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# Attributes, sets, partial sets and identity

Dedicated to A. Heyting on the occasion of his 70<sup>th</sup> birthday

by

P. C. Gilmore

## 1. Summary

In Section 2 it is argued contrary to Quine that classical set theory can better be interpreted following Russell as a theory of properties or attributes rather than as a theory of sets. A formalized set theory provides first a domain of attributes which may consistently co-exist if the theory is consistent, and second, a theory of identity for the domain. For classical set theories the theory of identity takes a simple form via the axiom of extensionality. However, that axiom need not be true of all consistent domains of attributes, and indeed to insist upon its truth can complicate the search for such domains.

In a theory of attributes  $\in$  is interpreted as the application relation. In Section 3 a second dual application relation  $\notin$  is introduced. Recognition of this second relation permits a simple and intuitive definition of a domain  $D'_c$  of attributes by means of a demonstrably consistent theory  $PST'$  formalized in classical quantificational logic. The domain  $D'_c$  is inclusive enough to contain members for any attribute of classical set theory defined by a formula of the theory.

A theory of identity for the domain  $D'_c$  presents new problems discussed in Section 4. The axiom of extensionality cannot provide a theory of identity for  $D'_c$  since it is inconsistent with  $PST'$ . However, a theory of identity  $PST$  is described elsewhere.

In Section 5, we briefly examine the intuitionist, predicativist and constructivist conceptions of sets and conclude that they are compatible with Russell's conception of sets as attributes. Finally we conclude that the theory  $PST'$  can be defended as predicative because of the domain  $D'_c$  of the interpretation provided for  $PST'$ .

## 2. Attributes, sets and identity

Russell describes in [1] two ways of defining a class or set, by an extensional definition and by an intensional definition. An extensional definition of a set is one which “enumerates” its members; that is, one which specifies the membership by naming each member. An intensional definition “mentions a defining property”, asserting that the membership consists of just those objects possessing the defining property. Russell concludes that intensional definitions suffice: “Of these two kinds of definition, the one by intension is logically more fundamental. This is shown by two considerations: (1) that the extensional definition can always be reduced to an intensional one; (2) that the intensional one often cannot even theoretically be reduced to the extensional one.”

Following Russell we take the concept of a property or attribute as more fundamental than the concept of set. Sets are the extensions of attributes. For the present, however, we need not follow Russell into the confusion of attributes with linguistic expressions.<sup>1</sup> Attributes are abstractions over which the variables of set theory are thought to range. In a language of set theory, certain linguistic expressions will denote attributes, but a confusion of one with the other must, for the present, be avoided. As for the future, we will advocate such a confusion at the end of Section 3.

We agree therefore with Quine when he writes on page 122 of [3] that the reduction of a theory of classes or sets to a theory of attributes is only “a reduction of certain universals to others” and does not lessen the dependence of the theory upon abstractions. But he then goes on to write, “Such reduction comes to seem pretty idle when we reflect that the underlying theory of attributes itself might better have been interpreted as a theory of classes all along, in conformity with the policy of identifying indiscernibles.” Strictly in the context of determining the dependence of theory upon abstractions, this assertion may be justified. But in wider contexts, we will see that this assertion cannot be defended.

Statements about sets are to be interpreted as statements about attributes. The epsilon relation of set theory is to be understood as the application relation which holds between an object and an attribute when the attribute applies to the object. Thus, “ $x \in y$ ” is shorthand for “the attribute  $y$  applies to  $x$ ”. Should  $y$  not be

<sup>1</sup> An account of which is given on page 254 of [5].

an attribute, then “ $x \in y$ ” is false. An attribute may apply to another attribute or possibly to itself so that “ $x \in x$ ” may be true or false depending upon  $x$  and is not a priori false. Consider, therefore, a first order language  $L$  in which  $\in$  and  $=$  are the only primitive predicates. The variables of  $L$  range over a domain  $D$  consisting of attributes and possibly other objects.

But what attributes are to be included in  $D$ ? We are not in a position to describe the domain  $D$ , but we can indicate, at least, some of its members. For corresponding to any formula  $P(x)$  of  $L$ , in which  $x$  is the only free variable, is the attribute of satisfying  $P(x)$  which we will denote by  $\{x : P(x)\}$ .<sup>2</sup> The name  $\{x : P(x)\}$  can be accepted as a term for the language  $L$ . More generally, given any formula  $P$  of  $L$  and any variable  $x$ ,  $\{x : P\}$  is a term of  $L$ .<sup>3</sup> The variable  $x$  is not free in the term  $\{x : P\}$  although any free variables of  $P$ , other than  $x$ , are free. A constant term of  $L$  is one in which no variables occur free and, therefore, denotes an attribute of the domain  $D$ . More generally, when the free variables of  $\{x : P(x)\}$  have been assigned attributes from the domain  $D$ , the term denotes the attribute of satisfying  $P(x)$ . Such terms should therefore satisfy

$$(2.1) \quad (\mathbf{u})(v)[v \in \{x : P(x)\} \equiv P(v)],$$

where  $v$  is any variable free to replace  $x$  in  $P(x)$  and where  $(\mathbf{u})$  is a sequence of universal quantifiers, one for each free variable of  $\{x : P(x)\}$ .

We cannot pretend that we have motivated the axiom scheme (2.1) in clearly evident steps. Given a particular domain  $D$  for the variables of  $L$  and an interpretation of  $\in$  and of  $=$  over  $D$ , one can ask whether (2.1) is true or false, but a motivation for (2.1) should be able to provide a model of (2.1), an interpretation in which (2.1) is true. It is evident, however, that a motivation in this strong sense cannot be provided. Confining the variables of  $L$  to attributes rather than sets is no protection from the paradoxes. The term  $\{x : \sim x \in x\}$  is just as disastrous to (2.1) when it denotes the attribute of being heterological as it is when it denotes the Russell set.

If a term  $\{x : P(x)\}$  satisfying (2.1) cannot be permitted for

<sup>2</sup> We will systematically drop quotes when no confusion can result.

<sup>3</sup> As is done in this sentence with the variable  $x$ , the variables of  $L$  will on occasion be regarded as metalinguistic variables ranging over the variables of  $L$ . Such confusions are infrequent and obvious enough as to make the introduction of special notation unnecessary.

every formula  $P(x)$ , for what formulae can it be permitted? How this question is answered distinguishes, to a degree, the various formulations of classical set theory. For example, the “constructive” axioms of Zermelo-Fraenkel set theory [6], the comprehension axioms of the simple theory of types [7] or of new foundations [3], and the axioms \*202 of mathematical logic [2] can all be interpreted as providing different answers to this question. To move on to the central questions of this paper, it will be assumed that this question has been answered in a definite form. The assumption will be made that an interpretation has been provided for some classical set theory with axioms consisting entirely of instances of (2.1). The domain  $D$  for this interpretation must include attributes denoted by constant terms of  $L$  as well as possibly other attributes. Let  $D_c$  be all the constant terms of  $L$  together with individual constants, one for each attribute in the domain of the interpretation not denoted by a constant term. The Skolem-Lowenheim theorem [8] assures us that only denumerably many individual constants need be added so that  $D_c$  can be thought of as a domain of syntactic objects. For the present we will continue to distinguish between  $D_c$  and the domain  $D$  of attributes for this hypothetical set theory, although, of course,  $D_c$  can be thought of as the domain of the interpretation.

The interpretation of the  $\in$  relation for the domain  $D$  is that of the application relation. How is identity to be interpreted? For the domain  $D_c$ , there are at least three different notions of identity:

(1) Syntactic identity.  $D_c$  is a domain of syntactic objects, namely constant terms of  $L$ , and hence syntactic identity is available as a relation on the domain.

(2) Extensional identity.  $a$  and  $b$  are extensionally identical, written  $a =_e b$ , when  $(z)(z \in a \equiv z \in b)$  is valid; that is, when  $a$  and  $b$  apply to the same things.

(3) The identity of indiscernibles.  $a$  and  $b$  are indiscernible, written  $a =_i b$ , when  $(z)(a \in z \equiv b \in z)$  is valid; that is, when exactly the same attributes apply to both  $a$  and  $b$ .

Of these three notions of identity, only the second two are meaningful for the domain  $D$  of attributes denoted by the terms of  $D_c$ . However, each of these three definitions of identity is reflexive and symmetric and the first satisfies all instances of the other axioms of identity as well for the domain  $D_c$ . These other axioms can be expressed as:

$$(2.2) \quad (x)(y)[(x = y \ \& \ p[x]) \equiv (x = y \ \& \ p[y])]$$

Where  $p[x]$  is any atomic formula in which  $x$  occurs at a single designated location and  $p[y]$  has been obtained from  $p[x]$  by replacing  $x$  at the designated location by  $y$ . That syntactic identity satisfies (2.2) for the domain  $D_e$  is immediate.

The identity of indiscernibles can be shown to satisfy (2.2) under mild assumptions about the terms satisfying (2.1). For the language  $L$  among the possibilities for  $p[x]$  are  $x \in a$  or  $a \in x$  with the indicated occurrence of  $x$  the designated occurrence and with  $a$  any term (possibly  $x$ ). Interpret  $x = y$  in (2.2) as  $(z)(x \in z \equiv y \in z)$ . Then when  $p[x]$  is  $x \in a$ , (2.2) is immediate, and when  $p[x]$  is  $a \in x$  (2.2) follows from

$$(2.3) \quad (u)(v)[v \in \{x : u \in x\} \equiv u \in v]$$

which we will assume is an acceptable instance of (2.1). Under this assumption, the identity of indiscernibles is a satisfactory definition of identity. Should the language  $L$  be extended to include primitive predicates other than  $\in$  and  $=$ , the assumption can be appropriately extended. From now on, when we speak of identity without qualification, we mean the identity of indiscernibles.

Quine objects on page 2 of [5] to a theory of attributes as a replacement for a set theory "Partly because of the vagueness of the circumstances under which the attributes attributed by two open sentences may be identified." This is not a sustainable objection. On page 209 of [4] he writes of the identity of sets of classes, "Classes raise no perplexities over identity, being identical if and only if their members are identical." In brief,

$$(2.4) \quad \text{Extensionally identical sets are identical,}$$

a transparent truth about sets which is expressed formally by the axiom of extensionality

$$(2.5) \quad (x)(y)(x =_e y \supset x =_i y),$$

since the converse of (2.5) follows quite simply from our conclusion that  $=_i$  satisfies (2.2). Quine appears to be arguing that since we have a simple necessary and sufficient condition for the identity of sets, namely extensional identity, and fail to have such a condition for the identity of attributes, we are better off speaking only of sets whenever possible. But whenever it is possible to speak of sets, it is also possible to speak only of attributes, and often with better insight.

When the variables of  $L$  are interpreted as ranging over a domain  $D$  of attributes, (2.5) is a profound assertion about the domain  $D$ :

- (2.6) When two attributes of  $D$  apply to the same members of  $D$ , then any member of  $D$  applying to the one applies to the other also.

To insist upon interpreting the variables of  $L$  as ranging over sets is to insist upon a domain of attributes satisfying (2.6).<sup>4</sup> The resulting simplification in the theory of identity for the domain may be useful in some contexts, but more than that simplification is necessary to justify the restriction.

Quine's objections to attributes do not end with identity. On page 209 of [4] he writes, "Not only is the use of classes instead of attributes, where possible, to be desired on account of the identity question. It is also important in that intensional abstraction is opaque whereas class abstraction is transparent." This objection too rests ultimately upon the acceptance of the axiom of extensionality, although to answer the objection properly requires arguments too lengthy for this occasion.

The axiom of extensionality is not a trivial assertion about sets, but a profound assertion about the domain of attributes admitted as the range of the variables for theories of sets. The insistence of its truth can complicate the problem of describing domains of attributes that can consistently co-exist, as will be seen in Section 4. Where the axiom of extensionality can be admitted for a domain, it does simplify the theory of identity for the domain, but the larger question of developing a theory of identity for other domains of attributes still remains.

### 3. Attributes re-examined

The epsilon relation has been taken as the application relation which holds between an object and an attribute when the attribute applies to the object. But equally fundamental with the epsilon relation is a second dual relation which we call the epsilon-stroke relation and denote by  $\notin$ . An attribute carries with it a range of

<sup>4</sup> Gandy's result [9] concerning the axiom of extensionality in the single theory of types and his result [10] concerning the axioms in Gödel-Bernays set theory [11] do not provide a contrary argument, for he showed that the axiom of extensionality could be dispensed with in these theories by showing that it is valid in certain inner models of the theories.

significance or meaningfulness; an attribute meaningfully applies or does not apply to an object in its range of significance. For example, the attribute of being an odd number can apply or not apply meaningfully only to integers; we say that 3 is odd, that 2 is not odd, but can give no answer when asked of  $\frac{1}{2}$  whether it is even or odd. The epsilon-stroke relation is the relation which holds between an object and an attribute when the object is in the range of significance of the attribute, but the attribute does not apply to the object. Thus, 3 is epsilon related to the attribute of oddness; 2 is epsilon-stroke related, while  $\frac{1}{2}$  is neither epsilon nor epsilon-stroke related.

There is no suggestion here that it is indefinite whether  $\frac{1}{2}$  is even or odd, or that it is possible that it is even or odd. There is no motivation to depart from classical logic either by going to a many-valued logic as in [12] or to a modal logic as in [14]. The law of the excluded middle is assumed to apply to both the epsilon and the epsilon-stroke relations.

The epsilon-stroke relation has been so named because it is related to  $\notin$  as it is defined by the equivalence

$$(x)(y)(x \notin y \equiv \sim x \in y).$$

This equivalence can be expressed as the conjunction of

$$(3.1) \quad (x)(y)(x \in y \vee x \notin y),$$

and

$$(3.2) \quad (x)(y) \sim (x \in y \ \& \ x \notin y).$$

From the example of  $\frac{1}{2}$  and the attribute of oddness, it is clear that (3.1) must be rejected. It is assumed only that  $\in$  and  $\notin$  satisfy (3.2); that is, that they are incompatible. In earlier publications [16] and [17], we chose to denote  $\notin$  by  $\nu$ , but the notation  $\notin$  is more suggestive and need not be confusing if it is kept in mind that the relation is no longer defined.

Instead of taking  $\notin$  along with  $\in$  as the second fundamental relation for attributes, we could have taken the relation of significance or meaningfulness as was suggested in [15]. But clearly, such a relation can be defined in terms of  $\in$  and  $\notin$  just as  $\notin$  can be defined in terms of that relation and  $\in$ . However, there are distinct technical advantages to taking  $\notin$  as the fundamental relation along with  $\in$ . These advantages will become evident shortly.

In Section 2 attributes were named by the constant terms of  $D_e$  which included terms  $\{x : P(x)\}$  of a first order language  $L$  in which  $\in$  and  $=$  were primitive predicates. Identity could have



been defined over an interpreted domain  $D_e$  as the identity of indiscernibles, but we chose to keep  $=$  as a primitive predicate. Now instead of  $L$ , we have a first order language  $L'$  in which  $\notin$  in addition to  $\in$  and  $=$  are primitive predicates and which has terms other than those of  $L$ .

When an attribute is described, it is necessary to assert what objects will be epsilon related to it and what objects will be epsilon-stroke related to it. Hence, terms in  $L'$  which are to denote attributes must be constructed from two formulae, not just one as with  $L$ . For any variable  $x$  and any formulae  $P$  and  $Q$  of  $L'$ ,  $\{x : P, Q\}$  will be a term of  $L'$ . Corresponding to the statement (2.1) indicating the attribute denoted by  $\{x : P(x)\}$  is the following pair of statements indicating the attribute denoted by  $\{x : P(x), Q(x)\}$ :

$$(3.3) \quad (\mathbf{u})(v)[(v \in \{x : P(x), Q(x)\} \vee (P(v) \& Q(v))) \equiv P(v)],$$

and

$$(3.4) \quad (\mathbf{u})(v)[(v \notin \{x : P(x), Q(x)\} \vee (P(v) \& Q(v))) \equiv Q(v)],$$

where  $v$  is free to replace  $x$  in  $P(x)$  and  $Q(x)$ . These do not take the immediately simple form of (2.1) because no restriction has been placed on the pairs of formulae  $P(x)$  and  $Q(x)$  for which terms  $\{x : P(x), Q(x)\}$  are permitted. The disjunct  $(P(v) \& Q(v))$  is needed to prevent unfortunate consequences of such liberty. For example, an immediate contradiction of (3.2) would otherwise result from the term  $\{x : x = x, x = x\}$ . From (3.3) and (3.4) we can conclude that  $a$  is epsilon related to the attribute  $\{x : P(x), Q(x)\}$  if  $P(a)$  is true and  $Q(a)$  is false, while it is epsilon-stroke related if  $P(a)$  is false and  $Q(a)$  is true.

But still, both (3.3) and (3.4) are self-contradictory as the term  $\{x : \sim x \in x, x \in x\}$  shows for (3.3) and the term  $\{x : x \notin x, \sim x \notin x\}$  shows for (3.4). What then has been gained by recognizing the  $\notin$  relation as well as the  $\in$  relation? Nothing as yet, but the question as to which formulae  $P(x)$  and  $Q(x)$  may define a term  $\{x : P(x), Q(x)\}$  satisfying (3.3) and (3.4) can be given a simple and intuitively appealing answer. It is only necessary to insist that  $P(x)$  and  $Q(x)$  be positive; that is, that they employ only disjunction, conjunction, and existential and universal quantification. For ease of reference, we will denote by  $PST'$  the theory with axiom (3.2), axiom schemes (3.3) and (3.4), where  $P(x)$  and  $Q(x)$  are any positive formulae, and axioms for identity. Further, we will restrict the terms of  $L'$  to those terms  $\{x : P, Q\}$  for which

$P$  and  $Q$  are positive and  $x$  is any variable.

The restriction to positive formulae  $P$  and  $Q$  for the admissible terms  $\{x : P, Q\}$  is a simple one; that it has an intuitive appeal as well requires justification. For any terms  $s$  and  $t$ ,  $s \notin t$  is a positive counterpart to the formula  $\sim s \in t$ . Also,  $(Ez)(s \in z \ \& \ t \notin z)$ , which we abbreviate by  $s \neq t$ , is a positive counterpart to  $\sim s = t$  as long as  $z$  does not occur free in  $s$  or  $t$ . Hence, any formula of the first order language  $L$  can be given a positive counterpart in  $L'$ . To find the positive counterpart to any formula of  $L$ , express it first using only negation, disjunction, conjunction, and existential and universal quantification with all negations applied to atomic formulae and then replace any formula part  $\sim s \in t$  by  $s \notin t$  and  $\sim s = t$  by  $s \neq t$ . It is a simple matter to show that if  $P^*$  is the positive counterpart of a formula  $P$  obtained in this way, then

$$(3.5) \quad (\mathbf{u})(P^* \supset P)$$

is a logical consequence of (3.2) and the axioms for identity. Hence, for any formula  $P(x)$  of  $L$ , the term  $\{x : (P(x))^*, (\sim P(x))^*\}$  is to satisfy (3.3) and (3.4) and consequently by (3.5):

$$(3.6) \quad (\mathbf{u})(v)[v \in \{x : (P(x))^*, (\sim P(x))^*\} \equiv (P(v))^*],$$

and

$$(3.7) \quad (\mathbf{u})(v)[v \notin \{x : (P(x))^*, (\sim P(x))^*\} \equiv (\sim P(v))^*].$$

In particular, for the troublesome formula  $\sim x \in x$  there is the term  $\{x : x \notin x, x \in x\}$  which will be abbreviated by  $R$ .  $R$  satisfies (3.6) and (3.7) but no contradiction results now but only the conclusion:

$$(3.8) \quad (\sim R \in R \ \& \ \sim R \notin R);$$

that is,  $R$  is not in its own range of significance.

In Section 2, an interpretation over a domain  $D$  of attributes was presumed for a classical set theory without an axiom of extensionality. By an appeal to the Skolem-Lowenheim theorem, we concluded that the theory could equally well have an interpretation over a denumerable domain  $D_c$  of syntactic objects including the constant terms  $\{x : P\}$  of  $L$ . Such an appeal is not necessary for  $PST'$ . Let now  $D'_c$  be just the domain of all constant terms  $\{x : P, Q\}$  of  $L'$  including no individual constants. It was shown in [18] that an interpretation of  $\in$ ,  $\notin$  and  $=$  can be provided over the domain  $D'_c$  for which all axioms of  $PST'$  are true. The recogni-

tion of  $\notin$  as well as  $\in$  and the restriction to positive formulae for the terms  $\{x : P(x), Q(x)\}$  satisfying (3.3) and (3.4) leads, therefore, to a simple solution for the problem of what domain of attributes to admit as the range of the variables for the language  $L'$ . It is true that  $D'_c$  is a denumerable domain of linguistic objects, namely constant terms of the language  $L'$ , and not a domain of abstract attributes like the domain  $D$  for  $L$ . But now the fiction of this distinction need no longer be maintained. Attributes are just constant terms of a language like  $L'$  which by virtue of their syntactic structure are epsilon and epsilon-stroke related to other constant terms. This point we argued at greater length in [17] but, because of the absence of a theory like  $PST'$  at that time, not very convincingly. Now we can claim a real reduction by Quine's standard of [3] in the ontological commitment of the theory  $PST'$  as compared with a classical set theory; the variables of  $PST'$  range over the constant terms of  $PST'$  and not over sets or attributes in their usual abstract sense.

#### 4. Partial sets and identity

If the extensions of the attributes  $\{x : P\}$  of Section 2 are sets, what then are the extensions of the attributes  $\{x : P, Q\}$  of Section 3? Each of the sets of Section 2 divides the domain  $D$  into two parts, one part consisting of the members of the set and the other part consisting of the members of the complement of the set. The extensions of the attributes  $\{x : P, Q\}$  also have two parts, the  $a$ 's epsilon related to  $\{x : P, Q\}$  and the  $a$ 's epsilon-stroke related to  $\{x : P, Q\}$ , which can be thought of respectively as a set and its complement. But then, a set and its complement need not exhaust the domain  $D'_c$ . The characteristic function of such a set over the domain  $D'_c$  is the function which takes value 1 for an element of the set and takes value 0 for an element of the complement of the set. Since the characteristic function of such sets may be only partial over the domain  $D'_c$ , they are called partial sets.

It is an interesting fact that most formalized set theories must be interpreted as theories of partial sets. For example, in the simple theory of types, the domain of the theory is the union of the domains for its types, while the complement of a set can be defined only relative to a domain of a particular type. Also, with Zermelo-Fraenkel set theory, the complement of a set can be defined only relative to another set and not relative to the domain of all sets. New foundations, on the other hand, is a theory of sets

rather than partial sets, which might just possibly account for some of its curious properties.

However suggestive the name partial sets might be,  $PST'$  is still a theory of attributes. All of the attributes of  $D'_e$  can co-exist in the sense that  $PST'$  is a consistent theory. But a consequence of the consistency proof for  $PST'$  given in [18] is that  $s = t$  is provable in  $PST'$  if and only if  $s$  and  $t$  are syntactically identical terms. Clearly, therefore, however adequate  $PST'$  may be for providing a consistent theory of attributes, it is inadequate as a theory of identity for these attributes if we take even a small fragment of classical set theory as a test for adequacy.

The inadequacy of  $PST'$  as a theory of identity is not a failure of the theory to provide sufficiently many instances of terms like the one assumed to satisfy (2.3). For it is a simple matter to prove

$$(4.1) \quad (x)(y)(x = y \equiv (z)(x \in z \equiv y \in z))$$

is a theorem of  $PST'$ ; the left to right implication of (4.1) is a consequence of the axioms for identity, while the right to left implication results by substituting  $\{u : u = x, u \neq x\}$  for  $z$  and noting that this term is an instance of (3.6). Identity for  $PST'$  will always be the identity of indiscernibles no matter how  $PST'$  is extended. What the theory lacks is the ability to be *not* discerning — to identify such syntactically distinct terms as  $\{x : x = x, x \neq x\}$  and  $\{y : y = y, y \neq y\}$ . The axiom of extensionality (2.5) supplies this ability to set theory, but we will see it cannot do likewise for  $PST'$ ; of course,  $a =_e b$  is redefined for  $PST'$  to be  $(z)[(z \in a \equiv z \in b) \ \& \ (z \notin a \equiv z \notin b)]$ .

Consider the following abbreviated terms in  $PST'$ :

$$\begin{array}{ll} (w \cap z) & \text{for } \{x : (x \in w \ \& \ x \in z), (x \notin w \vee x \notin z)\} \\ A^* & \text{for } \{u : u \neq u, u \neq u\} \\ R' & \text{for } \{u : u \in u, u \notin u\} \\ A(u) & \text{for } \{x : (x = R' \ \& \ u \in x), (x \neq R' \vee u \notin x)\} \\ T & \text{for } \{u : (A(u) \cap A^*) = A(u), (A(u) \cap A^*) \neq A(u)\} \end{array}$$

$A^*$ ,  $A(u)$  and  $T$  are as they were defined in [18] with the exception that there  $\subseteq$  incorrectly took the place of  $=$  in the definition of  $A(u)$  and subsequent proofs. In [18] it was shown that

$$(4.2) \quad (A(T) \cap A^*) =_e A(T)$$

is provable in  $PST'$ . The attributes  $(A(T) \cap A^*)$  and  $A(T)$  apply, therefore, to exactly the same members of  $D'_e$ . It is not the case,

however, that any attribute of  $D'_e$  applying to  $(A(T) \cap A^*)$  also applies to  $A(T)$  since

$$(4.3) \quad (A(T) \cap A^*) = A(T)$$

is not true in the interpretation of  $PST'$  provided in [18] over the domain  $D'_e$ , for there identity was interpreted as syntactic identity. That (4.3) is not true in the interpretation is of no consequence; the important conclusion of [18] is that no interpretation of  $PST'$  is possible in which (4.3) is true, for the negation of that identity is a theorem of  $PST'$ . The terms  $(A(T) \cap A^*)$  and  $A(T)$  are counterexamples to the axiom of extensionality; that axiom is inconsistent with  $PST'$ . With the interpretation (2.6) of the axiom of extensionality in mind, this latter conclusion should not be too alarming.

A natural substitute for the axiom of extensionality as a means for extending  $PST'$  to a suitable theory of identity is a rule of extensionality:

$$(4.4) \quad \text{When } (\mathbf{u})(s =_e t) \text{ is a theorem of } PST', \text{ then so is } (\mathbf{u})(s = t).$$

But this rule, too, is blocked by the above example since (4.2) and the negation of (4.3) are both theorems of  $PST'$ .

The inconsistency of (4.3) with  $PST'$  is itself not an unwanted result. For from (4.3) follows  $T \in T$  by (3.6) for the term  $T$ ; hence,  $T \in R'$  by (3.6) for the term  $R'$ ; hence,  $(R' = R' \ \& \ T \in R')$  since  $R' = R'$  is provable in  $PST'$ , and, therefore,  $R' \in A(T)$  by (3.6) for the term  $A(T)$ . But if  $R' \in A(T)$  is a consequence of (4.3), then so is  $R' \in (A(T) \cap A^*)$  by (2.2), and hence,  $R' \in A^*$  by (3.6) for  $(A(T) \cap A^*)$ , and finally,  $R' \neq R'$  by (3.3) for  $A^*$ , the definition of  $\neq$  and (3.2). But  $R' \neq R'$  contradicts (3.2) and, therefore, the negation of (4.3) is a theorem of  $PST'$  by a simple and direct proof.

We have found, however, no simple and direct proof of (4.2). The one offered in [19], which is simpler than the one offered in [18], still has a curious feature: The formula (4.3) occurs as a subformula of the proof of (4.2). Specifically, that proof of (4.2) requires a substitution instance of the axiom (2.2) with  $x$  replaced by  $(A(T) \cap A^*)$  and  $y$  replaced by  $A(T)$ . Accepting such a proof for the premise (4.2) of the rule (4.4) to permit the conclusion (4.3) would mean that (4.3) would appear as a proper subformula of a theorem appearing in its own proof! This suggests that the rule (4.4) can be consistently added to  $PST'$  if proofs for a premise of that rule were restricted in such a way as to enjoy some kind

of subformula property, as is the case, for example, with the normal proofs of [22].

A method for extending  $PST'$  to a theory of identity  $PST$  is described in detail in [20]. The method uses an extension of Gentzen's formulation  $G1$  of classical logic described in [21]. The consistency of the theory  $PST$  is still open and is not trivial since Peano's axioms for arithmetic are theorems of  $PST$ . But perhaps a few speculative remarks are not out of order. If  $PST$  is consistent, it is not unreasonable to expect a model of it to be found, as a model of  $PST'$  was found, with the domain  $D'_c$ . That  $D'_c$  is denumerable is itself no obstacle to being a satisfactory set theory as a relative interpretation of denumerability, as Skolem in [8] and we in [17] have argued, is a satisfactory way of interpreting nondenumerability over denumerable domains. But that the structure of  $D'_c$  is so simple is probably an obstacle to the proof of nonexistence of a mapping of the integers onto the power set of the integers. This is also strongly suggested by the close resemblance of the Cantor diagonal argument to the Russell paradox; that argument is likely to lead to no conclusion for the same reason that the paradox was disarmed. We are not suggesting that  $PST$  is like Wang's theory  $\Sigma$  described in [23] in which all sets are provably denumerable. Rather, we believe the nondenumerability of the power set of the integers will prove to be independent in  $PST$  — the first of a series of independent axioms required to develop a theory of nondenumerable cardinals.

## 5. Other views

It would be impossible to adequately present in one section all conceptions of sets in opposition to those presented in Section 2. But we do want to remark upon related opposing conceptions which have received considerable attention, the ones presented by the intuitionists, predicativists and constructivists. In treating these views together, no suggestion is made that they are identical; indeed, Feferman's distinction made in [24] between the latter two groups will be relied upon.

Common to the three groups is the belief that sets are formed or generated in some sense from pre-existing objects. The definition of a species, the closest intuitionistic counterpart to a set, reads on page 37 of [27], "A species is a property which mathematical entities can be supposed to possess", and on the next page the intuitionistic protagonist remarks, "Circular definitions

are excluded by the condition that the members of a species  $S$  must be definable independently of the definition of  $S$ ; this condition is obvious from the constructive point of view." On page 2 of [24], Feferman writes, "In order, for example, to predicatively introduce a set  $S$  of natural numbers  $x$  we must have before us a condition  $\mathcal{F}(x)$ , in terms of which we define  $S$  by

$$(5.1) \quad \Delta x[x \in S \leftrightarrow \mathcal{F}(x)].$$

However, before we can assert the existence of such  $S$ , it should already have been realized that the defining condition  $\mathcal{F}(x)$  has a well-defined meaning which is independent of whether or not there exists a set  $S$  satisfying (1.1) (but which can depend upon what sets have been previously realized to exist)." On page 245 of [23], Wang writes of a definition just given, "Hence in order to define  $N$ ,  $N$  must already be there. This is clearly unacceptable from a constructive viewpoint . . .".

To have some pre-existing objects to begin the process of generating sets, the intuitionists, predicativists and constructivists generally accept the integers or their equivalents as given. The intuitionists justify their acceptance of the integers by indicating some process for generating them, but the others are generally content with accepting them a priori. We believe that there is a certain arbitrariness in this acceptance, especially on the part of the predicativists. Assuming 0 and the successor relation, the set of integers in classical set theory is defined to be the intersection of all sets which contain 0 and are closed under the successor relation. This is an impredicative definition in the same sense as the definition of the least upper bound of a set of reals given originally in [28] and again in [24]. Why, therefore, should the set of integers be accepted when the least upper bound definition is rejected?

Leaving this question, let us examine the concept of impredicative definitions for the theory  $PST'$ . Both the above two quoted conceptions of sets are compatible with the interpretation of a theory of sets as a theory of attributes. With Heyting, who has quoted Brouwer's definition of species, the identification of species with attributes is explicit (although, of course, it does not follow that the law of the excluded middle is necessarily acceptable). With Feferman it is difficult to see why he would prefer speaking of the set  $S$  rather than the attribute  $\{x : \mathcal{F}(x)\}$ , especially in the light of his earlier assertion from [24], "... sets are created by man to act as convenient abstractions (façons de parler) from

particular conditions or definitions." With Wang, his abstraction scheme  $C$  on page 248 of [23] has the same form as Feferman's (1.1).

Interpreting set theory as a theory of attributes does not lessen the requirement that attributes are defined after the objects in their range of significance have been generated or seen to exist. But in what way is that requirement not met by the theory  $PST'$ ? Recall that  $PST'$  has a model over the domain  $D'_c$  of constant terms of  $PST'$ . The domain  $D'_c$  is the pre-existing range of objects for which "attributes" are defined in our sense of attribute; the axiom schemes (3.3) and (3.4) "define" the attributes in the same sense in which Feferman's (5.1) defines the set  $S$ . If we think of the instances of (3.3) and (3.4) being introduced one at a time, then the defining conditions  $P(v)$  and  $Q(v)$  have well-defined meanings independent of any instance of (3.3) and (3.4).

A brief technical remark concerning Feferman's system  $PS$  of [25] and  $PST'$  is not out of place. Feferman exploits in his system the  $\leq$ -persistence of  $\Sigma$ -formulae established in [26]. It is interesting that the consistency proof of  $PST'$  given in [18] exploits in almost the same way another persistence theorem established by Lyndon [29] for positive formulae.

#### BIBLIOGRAPHY

##### BERTRAND RUSSELL

- [1] Introduction to Mathematical Philosophy, George Allen and Unwin, Ltd., London, 1919.

##### WILLARD VAN ORMAN QUINE

- [2] Mathematical Logic, Harvard University Press, Cambridge, revised edition, 1951.

##### WILLARD VAN ORMAN QUINE

- [3] From a Logical Point of View, Harvard University Press, Cambridge, 1953.

##### WILLARD VAN ORMAN QUINE

- [4] Word and Object, The Technology Press of MIT and John Wiley and Sons, Inc., New York and London, 1960.

##### WILLARD VAN ORMAN QUINE

- [5] Set Theory and Its Logic, The Belknap Press of Harvard University Press, Cambridge, 1963.

##### ABRAHAM A. FRAENKEL

- [6] Abstract Set Theory, North-Holland Publishing Co., Amsterdam, 1953.

##### ALONZO CHURCH

- [7] "A formulation of the simple theory of types", Journal of Symbolic Logic, Vol. 5 (1940), 56-68.



**THORALF SKOLEM**

- [8] "Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre", *Conférences au 5e. Congrès [1922] des Mathématiciens Scandinaves (Helsingfors, 1923)* 218–232.

**R. O. GANDY**

- [9] "On the axiom of extensionality — Part I", *Journal of Symbolic Logic*, Vol. 21 (1956) 36–48.

**R. O. GANDY**

- [10] "On the axiom of extensionality — Part II", *Journal of Symbolic Logic*, Vol. 24 (1959) 287–300.

**KURT GÖDEL**

- [11] *The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory*, Princeton University Press, Princeton, 1940; second printing, 1951.

**DANA SCOTT**

- [12] "Lectures on Boolean-valued models for set theory", lecture notes prepared in connection with the Summer Institute on Axiomatic Set Theory, UCLA, Los Angeles, July 10–August 4, 1967.

**FREDERIC BENTON FITCH**

- [13] *Symbolic Logic, An Introduction*, The Ronald Press Co., New York, 1952.

**FREDERIC BENTON FITCH**

- [14] "A complete and consistent modal set theory", *Journal of Symbolic Logic*, Vol. 32 (1967) 93–103.

**P. C. GILMORE**

- [15] "On the epsilon relation", Abstract of paper presented to the Association for Symbolic Logic, Vol. 21 (1956) 219.

**P. C. GILMORE**

- [16] "The monadic theory of types in the lower predicate calculus", *Proceedings of the Summer Institute for Symbolic Logic*, Cornell University (1957) 309–312.

**P. C. GILMORE**

- [17] "An alternative to set theory", *American Mathematical Monthly*, Vol. 67 (1960) 621–632.

**P. C. GILMORE**

- [18] "Partial set theory", lecture notes prepared in connection with the Summer Institute on Axiomatic Set Theory, UCLA, Los Angeles, July 10–August 4, 1967.

**P. C. GILMORE**

- [19] "The consistency of partial set theory without extensionality", IBM Research Report RC1973, Dec. 21, 1967.

**P. C. GILMORE**

- [20] "A theory of identity for partial set theory", a forthcoming manuscript.

**S. C. KLEENE**

- [21] *Introduction to Metamathematics*, Nort-Holland, Amsterdam, 1952.

**DAG PRAWITZ**

- [22] *Natural Deduction, a Proof-Theoretical Study*, Almquist and Wiksell, Stockholm, 1965.

**HAO WANG**

- [23] "The formalization of mathematics", *Journal of Symbolic Logic*, Vol. 19 (1954) 241–266.

**SOLOMON FEFERMAN**

- [24] "Systems of predicative analysis", *Journal of Symbolic Logic*, Vol. 29 (1964) 1–30

**SOLOMON FEFERMAN**

- [25] "Predicative provability in set theory", *Bulletin of the American Mathematical Society*, Vol. 72 (1966) 486–489.

**SOLOMON FEFERMAN and G. KREISEL**

- [26] "Persistent and invariant formulas relative to theories of higher order," *Bulletin of the American Mathematical Society*, Vol. 72 (1966) 480–485.

**A. HEYTING**

- [27] *Intuitionism, an Introduction*, North-Holland Publishing Co., Amsterdam, 1946.

**H. WEYL**

- [28] *Das Koninuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Leipzig, 1918.

**RODGER C. LYNDON**

- [29] "Properties preserved under homomorphism", *Pacific Journal of Mathematics*, Vol. 9 (1959) 143–154.

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