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Sequential properties of ordered topological spaces

by

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The concepts of Fréchet space and sequential space are known [1] to provide successive proper generalizations of first countable spaces. It is also known that neither the Fréchet property nor the sequential property is productive; in fact a product of two Fréchet spaces need not be sequential. In the present paper we show that for products of ordered topological spaces (products in which each coordinate space has the order topology arising from a total order), the situation is quite different. All three properties (first countable, Fréchet, sequential) are equivalent, and these properties are preserved under the formation of products of a countably infinite number of coordinate spaces. Furthermore, this approach yields higher cardinality versions of the above statements so as to apply to arbitrary products of ordered spaces without countability restrictions.

Background

We begin by summarizing some material from [5] which will be needed here. It is sometimes of interest to know the extent to which a topology is determined by its convergent $\mathfrak{m}$-nets. (An $\mathfrak{m}$-net is a net whose directed set has cardinality $\leqq \mathfrak{m}$, where $\mathfrak{m}$ is an infinite cardinal number. For nets, as well as other topological concepts, we follow the terminology and notation of Kelley [2].) In a topological space (X, t) we form the $\mathfrak{m}$-closure of a subset A (denoted $\mathfrak{m}$-cl A) by adding to A all limits of convergent $\mathfrak{m}$-nets in A. If $\mathfrak{m}$-cl $A = t$-cl A for all subsets A of X, we say that (X, t) is an $\mathfrak{m}$-*Fréchet space*. More generally, if we can obtain the t-closure operator by iteration of the $\mathfrak{m}$-closure operator, we say that (X, t) is an $\mathfrak{m}$-sequential space. For the case $\mathfrak{m} = \aleph_0$, these definitions are equivalent to the usual ones [1], because every non-trivial $\aleph_0$-net has a cofinal sequence.

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This paper grew out of earlier work on spaces of real-valued functions. For several classes of function spaces it is known that the relative product topology (= pointwise topology) is sequential iff it is Fréchet. Higher cardinality versions of this might have bearing on the Aleksandrov-Urysohn problem about the cardinality of first countable compact Hausdorff spaces. For details see [3] and [4].

THEOREM. *If X is a (totally) ordered topological space or is a product of such spaces, then the following are equivalent:*

a) *X is $\mathfrak{m}$-sequential.*

b) *X is $\mathfrak{m}$-Fréchet.*

c) *The point character of X, $\chi(X)$, is $\leqq \mathfrak{m}$ (i.e., each point of X has a neighborhood base of cardinality $\leqq \mathfrak{m}$). Furthermore, if X is the product of a family of (totally) ordered spaces $\{X_i : i \in I\}$, then the least $\mathfrak{m}$ for which the preceding statement holds is the larger of the two numbers card I and* $\sup \{\chi(X_i) : i \in I\}$.

REMARK. The proof also yields the following. Let X be the product of a family of non-trivial topological spaces $\{X_i : i \in I\}$. In order that X be $\mathfrak{m}$-sequential it is necessary that each X_i be $\mathfrak{m}$-sequential and card $I \leqq \mathfrak{m}$. If, in addition, each X_i is an ordered topological space, then these conditions are also sufficient.

PROOF. For arbitrary topological spaces it is true that c) $\Rightarrow$ b) [5, Prop. 3.1] and b) $\Rightarrow$ a) (trivial). We prove the opposite implications first for a single ordered space and then for a product.

Assume that X is an ordered topological space; we show first that b) $\Rightarrow$ c). For an arbitrary point x in X, we construct a neighborhood base of cardinality $\leqq \mathfrak{m}$. Assume x is not isolated from above or below (the other cases are easier). Then by *b*) there exists a net $\{x_\nu, \nu \in D\}$ converging to x with $x_\nu < x$ for each ν and card $D \leqq \mathfrak{m}$. Similarly, there is a net $\{y_\mu, \mu \in E\}$ converging to x with $y_\mu > x$ and card $E \leqq \mathfrak{m}$. The open intervals $\{z : x_\nu < z < y_\mu\}$ with $(\nu, \mu) \in D \times E$ form a neighborhood base at x and card $D \times E \leqq \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$.

Still assuming that X is an ordered space, we now show that a) $\Rightarrow$ b). Suppose that nets $S_\nu = \{x_{\nu\mu}, \mu \in E_\nu\}$ converge to x_ν and $S = \{x_\nu, \nu \in D\}$ converges to x, with card E_ν and card D not greater than $\mathfrak{m}$. It certainly suffices to construct a net in the set formed by the union of the ranges of the nets S_ν, directed by D or a subset of D, and converging to x. We may assume that all of the given nets are strictly monotone and directed by ordinal numbers. (This simplifying assumption is justified in Lemma 1 below.) There are essentially two cases. For the first case, assume that the nets S_ν are increasing and that the net S is decreasing. Since $x_\nu > x$ for each ν, we can choose $\mu(\nu)$ such that

$$x_\nu > x_{\nu\mu(\nu)} \geqq x.$$

Then $\{x_{\nu\mu(\nu)}, \nu \in D\}$ is the desired net; it converges to x since it is bounded above by net S which converges down to x. For the second case, assume that all of the nets are strictly increasing. Let D' denote the set of all isolated ordinals in D; clearly D' is cofinal in D. For ν in D', $x_{\nu-1} < x_\nu$ and hence we may choose $\mu(\nu)$ such that $x_{\nu-1} < x_{\nu\mu(\nu)}$. Then $\{x_{\nu\mu(\nu)}, \nu \in D'\}$ is the desired net which converges to x. This completes the proof for the case in which X consists of one ordered space.

We now assume that X is the product of a family $\{X_i : i \in I\}$ and that each X_i is ordered. We prove directly that a) $\Rightarrow$ c) for such X. By a) and the fact that every quotient of an $\mathfrak{m}$-sequential space is $\mathfrak{m}$-sequential [5, Cor. 2.2], it follows that each X_i is $\mathfrak{m}$-sequential. By our earlier work, it follows that $\chi(X_i) \leqq \mathfrak{m}$ for each i. From Lemma 2 below we have card $I \leqq \mathfrak{m}$. Thus, by Lemma 3, $\chi(X) \leqq \mathfrak{m}$, and c) holds. The proof of the theorem will be complete if we establish the lemmas.

Lemma 1. *Let X be an ordered topological space and A a subset of X. If an $\mathfrak{m}$-net in A converges to x (with x in X but not in A), then there is a strictly monotone $\mathfrak{m}$-net in A which is directed by an ordinal and converges to x. (However, the monotone net is not in general a subnet of the original net — such a subnet does not always exist.)*

Proof. Let A_0 be the intersection of A and the range of the $\mathfrak{m}$-net in the hypothesis. This replacement is made to obtain the inequality card $A_0 \leqq \mathfrak{m}$. Let

$$A_1 = \{y \in A_0 : y < x\} \quad \text{and} \quad A_2 = \{y \in A_0 : y > x\}.$$

It follows that $x \in \mathfrak{m}\text{-cl}\, A_i$ for $i = 1$ or $i = 2$; assume the former. A_1 is directed by the order inherited from X, and thus the identity mapping $A_1 \to X$ is a net. (A_1 is both the directed set and the range of this net.) This net is strictly monotone and converges to x. Since A_1 is totally ordered, Zorn's lemma can be used to show that A_1 has a cofinal well ordered subset A_3. A_3 is an ordinal and the identity mapping $A_3 \to X$ is the desired net; it is an $\mathfrak{m}$-net since card $A_3 \leqq$ card $A_0 \leqq \mathfrak{m}$.

For the next lemma in the case $\mathfrak{m} = \aleph_0$, see J. Novák [6, page 23].

Lemma 2. *Let X be a product of a family $\{X_i : i \in I\}$ of non-trivial topological spaces. (Here the spaces need not be ordered; we assume only that each has at least one non-void proper open set.) If card $I > \mathfrak{m}$, then X is not $\mathfrak{m}$-sequential. In particular, no uncountable product of non-trivial spaces is a sequential space.*

Proof. The proof is patterned after a familiar argument about sequences in spaces of real-valued functions. We may assume that each X_i contains two points, which we denote by 0 and 1, and a neighborhood of 1 not containing 0. This notation is chosen so that some function space terminology can be used. Let e denote the function in X whose i'th value is 1 for each i. Let S be the subset of X consisting of all characteristic functions of finite subsets of I (i.e., all functions which take the value 1 at a finite number of coordinates and the value 0 elsewhere). Then e is a limit point of S, but we can show that no iteration of $\mathfrak{m}$-nets in S can converge to e. The functions in S have finite cozero sets. (The cozero set of a function f is $\{i \in I : f(i) \neq 0\}$; we denote it by coz f.) If we show that by forming iterated limits of $\mathfrak{m}$-nets in S we get only functions whose cozero sets have cardinality $\leqq \mathfrak{m}$, it follows that e cannot be so approached and the product topology is not $\mathfrak{m}$-sequential. The assertion is verified by the following: If $\{f_\nu, \nu \in D\}$ converges to f with card $D \leqq \mathfrak{m}$ and card (coz f_ν) $\leqq \mathfrak{m}$ for each ν, then

$$\text{card}\,(\text{coz}\, f) \leqq \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$$

(because $\text{coz}\, f \subset \bigcup \{\text{coz}\, f_\nu : \nu \in D\}$).

Lemma 3. *Assume as in Lemma 2 that X is the product of a family $\{X_i : i \in I\}$ of non-trivial topological spaces (not necessarily ordered spaces). Then the point character of X, $\chi(X)$, is the larger of the two numbers card I and* sup $\{\chi(X_i) : i \in I\}$.

Proof. This is a generalization of the fact that a countable product of first countable spaces is first countable, and the proof is similar. We omit the details.

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