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On Grothendieck universes


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The purpose of this note is to observe that by using the full set theory of Gödel [2], i.e. including the axiom D (Foundation), Grothendieck universes may be characterized as follows: $U$ is a Grothendieck universe if and only if for some inaccessible cardinal $\alpha$, $U$ is the collection of all sets of rank $\alpha$.

Define a set $U$ to be a Grothendieck universe, after Gabriel [1], if

UA1. For each $x$, $x \in U \Rightarrow x \subseteq U$,
UA2. For each $x$, $x \in U \Rightarrow P(x) \in U$,
UA3. For each $x$, $x \in U \Rightarrow \{x\} \in U$,
UA4. For each family $\{x_i\}_{i \in I}$ such that $I \in U$ and such that $x_i \in U$ for each $i \in I$, $\bigcup \{x_i : i \in I\} \in U$,
UA5. $U$ is non-empty.

A cardinal number is to be understood as an ordinal number which is equipollent to no smaller ordinal number.

Call a cardinal number $\alpha$ inaccessible in the narrower sense of Tarski [6] (henceforth referred to as inaccessible) if

IA1. $\text{card} \left( \bigcup \{x_i : i \in I\} \right) < \alpha$ for each family $\{x_i\}_{i \in I}$ of sets with $\text{card} (I) < \alpha$ and with $\text{card} (x_i) < \alpha$ for each $i \in I$,
IA2. For any two cardinals $\xi$ and $\eta$, $\xi^\eta < \alpha$ whenever $\xi < \alpha$ and $\eta < \alpha$.

Use transfinite induction to define a function $\Psi$ over the class of all ordinals (after von Neumann [4]) by

(1) $\Psi(0) = 0$,
(2) $\Psi(\beta + 1) = P(\Psi(\beta))$,
(3) if $\lambda$ is a limit ordinal, $\Psi(\lambda) = \bigcup \{\Psi(\beta) : \beta < \lambda\}$

Thus $\Psi$ is the rank function: $\Psi(\beta)$ is the collection of all sets of rank $\leq \beta$. Then the following results are well known (e.g. Shepherdson [5]):
(a) For each set \( x \), there is an ordinal \( \beta \) such that
\[ x \in \mathcal{P}(\beta), \]
(b) If \( \alpha \) is an inaccessible cardinal, then
\[ \beta < \alpha \Rightarrow \text{card}(\mathcal{P}(\beta)) < \alpha, \]
(c) If \( \alpha \) is an inaccessible cardinal, then
\[ x \in \mathcal{P}(\alpha) \iff x \subseteq \mathcal{P}(\alpha) \& \text{card}(x) < \alpha. \]

It is shown in Kruse [3] that \( U \) is a Grothendieck universe to which an infinite set belongs if and only if \( U \) is a super-complete model in the sense of Shepherdson [5]. In Shepherdson [5], it is shown that all the super-complete models are of the form \( \mathcal{P}(\alpha) \) for some uncountable inaccessible cardinal \( \alpha \). It is the purpose of this note to distinguish to which \( \mathcal{P}(\alpha) \) a given universe \( U \) corresponds.

The following result is proved in Kruse [3]:
\( U \) is a Grothendieck universe if and only if there exists an inaccessible cardinal \( \alpha \) such that \( x \in U \iff x \subseteq U \& \text{card}(x) < \alpha. \)

For a given universe \( U \), this inaccessible cardinal is clearly uniquely determined. Write it as \( \alpha(U) \). Then the main result is:

**Theorem 1.** \( U = \mathcal{P}(\alpha(U)) \).

**Proof.** First, to show that \( \mathcal{P}(\alpha(U)) \subseteq U \). For this, use transfinite induction on the ordinal \( \beta \) to prove
\[ A(\beta) : \beta < \alpha(U) \Rightarrow \mathcal{P}(\beta) \subseteq U. \]

(i) Trivially \( A(0) \).

(ii) Assume \( A(\beta) \); to show \( A(\beta + 1) \). Let \( \beta + 1 < \alpha(U) \), then \( \beta < \alpha(U) \) and so \( \mathcal{P}(\beta) \subseteq U \). But \( \beta < \alpha(U) \Rightarrow \text{card}(\mathcal{P}(\beta)) < \alpha(U) \), by (b). Hence \( \mathcal{P}(\beta) \subseteq U \& \text{card}(\mathcal{P}(\beta)) < \alpha(U) \), thus \( \mathcal{P}(\beta) \in U \).

But then \( \mathcal{P}(\beta + 1) = \mathcal{P}(\mathcal{P}(\beta)) \in U \), by \( \text{UA2} \), and so \( \mathcal{P}(\beta + 1) \subseteq U \), by \( \text{UA1} \). Hence \( A(\beta + 1) \).

(iii) Let \( \lambda \) be a limit ordinal, and assume \( \beta < \lambda \Rightarrow A(\beta) \); to show \( A(\lambda) \). Let \( \lambda < \alpha(U) \), and then \( \beta < \lambda \Rightarrow \mathcal{P}(\beta) \subseteq U \). Hence \( \mathcal{P}(\lambda) = \bigcup \{ \mathcal{P}(\beta) : \beta < \lambda \} \subseteq U \); thus \( A(\lambda) \).

Hence, for any ordinal \( \beta \), \( \beta < \alpha(U) \Rightarrow \mathcal{P}(\beta) \subseteq U \). But
\[ \mathcal{P}(\alpha(U)) = \bigcup \{ \mathcal{P}(\beta) : \beta < \alpha(U) \}, \] hence \( \mathcal{P}(\alpha(U)) \subseteq U \).

Now suppose that \( U \setminus \mathcal{P}(\alpha(U)) \neq 0 \).

Then, by the axiom of foundation, there is a set \( x \) such that
\[ x \in U \setminus \mathcal{P}(\alpha(U)) \& x \cap [U \setminus \mathcal{P}(\alpha(U))] = \emptyset. \]
Since \( x \in U \Rightarrow (x \subseteq U \& \text{card}(x) < \alpha(U)) \), and
\[
(x \subseteq U \& x \cap [U \setminus \mathcal{P}(\alpha(U))] = 0 ) \Rightarrow x \subseteq \mathcal{P}(\alpha(U)),
\]
\( x \subseteq \mathcal{P}(\alpha(U)) \& \text{card}(x) < \alpha(U), \)

hence \( x \in \mathcal{P}(\alpha(U)) \), by (c). But this contradicts \( x \in U \setminus \mathcal{P}(\alpha(U)) \),
and hence \( U = \mathcal{P}(\alpha(U)) \).

**Theorem 2.**
If \( U \) is a Grothendieck universe, then \( \text{card}(U) = \alpha(U) \).

This follows from the fact that for inaccessible cardinals \( \alpha \),
\( \text{card}(\mathcal{P}(\alpha)) = \alpha \), (which may be shown by relativizing to \( \mathcal{P}(\alpha) \)
the proof of the existence of a one-to-one mapping from the class
of all ordinals to the class of all sets).

Note that the equivalence of the two statements:

(A) For every cardinal \( \beta \), there is an inaccessible cardinal \( \alpha \)
such that \( \beta < \alpha \),

(B) For every set \( x \), there is a Grothendieck universe \( U \) such
that \( x \in U \),

follows immediately from (a) and Theorem 1.

**References**

P. Gabriel


K. Gödel


A. H. Kruse


J. Von Neumann,


J. C. Shepherdson


A. Tarski


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