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by

A. K. Varma ¹

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A systematic study of Lacumary interpolation² was first initiated by Suranyi, J. and Turán, P. [8] and Balázs and Turán, P. [1, 2, 3] in the special case when the values and second derivatives are prescribed on the zeros of \( \pi_n(x) = (1-x^2)P'_{n-1}(x) \) where \( P_{n-1}(x) \) is the Legendre polynomial of degree \( \leq n-1 \) while the existence and uniqueness have been shown for the abscissas as the zeros of ultraspherical polynomials \( P^{(\lambda)}_n(x), \lambda \geq -\frac{1}{2} \) the explicit representation and convergence theorems have been proved for \( \pi \) abscissas only. Later the convergence theorem of Balázs and Turán [3] is sharpened by Freud, G. [4] in the sense that the interpolatory polynomials of Balázs and Turán converges uniformly to given \( f(x) \) in \([-1, +1]\) if \( f(x) \) satisfies the Zygmund condition

\[
|f(x+h) - 2f(x) + f(x-h)| = o(h) \quad \text{in \([-1, +1]\).}
\]

Other interesting results are due to Saxena and Sharma [7, 8], Kis [5, 6], Varma and Sharma [11, 12] and Varma [15, 16].

The object of this paper is to consider the problem of existence, uniqueness, explicit representation and convergence of the sequence \( R_n(x) \) of polynomials of degree \( \leq 3n+3 \) such that \( R_n(x), R'_n(x) \) are prescribed at the zeros of \( (1-x^2)u_n(x) \) where

\[
u_n(x) = \frac{\sin \left( n+1 \right) \theta}{\sin \theta}, \quad x = \cos \theta,
\]

while \( R''_n(x) \) is prescribed at all the above abscissas except at \(-1\) and \(+1\). We shall call this “modified” \((0, 1, 3)\) interpolation.

In § 2 we state the existence theorem and give the explicit rep-

¹ The author is thankful to Prof. P. Turán and Prof. A Sharma for some valuable suggestions.

² They called it \((0, 2)\) case.
presentation of these polynomials in a most suitable form and in § 3 and onwards we prove convergence theorem. It is interesting to remark that in modified \((0, 2)\) interpolation \([15]\) we require for the uniform convergence of the sequence of polynomials \(R_n(x)\) to \(f(x)\) is that \(f'(x) \in \text{Lip } \alpha, \alpha > \frac{1}{2}\), and this is best possible in a certain sense. So one would be inclined to think that in modified \((0, 1, 3)\) interpolation, we may require \(f(x)\) to be twice differentiable or at least \(f'(x) \in \text{Lip } \alpha, \alpha > \frac{1}{2}\) [compare corresponding theorem of Saxena and Sharma \([7]\)]. But our theorem 3.1 asserts that this is not really the case. Here we need only \(f'(x) \in \text{Lip } \alpha, \alpha > 0\). Although we could not prove that this is best possible, it seems quite plausible that this is really so in view of other known results in this direction \([7, 8]\).

2

Let us consider the set of numbers
\begin{equation}
-1 = x_{n+2} < x_{n+1} < \ldots < x_2 < x_1 = +1
\end{equation}
by which we shall denote the zeros of \((1-x^2)u_n(x)\), where
\[ u_n(x) = \frac{\sin (n+1)\theta}{\sin \theta}, \quad x = \cos \theta. \]
Then we have the following

**Theorem 2.1** If \( n = 2k \), then to prescribed values \( f(x_i), f'(x_i) \) \((i = 1, 2, \ldots, n+2)\) and \( \delta_i \) \( (i = 2, 3, \ldots, n+1) \) there is a uniquely determined polynomial \( R_n(x) \) of degree \( \leq 3n+3 \) such that
\begin{equation}
R_n(x_i) = f(x_i), \quad R'_n(x_i) = f'(x_i), \quad i = 1, 2, \ldots, n+2
\end{equation}
\begin{equation}
R''_n(x_i) = \delta_i, \quad i = 2, 3, \ldots, n+1.
\end{equation}

But if \( n \) is odd, \( n = 2k+1 \) there is in general no polynomial \( R_n(x) \) of degree \( \leq 3n+3 \) which satisfies (2.2) and (2.3) and if there exists such a polynomial then there is an infinity of them.

From the uniqueness theorem it follows that \( R_n(x) \) is given by
\begin{equation}
R_n(x) = \sum_{i=1}^{n+2} f(x_i) A_i(x) + \sum_{i=1}^{n+2} f'(x_i) B_i(x) + \sum_{i=2}^{n+1} \delta_i C_i(x)
\end{equation}
where the polynomials \( A_i(x) \), \( B_i(x) \) and \( C_i(x) \) are the fundamental polynomials of degree \( \leq 3n+3 \). Their explicit forms are given by the following
THEOREM 2.2 For $n$ even, the fundamental polynomials have the following representation

(a) For $i = 2, 3, \ldots, n+1$ we have

\begin{equation}
(2.5) \quad C_i(x) = \frac{(1-x^2)^{\frac{3}{2}}u_n^2(x)}{6(1-x^2)u_n^2(x_i)} \left[ a_i \int_{-1}^{x} \frac{u_n(t)}{(1-t^2)^{\frac{1}{2}}} \, dt + \int_{-1}^{x} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right]
\end{equation}

where

\begin{equation}
(2.6) \quad xa_i = -\int_{-1}^{t} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt, \quad l_i(t) = \frac{u_n(t)}{(t-x_i)u_n'(x_i)}.
\end{equation}

(b)

\begin{equation}
(2.7) \quad B_1(x) = \frac{(1-x^2)u_n^2(x)}{4(n+1)^3} \left[ (1+x)u_n(x) + (1-x^2)u_n'(x) \right] + (1-x^2)^{\frac{1}{2}} \int_{-1}^{x} \frac{u_n'(t)}{(1-t^2)^{\frac{1}{2}}} \, dt
\end{equation}

\begin{equation}
(2.8) \quad B_{n+2}(x) = \frac{(1-x^2)u_n^2(x)}{4(n+1)^3} \left[ (1-x)u_n(x) - (1-x^2)u_n'(x) \right] + (1-x^2)^{\frac{1}{2}} \int_{-1}^{x} \frac{u_n'(t)}{(1-t^2)^{\frac{1}{2}}} \, dt
\end{equation}

and for $i = 2, 3, \ldots, n+1$ we have

\begin{equation}
(2.9) \quad B_i(x) = \frac{(1-x^2)^3l_i^2(x)u_n(x)}{(1-x_i)^2u_n^2(x_j)} + \frac{(1-x^2)^{\frac{3}{2}}u_n^2(x)}{(n+1)^2} \left[ b_i \int_{-1}^{x} \frac{u_n(t)}{(1-t^2)^{\frac{1}{2}}} \, dt + c_i \int_{-1}^{x} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt + \int_{-1}^{x} \frac{p_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right]
\end{equation}

where

\begin{equation}
(2.10) \quad p_i(t) = \frac{3x_i^2l_i(t) - 2(1-t^2)l_i'(t)}{2(t-x_i)} = \frac{2(1-x_i)^2}{(n+1)} \sum_{r=0}^{n-1} \alpha_r u_r(x_i)u_r(t)
\end{equation}

\begin{equation}
(2.11) \quad \alpha_r = n(n+2) - r(r+2) - 3
\end{equation}

\begin{equation}
(2.12) \quad x^2c_i = \frac{(n+3)(n-1)}{6} + \frac{a_i}{2(1-x_i)^2} + \frac{x_i}{2(1-x_i^2)}
\end{equation}

\begin{equation}
(2.13) \quad x^2b_i = -\int_{-1}^{x} \frac{p_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt - c_i \int_{-1}^{x} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt
\end{equation}

(c)

\begin{equation}
(2.14) \quad A_1(x) = \frac{(1+x^2)u_n^2(x)}{4(n+1)^3} - (n+1)^2 B_1(x) - \frac{(1-x^2)^2u_n^2(x)u_n'(x)}{8(n+1)^3} - \frac{(1-x^2)^{\frac{3}{2}}u_n^2(x)}{4(n+1)^3} \int_{-1}^{x} \frac{(1+t^2)u_n'(t)}{(1-t^2)^{\frac{1}{2}}} \, dt
\end{equation}
and for $i = 2, 3, \ldots, n+1$ we have

$$A_i(x) = \frac{(1-x^2)^2 l_i^3(x)}{(1-x_i^2)^2} + \frac{x_i B_i(x)}{2(1-x_i^2)} + \frac{(1-x^2)^{3/2} u_n^3(x)}{(n+1)^2}$$

$$\cdot \left[ d_i \int_{-1}^{x} \frac{u_n(t)}{(1-t^2)^{3/2}} dt + c'_i \int_{-1}^{x} \frac{l_i(t)}{(1-t^2)^{3/2}} dt + \int_{-1}^{x} \frac{q_i(t)}{(1-t^2)^{3/2}} dt \right]$$

where

$$c_i = 3x_i \left[ \frac{(n+3)(n-1)-6}{4(1-x_i^2)} + \frac{37x_i^2+36}{12(1-x_i^2)^2} \right]$$

$$2x_i = -\int_{-1}^{x} \frac{q_i(t)}{(1-t^2)^{3/2}} dt - c'_i \int_{-1}^{x} \frac{l_i(t)}{(1-t^2)^{3/2}} dt$$

$$q_i(t) = \frac{(a'_i t+b'_i) l_i(t) - (1-t^2) l'_i(t)}{(t-x_i)^2}$$

$$a'_i x_i + b'_i = \frac{3x_i}{2}$$

$$a'_i = \left[ \frac{-x_i^2}{4(1-x_i^2)} - \frac{(n+3)(n-1)}{3} \right].$$

In view of the uniqueness theorem 2.1, it remains to verify that the fundamental functions $A_i(x)$, $B_i(x)$ and $C_i(x)$ are polynomials of degree $\leq 3n+3$ and satisfy the following conditions.

$$A_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad A'_i(x_j) = 0 \quad i, j = 1, 2, \ldots, n+2$$

$$A''''(x_j) = 0$$

$$B_i(x_j) = 0, \quad B'_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad i, j = 1, 2, \ldots, n+2$$

$$B''''(x_j) = 0$$
To verify that the above conditions are satisfied by \( A_i(x), B_i(x), \) and \( C_i(x) \) as given in the above theorem, we proceed on the lines of [15] and show that they are polynomials of degree \( \leq 3n+3 \).

3

Let \( f(x) \) be continuously differentiable in \([-1, +1]\) and consider the sequence of polynomials

\[
R_n(x, f) = \sum_{i=1}^{n+2} f(x_{in})A_{in}(x) + \sum_{i=1}^{n+2} f'(x_{in})B_{in}(x) + \sum_{i=2}^{n+1} \delta_{in} C_{in}(x)
\]

with arbitrary numbers \( \delta_{in} \). We shall prove the following

**Theorem 3.1** Let \( f(x) \) have a continuous first derivative in \([-1, +1]\) and let \( f'(x) \in \text{Lip}\ x, x > 0 \) and if

\[
|\delta_{in}| = \frac{o(n^2)}{(1-x_{in}^2)} \quad i = 2, 3, \ldots, n+1
\]

then the sequence \( R_n(x, f) \) converges uniformly to \( f(x) \) in \([-1, +1]\).

4. Preliminary results

**Lemma 4.1** For \( x = \cos \theta \) we have

\[
\int_{-1}^{x} \frac{u_{2r}(t)}{(1-t^2)^{\frac{3}{2}}} dt = \pi - \theta - 2 \sum_{j=1}^{r} \sin \left(\frac{2j-1}{2j}\right) \theta;
\]

\[
\int_{-1}^{x} \frac{u_{2r-1}(t)}{(1-t^2)^{\frac{3}{2}}} dt = -2 \sum_{j=1}^{r} \sin \left(\frac{2j-1}{2j-1}\right) \theta.
\]

**Lemma 4.2** For \(-1 \leq x \leq +1, x = \cos \theta \) we have

\[
\left| \int_{-1}^{x} \frac{u_{2}(t)}{(1-t^2)^{\frac{3}{2}}} dt \right| \leq 12.
\]

\[
\left| \int_{-1}^{x} \frac{l_{2n}(t)}{(1-t^2)^{\frac{3}{2}}} dt \right| \leq \frac{35}{n} \quad i = 2, 3, \ldots, n+1.
\]
where $\pi(t)$ is defined by (2.10).

**PROOF.** Inequality (4.6) is due to P. Szasz [9]. (4.3) is immediate from (4.1) and (4.2). Since

\[
\int_{-1}^{1} \frac{u_n'(t)}{(1-t^2)^{1/2}} dt \leq 6(n+1)^2
\]

and using (4.3) we get (4.4) and (4.5). From (2.10) we have

\[
0 \leq \int_{-1}^{1} \frac{l_n(t)}{(1-t^2)^{1/2}} dt = \frac{\pi}{(n+1)} \left( 1 - \cos n\theta \right) \leq \frac{7}{n}
\]

Now using (4.3) we get (4.8).

In the estimation of $\sum_{i=1}^{n+1} |A_i(x)|$ we need the following results.

Let

\[
k_1(t) = k_1(x_n, t) = \sum_{r=0}^{n-1} u_r(x_n)u_r'(t)
\]

\[
k_2(t) = k_2(x_n, t) = \sum_{r=0}^{n-1} (e_r-1)u_r(x_n)u_r(t)
\]

Further using (4.3) we get (4.8).
where
\begin{equation}
(4.15) \quad e_r = n(n+2)-r(r+2).
\end{equation}

Now we prove the following

**Lemma 4.3** For $-1 \leq x \leq +1$, we have
\begin{align}
(4.16) \quad & \left| \int_{-1}^{x} \frac{k_1(t)}{(1-t^2)^{1/4}} \ dt \right| \leq \frac{36n^2}{(1-x_{in}^2)}, \quad i = 2, 3, \ldots, n+1 \\
(4.17) \quad & \left| \int_{-1}^{x} \frac{k_2(t)}{(1-t^2)^{1/4}} \ dt \right| \leq \frac{36n^2}{(1-x_{in}^2)}, \quad i = 2, 3, \ldots, n+1
\end{align}

and
\begin{equation}
(4.18) \quad \left| \int_{-1}^{x} \frac{k_3(t)}{(1-t^2)^{1/4}} \ dt \right| \leq \frac{70(n+1)^3}{\sqrt{1-x_{in}^2}}, \quad i = 2, 3, \ldots, n+1.
\end{equation}

**Proof.** Proof of (4.16) and (4.17) are similar to (4.8). A simple computation leads

\[ 4k_3(t) = \sum_{r=1}^{n-1} \beta_r(t)u_r(x_{in}) + k_4(t) \]

where
\begin{align*}
k_4(x_{in}, t) &= k_4(t) \\
&= -e_2u_2'(t) - \cos (n+1) \theta_{in}(e_{n+1}u_{n+1}'(t) + e_{n-1}u_{n-1}'(t))
\end{align*}

and
\[ \beta_r(t) = 8u_r'(t) - 2(r+2)e_2T_{r+1}(t) + 4(5r+2) - n(n+2) + 8)u_{r-1}(t). \]

Using (4.15), (4.7) and (4.3) we have
\begin{equation}
(4.19) \quad \left| \int_{-1}^{x} \frac{k_4(t)}{(1-t^2)^{1/4}} \ dt \right| \leq 72(n+1)^3
\end{equation}

and
\begin{align*}
\left| \sum_{r=1}^{n-1} u_r(x_{in}) \int_{-1}^{x} \frac{\beta_r(t)}{(1-t^2)^{1/4}} \ dt \right| &\leq \frac{16(n+1)^3}{\sqrt{1-x_{in}^2}} + \frac{2(n+1)^3}{\sqrt{1-x_{in}^2}} + \frac{192(n+1)^3}{\sqrt{1-x_{in}^2}} \\
&= \frac{210(n+1)^3}{\sqrt{1-x_{in}^2}}.
\end{align*}

Therefore
\begin{equation}
\left| \int_{-1}^{x} \frac{k_3(t)}{\sqrt{1-t^2}} \ dt \right| \leq \frac{70(n+1)^3}{\sqrt{1-x_{in}^2}}.
\end{equation}
LEMMA 4.4 For $-1 \leq x \leq +1$ we have

\begin{equation}
\left| \int_{-1}^{x} \frac{q_{in}(t)}{\sqrt{1-t^2}} \, dt \right| \leq \frac{36(n+1)}{1-x_{in}^2} + \frac{23(n+1)^2}{(1-x_{in}^2)^{\frac{3}{2}}} \quad i = 2, 3, \ldots, n+1
\end{equation}

where $q_{in}(t)$ is defined by (2.18).

PROOF. It can be shown that [15]

\begin{equation}
q_{in}(t) = \frac{x_{in}}{6(1-x_{in}^2)} \frac{u_n'(t)}{u_n(x_{in})} + \frac{1}{3(n+1)} \cdot [2x_{in}k_1(t)-x_{in}k_2(t)+k_3(t)].
\end{equation}

Now using (4.7) and Lemma (4.3) we get (4.20).

5

Here we shall investigate the estimation of the fundamental polynomials.

LEMMA 5.1 For $-1 \leq x \leq +1$ we have

\begin{equation}
|C_{i,n}(x)| \leq \frac{10(1-x_{in}^2)}{n^3}, \quad i = 2, 3, \ldots, n+1, \quad n = 4, 6, \ldots
\end{equation}

\begin{equation}
\sum_{i=2}^{n+1} (1-x_{in}^2)^{-1} |C_{i,n}(x)| \leq 10n^{-2}.
\end{equation}

LEMMA 5.2 For $n = 4, 6, \ldots$ and for $-1 \leq x \leq +1$ we have

\begin{equation}
|B_{1,n}(x)| \leq \frac{3}{n}, \quad |B_{n+2,n}(x)| \leq \frac{3}{n}
\end{equation}

\begin{equation}
|B_{i,n}(x)| \leq \frac{(1-x_{in}^2)^2}{n(1-x_{in}^2)} + \frac{80}{n} \quad i = 2, 3, \ldots, n+1
\end{equation}

and

\begin{equation}
\sum_{i=1}^{n+2} |B_{i,n}(x)| \leq 88.
\end{equation}

The proof of these two lemmas follows very easily from Lemma 4.2 and (2.5)—(2.12).

LEMMA 5.3 For $-1 \leq x \leq +1$ we have

\begin{equation}
|A_{1,n}(x)| \leq 7n, \quad |A_{n+2,n}(x)| \leq 7n
\end{equation}

\begin{equation}
|A_{i,n}(x)| \leq \frac{6n(1-x_{in}^2)^2}{1-x_{in}^2} + \frac{1400}{(n+1)(1-x_{in}^2)} + \frac{119}{\sqrt{1-x_{in}^2}}
\end{equation}
and

\[ (5.8) \quad \sum_{i=1}^{n+2} |A_{in}(x)| \leq 1626 n \log n. \]

**Proof.** (5.6) follows easily from (2.13), (2.14) and (4.7). From (2.16)-(2.18) and (4.20) we have

\[ |c'_i| \leq \frac{20(n+1)^2}{1-a_{in}^2} \quad i = 2, 3, \ldots, n+1 \]

\[ |d'_i| \leq \frac{52(n+1)}{1-a_{in}^2} + \frac{8(n+1)^2}{\sqrt{1-a_{in}^2}} \]

\[ \left| \frac{(1-a^2)l_{in}(x)}{1-a_{in}^2} \right| \leq 5 \quad -1 \leq x \leq +1. \]

Therefore using (5.4), lemma 4.2

\[ |A_{in}(x)| \leq 6n \frac{(1-a^2)^2_{in}(x)}{1-a_{in}^2} + \frac{1400}{(n+1)(1-a_{in}^2)} + \frac{119}{\sqrt{1-a_{in}^2}} \]

\[ \cdot \sum_{i=2}^{n+1} |A_{in}(x)| \leq 12n + 1400(n+1) + 119n \log n \leq 1612n \log n. \]

**Lemma 5.4** Let \( f'(x) \in \text{Lip } \alpha, \ 0 < \alpha < 1 \) in \([-1, +1]\). Then there exists a sequence of polynomial \( \{q_n(x)\} \) of degree at most \( n \) with \(-1 \leq x \leq +1\)

\[ (5.12) \quad |f(x) - q_n(x)| \leq \frac{c}{n^{1+\alpha}} \left[ (\sqrt{1-a^2})^{1+\alpha} + \frac{1}{n^{1+\alpha}} \right] \]

\[ (5.13) \quad |f'(x) - q'_n(x)| \leq \frac{c_1}{n^{\alpha}} \left[ (\sqrt{1-a^2})^{2+\alpha} + \frac{1}{n^{\alpha}} \right] \]

\[ (5.14) \quad |q''_n(x)| \leq \frac{n^{2-\alpha}}{1-a^2} \text{ in } -1 < x < +1. \]

**Proof.** The existence of \( q_n(x) \) satisfying (5.12) and (5.13) are well known [see Timan [14]]. (5.14) follows closely on the lines of Freud G. [4].

6

**Proof of Theorem 3.1** From the uniqueness theorem we have

\[ R_n(x,f) - f(x) = \sum_{i=1}^{n+2} [f(x_i) - q_n(x_i)]A_{in}(x) \]

\[ + \sum_{i=1}^{n+2} [f'(x_i) - q'_n(x_i)]B_{in}(x) + \sum_{i=2}^{n+1} [\delta_{in} - q'''_n(x_i)]C_{in}(x) \]

\[ + q_n(x) - f(x) = I_1 + I_2 + I_3 + I_4 \text{ say.} \]
From (5.8) and (5.12) we have
\[ I_1 = c/n^{1+\alpha} \quad 1616 \quad n \log n = o(1). \]
From (5.13) and (5.5) we have
\[ I_2 = o(1). \]
Using (5.2) and (3.2), and (5.14) we have immediately
\[ I_3 = \sum_{i=2}^{n+1} \left[ \frac{o(n^2)}{(1-x_{in}^2)} + \frac{n^{2-\alpha}}{(1-x_{in}^2)} \right] |C_i(x)| \]
\[ = o(1) \quad \text{for} \quad 0 < \alpha < 1. \]
and lastly from (5.12) \[ I_4 = o(1). \]
Thus \[ R_n(f, x) - f(x) = o(1) \] which proves the theorem.

7

Here we shall consider existence and uniqueness theorem for modified \( (0, 1, 3) \) interpolation when nodes are taken as zeros of ultraspherical polynomials. Since \( \lambda \) and \( n \) are fixed, we shall denote \( P_n^{(\lambda)}(x) = \varphi_n(x) \). It is well known [11] that the differential equation for \( \varphi_n(x) \) for all non-negative integers \( n \)'s is given by
\[ (1-x^2)\varphi_n''(x) - (2\lambda+1)x\varphi_n'(x) + n(n+2\lambda)\varphi_n(x) = 0. \]
It is also known [11] that all the zeros of \( \varphi_n(x) \) are real, simple and lying in \(-1 < x < 1\).

We shall prove the following theorem (a special case of which is theorem 2.1 which corresponds to \( \lambda = 1 \)).

**Theorem 7.1** Let \( n = 2k \), and \( \lambda \neq \pm \frac{1}{4} \),
\[ \lambda \neq \frac{m-1}{2}, \quad m = 1, 2, \ldots, n+2, \]
then to prescribed values \( a_i, b_i, (i = 1, 2, \ldots, n+2), c_i (i = 2, 3, \ldots, n+1) \) there is a uniquely determined polynomial \( f(x) \) of degree \( \leq 3n+3 \) such that
\[ f(x_i) = a_i, \quad f'(x_i) = b_i, \quad i = 1, 2, \ldots, n+2 \]
\[ f'''(x_i) = c_i \quad i = 2, 3, \ldots, n+1. \]
Here \( x_i \)'s are zeros of \( (1-x^2)\varphi_n(x) \)
\[ +1 = x_1 > x_2 > \ldots > x_{n+1} > x_{n+2} = -1 \]

**Proof.** Proof of theorem 7.1 is on the lines of the paper [9] by J. Suranyi and P. Turán. We shall show that in the case
\[ f(x_i) = f'(x_i) = 0, \quad i = 1, 2, \ldots, n+2, \]
\[ f'''(x_i) = 0, \quad i = 2, 3, \ldots, n+1 \]
the only polynomial of degree \( \leq 3n+3 \) is \( f(x) = 0 \). Thus, using first part of (7.5) we have

\[
(7.6) \quad f(x) = (1-x^2)^2 \varphi_n^2(x) r_{n-1}(x)
\]

where \( r_{n-1}(x) \) is a polynomial in \( x \) of degree \( \leq n-1 \). Also, \( f'''(x_i) = 0, \ i = 2, 3, \ldots, n+1 \) and since the zeros of \( \varphi_n(x) \) are simple, we obtain

\[
(7.7) \quad (1-x^2) r_{n-1}'(x_i) + (2\lambda-3) x_i r_{n-1}(x_i) = 0 \quad i = 2, 3, \ldots, n+1.
\]

Since the polynomial

\[
(1-x^2) r_{n-1}'(x) + (2\lambda-3) x r_{n-1}(x)
\]

is of degree \( \leq n \), and by (7.7) all its zeros are the same as those of \( \varphi_n(x) \), we obtain

\[
(7.8) \quad (1-x^2) r_{n-1}'(x) + (2\lambda-3) x r_{n-1}(x) = c\varphi_n(x)
\]

with numerical \( c \).

Now we have to investigate whether or not the equation (7.8) has a polynomial solution of degree \( \leq n-1 \) \( (n \text{ even}) \). We try to solve the equation by

\[
(7.9) \quad r_{n-1}(x) = \sum_{i=0}^{n-1} c_i \varphi_i(x).
\]

We shall be using the identities for \( i \geq 1 \) [see Szego [11]].

\[
(7.10) \quad (1-x^2) \varphi_i'(x) = \frac{1}{2(i+\lambda)} \{ (i+2\lambda-1)(i+2\lambda) \varphi_{i-1}(x) - i(i+1) \varphi_{i+1}(x) \}
\]

and

\[
(7.11) \quad x \varphi_i(x) = \frac{1}{2(i+\lambda)} \{ (i+1) \varphi_{i+1}(x) + (i+2\lambda-1) \varphi_{i-1}(x) \}.
\]

Substituting (7.9) in (7.8) we obtain

\[
c\varphi_n(x) = \sum_{i=1}^{n-1} c_i (1-x^2) \varphi_i'(x) + \sum_{i=0}^{n-1} c_i (2\lambda-3) x \varphi_i(x)
\]

and using (7.10) (7.11)
We have to compare the coefficients of $C\varphi_i(x)$ in (7.12). Comparing the coefficient of $C\varphi_{n-1}(x)$ we find for $n > 4$

$$0 = \frac{c_{n-2}(n-1)(2\lambda-n-1)}{2}.$$

If $n \geq 4$ and $n \neq 2\lambda-1$ then (7.13) implies

$$c_{n-2} = 0.$$  

Comparing the coefficients of $\varphi_0(x)$ in (7.12) we obtain

$$0 = \frac{c_1(2\lambda)(2\lambda-1)}{\lambda+1}.$$  

From the conditions imposed in $\lambda$ in theorem 7.1

$$(\lambda \neq 0, \lambda \neq \frac{1}{2}, \lambda > -\frac{1}{2})$$

we have

$$c_1 = 0.$$  

If $n \geq 4$ and $i = 1, 2, \ldots, n-2$, the comparison of the coefficients of $\varphi_i(x)$ in (7.12) gives

$$0 = \frac{ic_{i-1}(2\lambda-i-2)}{2(i-1+\lambda)} + \frac{(i+2\lambda)(i+4\lambda-2)}{2(i+1+\lambda)} c_{i+1}.$$  

Evidently $c_{i+1}$ can be expressed always by $c_{i-1}$ from (7.16).

Starting from $c_1 = 0$ we have

$$c_1 = c_3 = \ldots = c_{n-1} = 0 \quad (n \text{ even}).$$
Similarly starting from (7.14) (i.e. $c_{n-2} = 0$) and using (7.16) we have

(7.19) \[ c_{n-2} = c_{n-4} = \ldots = c_2 = c_0 = 0 \quad (n \text{ even}). \]

Here we remark that (7.19) was possible owing to the condition $\lambda \neq (m-1)/2$ $m = 1, 2, n+2$. Therefore we conclude from (7.10) and (7.19) that $r_{n-1}(x) \equiv 0$. This implies that $f(x) \equiv 0$. Therefore in general equations (7.2) and (7.3) determines a unique polynomial $f(x)$ degree $\leq 3n+3$. This proves the theorem.

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