are being careful *not* to assume that these species are *extensional* in the sense of the validity of:

$$\forall x, y \forall X [x = y \wedge x \in X \rightarrow y \in X],$$

because remarks in [4] and elsewhere seem to indicate that non-extensional predicates may be of interest and even of importance.

In [3], free (i.e., universally quantified) species variables are used at several places; in particular, the obviously fundamental ***R14.14** [3, p. 171] expressing a kind of Dedekind completeness (called being "freely connected") uses them, and they enter at many other points. Note that there is no reason to stop at second-order species: we can easily continue upward to higher-order species and even to a kind of transfinite intuitionistic set theory. That last sounds almost like a contradiction in terms, but the *model* certainly can be defined in classical set theory.

BIBLIOGRAPHY

E. Bishop

[1] Foundations of Constructive Analysis, New York, 1967.

A. Heyting

[2] Intuitionism. An Introduction, Amsterdam, 1956.

S. C. Kleene and R. E. Vesley

[3] Foundations of Intuitionistic Mathematics, Amsterdam, 1965.

G. Kreisel

[4] Mathematical Logic, in Lectures on Mathematics, vol. III, (ed. T. L. Saaty), New York, 1965, pp. 95—195.

H. Rasiowa and R. Sikorski

[5] The Mathematics of Metamathematics, Warsaw, 1963.

A. S. Troelstra

[6] The Theory of Choice Sequences, to appear.

(Oblatum 3–1–'68) Stanford University