COMPOSITIO MATHEMATICA

A. K. VARMA

Convergence of interpolatory polynomials on Tchebycheff abscissas

Compositio Mathematica, tome 21, nº 1 (1969), p. 7-20
<http://www.numdam.org/item?id=CM_1969__21_1_7_0>

Convergence of interpolatory polynomials on Tchebycheff abscissas

by

A. K. Varma [1]

1

A systematic study of Lacumary interpolation[2] was first initiated by Suranyi, J. and Turán, P. [8] and Balázs and Turán, P. [1, 2, 3] in the special case when the values and second derivatives are prescribed on the zeros of $\pi_n(x) = (1-x^2)P'_{n-1}(x)$ where $P_{n-1}(x)$ is the Legendre polynomial of degree $\leqq n-1$ while the existence and uniqueness have been shown for the abscissas as the zeros of ultraspherical polynomials $P_n^{(\lambda)}(x)$, $\lambda \geqq -\frac{1}{2}$ the explicit representation and convergence theorems have been proved for π abscissas only. Later the convergence theorem of Balázs and Turán [3] is sharpened by Freud, G. [4] in the sense that the interpolatory polynomials of Balázs and Turán converges uniformly to given $f(x)$ in $[-1, +1]$ if $f(x)$ satisfies the Zygmund condition

$$|f(x+h)-2f(x)+f(x-h)| = o(h) \quad \text{in} \quad [-1, +1].$$

Other interesting results are due to Saxena and Sharma [7, 8], Kis [5, 6], Varma and Sharma [11, 12] and Varma [15, 16].

The object of this paper is to consider the problem of existence, uniqueness, explicit representation and convergence of the sequence $R_n(x)$ of polynomials of degree $\leqq 3n+3$ such that $R_n(x)$, $R'_n(x)$ are prescribed at the zeros of $(1-x^2)u_n(x)$ where

$$u_n(x) = \frac{\sin (n+1)\theta}{\sin \theta}, \qquad x = \cos \theta,$$

while $R'''_n(x)$ is prescribed at all the above abscissas except at -1 and $+1$. We shall call this "modified" (0, 1, 3) interpolation.

In § 2 we state the existence theorem and give the explicit rep-

[1] The author is thankful to Prof. P. Turán and Prof. A Sharma for some valuable suggestions.

[2] They called it (0, 2) case.

resentation of these polynomials in a most suitable form and in § 3 and onwards we prove convergence theorem. It is interesting to remark that in modified (0, 2) interpolation [15] we require for the uniform convergence of the sequence of polynomials $R_n(x)$ to $f(x)$ is that $f'(x) \in \operatorname{Lip} \alpha$, $\alpha > \frac{1}{2}$, and this is best possible in a certain sense. So one would be inclined to think that in modified (0, 1, 3) interpolation, we may require $f(x)$ to be twice differentiable or at least $f'(x) \in \operatorname{Lip} \alpha$, $\alpha > \frac{1}{2}$ [compare corresponding theorem of Saxena and Sharma [7]]. But our theorem 3.1 asserts that this is not really the case. Here we need only $f'(x) \in \operatorname{Lip} \alpha$, $\alpha > 0$. Although we could not prove that this is best possible, it seems quite plausible that this is really so in view of other known results in this direction [7, 8].

2

Let us consider the set of numbers

$$(2.1) \qquad -1 = x_{n+2} < x_{n+1} < \dots < x_2 < x_1 = +1$$

by which we shall denote the zeros of $(1-x^2)u_n(x)$, where

$$u_n(x) = \frac{\sin (n+1)\theta}{\sin \theta}, \qquad x = \cos \theta.$$

Then we have the following

THEOREM 2.1 *If $n = 2k$, then to prescribed values $f(x_i)$, $f'(x_i)$ $(i = 1, 2, \dots n+2)$ and δ_i $(i = 2, 3, \dots, n+1)$ there is a uniquely determined polynomial $R_n(x)$ of degree $\leqq 3n+3$ such that*

$$(2.2) \qquad R_n(x_i) = f(x_i), \quad R_n'(x_i) = f'(x_i), \qquad i = 1, 2, \dots, n+2$$

$$(2.3) \qquad R_n'''(x_i) = \delta_i, \qquad i = 2, 3, \dots, n+1.$$

But if n is odd, $(n = 2k+1)$ there is in general no polynomial $R_n(x)$ of degree $\leqq 3n+3$ which satisfies (2.2) *and* (2.3) *and if there exists such a polynomial then there is an infinity of them.*

From the uniqueness theorem it follows that $R_n(x)$ is given by

$$(2.4) \qquad R_n(x) = \sum_{i=1}^{n+2} f(x_i)\, A_i(x) + \sum_{i=1}^{n+2} f'(x_i) B_i(x) + \sum_{i=2}^{n+1} \delta_i C_i(x)$$

where the polynomials $A_i(x)$, $B_i(x)$ and $C_i(x)$ are the fundamental polynomials of degree $\leqq 3n+3$. Their explicit forms are given by the following

THEOREM 2.2 *For n even, the fundamental polynomials have the following representation*

(a) *For $i = 2, 3, \ldots, n+1$ we have*

$$(2.5)\quad C_i(x) = \frac{(1-x^2)^{\frac{3}{2}}u_n^2(x)}{6(1-x_i^2)u_n'^2(x_i)}\left[a_i\int_{-1}^{x}\frac{u_n(t)}{(1-t^2)^{\frac{1}{2}}}\,dt + \int_{-1}^{x}\frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\,dt\right]$$

where

$$(2.6)\quad \pi a_i = -\int_{-1}^{+1}\frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\,dt, \qquad l_i(t) = \frac{u_n(t)}{(t-x_i)u_n'(x_i)}.$$

(b)

$$(2.7)\quad B_1(x) = \frac{(1-x^2)u_n^2(x)}{4(n+1)^3}\left[(1+x)u_n(x)+(1-x^2)u_n'(x) + (1-x^2)^{\frac{1}{2}}\int_{-1}^{x}\frac{u_n'(t)}{(1-t^2)^{\frac{1}{2}}}\,dt\right]$$

$$(2.8)\quad B_{n+2}(x) = \frac{(1-x^2)u_n^2(x)}{4(n+1)^3}\left[(1-x)u_n(x)-(1-x^2)u_n'(x) + (1-x^2)^{\frac{1}{2}}\int_{-1}^{x}\frac{u_n'(t)}{(1-t^2)^{\frac{1}{2}}}\,dt\right]$$

and for $i = 2, 3, \ldots, n+1$ we have

$$(2.9)\quad B_i(x) = \frac{(1-x^2)^2 l_i^2(x)u_n(x)}{(1-x_i^2)^2u_n'(x_j)} + \frac{(1-x^2)^{\frac{3}{2}}u_n^2(x)}{(n+1)^2}\left[b_i\int_{-1}^{x}\frac{u_n(t)}{(1-t^2)^{\frac{1}{2}}}\,dt + c_i\int_{-1}^{x}\frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\,dt + \int_{-1}^{x}\frac{p_i(t)}{(1-t^2)^{\frac{1}{2}}}\,dt\right]$$

where

$$(2.10)\quad p_i(t) = \frac{3x_il_i(t)-2(1-t^2)l_i'(t)}{2(t-x_i)} = \frac{2(1-x_i^2)}{(n+1)}\sum_{r=0}^{n-1}\alpha_r u_r(x_i)u_r(t)$$

$$\alpha_r = n(n+2)-r(r+2)-3$$

$$(2.11)\quad c_i = \frac{(n+3)(n-1)}{6} + \frac{x_i^2}{2(1-x_i^2)}$$

$$(2.12)\quad \pi b_i = -\int_{-1}^{+1}\frac{p_i(t)}{(1-t^2)^{\frac{1}{2}}}\,dt - c_i\int_{-1}^{+1}\frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\,dt$$

(c)

$$(2.13)\quad A_1(x) = \frac{(1+x)^2u_n^3(x)}{4(n+1)^3} - (n+1)^2B_1(x) - \frac{(1-x^2)^2u_n^2(x)u_n'(x)}{8(n+1)^3} - \frac{(1-x^2)^{\frac{3}{2}}u_n^2(x)}{4(n+1)^3}\int_{-1}^{x}\frac{(1+t^2)u_n'(t)}{(1-t^2)^{\frac{1}{2}}}\,dt$$

$$(2.14)\quad A_{n+2}(x) = \frac{(1-x)^2 u_n^3(x)}{4(n+1)^3} + (n+1)^2 B_1(x) + \frac{(1-x^2)^2 u_n^2(x) u_n'(x)}{8(n+1)^3} - \frac{(1-x^2)^{\frac{3}{2}} u_n^2(x)}{4(n+1)^3} \int_{-1}^{x} \frac{(1+t^2)u_n'(t)}{(1-t^2)^{\frac{1}{2}}}\, dt$$

and for $i = 2, 3, \ldots, n+1$ we have

$$(2.15)\quad A_i(x) = \frac{(1-x^2)^2 l_i^3(x)}{(1-x_i^2)^2} + \frac{x_i B_i(x)}{2(1-x_i^2)} + \frac{(1-x^2)^{\frac{3}{2}} u_n^2(x)}{(n+1)^2} \cdot \left[d_i \int_{-1}^{x} \frac{u_n(t)}{(1-t^2)^{\frac{1}{2}}}\, dt + c_i' \int_{-1}^{x} \frac{l_i(t)dt}{(1-t^2)^{\frac{1}{2}}} + \int_{-1}^{x} \frac{q_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt \right]$$

where

$$(2.16)\quad c_i = 3x_i \left[\frac{(n+3)(n-1)-6}{4(1-x_i^2)} + \frac{37x_i^2+36}{12(1-x_i^2)^2} \right]$$

$$(2.17)\quad \pi d_i = -\int_{-1}^{+1} \frac{q_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt - c_i' \int_{-1}^{+1} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt$$

$$(2.18)\quad q_i(t) = \frac{(a_i't + b_i')l_i(t) - (1-t^2)l_i'(t)}{(t-x_i)^2}$$

$$(2.19)\quad a_i'x_i + b_i' = \frac{3x_i}{2}$$

$$(2.20)\quad a_i' = \left[\frac{-x_i^2}{4(1-x_i^2)} - \frac{(n+3)(n-1)}{3} \right].$$

In view of the uniqueness theorem 2.1, it remains to verify that the fundamental functions $A_i(x)$, $B_i(x)$ and $C_i(x)$ are polynomials of degree $\leqq 3n+3$ and satisfy the following conditions.

$$(2.21)\quad A_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad A_i'(x_j) = 0 \qquad i, j = 1, 2, \ldots, n+2$$

$$(2.22)\quad A_i'''(x_j) = 0 \qquad \begin{matrix} i = 1, 2, \ldots, n+2 \\ j = 2, 3, \ldots, n+1 \end{matrix}$$

$$(2.23)\quad B_i(x_j) = 0, \quad B_i'(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad i, j = 1, 2, \ldots, n+2$$

$$(2.24)\quad B_i'''(x_j) = 0 \qquad \begin{matrix} i = 1, 2, \ldots, n+2 \\ j = 2, 3, \ldots, n+1 \end{matrix}$$

$$(2.25)\quad C_i(x_j) = C_j'(x_j) = 0 \qquad \begin{array}{l} i = 2, 3, \ldots, n+1 \\ j = 1, 2, \ldots, n+2 \end{array}$$

$$(2.26)\quad C_i'''(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad j, i = 2, 3, \ldots, n+1.$$

To verify that the above conditions are satisfied by $A_i(x)$, $B_i(x)$, and $C_i(x)$ as given in the above theorem, we proceed on the lines of [15] and show that they are polynomials of degree $\leqq 3n+3$.

3

Let $f(x)$ be continuously differentiable in $[-1, +1]$ and consider the sequence of polynomials

$$(3.1)\quad R_n(x, f) = \sum_{i=1}^{n+2} f(x_{in})A_{in}(x) + \sum_{i=1}^{n+2} f'(x_{in})B_{in}(x) + \sum_{i=2}^{n+1} \delta_{in} C_{in}(x)$$

with arbitrary numbers δ_{in}. We shall prove the following

THEOREM 3.1 *Let $f(x)$ have a continuous first derivative in $[-1, +1]$ and let $f'(x) \in \mathrm{Lip}\ \alpha$, $\alpha > 0$ and if*

$$(3.2)\qquad |\delta_{in}| = \frac{o(n^2)}{(1-x_{in}^2)} \qquad i = 2, 3, \ldots, n+1$$

then the sequence $R_n(x, f)$ converges uniformly to $f(x)$ in $[-1, +1]$.

4. Preliminary results

LEMMA 4.1 *For $x = \cos\theta$ we have*

$$(4.1)\qquad \int_{-1}^{x} \frac{u_{2r}(t)}{(1-t^2)^{\frac{1}{2}}}\,dt = \pi - \theta - 2\sum_{j=1}^{r} \frac{\sin(2j-1)\theta}{2j};$$

$$(4.2)\qquad \int_{-1}^{x} \frac{u_{2r-1}(t)}{(1-t^2)^{\frac{1}{2}}}\,dt = -2\sum_{j=1}^{r} \frac{\sin(2j-1)\theta}{2j-1}.$$

LEMMA 4.2 *For $-1 \leqq x \leqq +1$, $x = \cos\theta$ we have*

$$(4.3)\qquad \left|\int_{-1}^{x} \frac{u_p(t)}{(1-t^2)^{\frac{1}{2}}}\,dt\right| \leqq 12.$$

$$(4.4)\qquad \left|\int_{-1}^{x} \frac{l_{in}(t)}{(1-t^2)^{\frac{1}{2}}}\,dt\right| \leqq \frac{35}{n} \qquad i = 2, 3, \ldots, n+1.$$

$$(4.5)\qquad 0 \leqq \int_{-1}^{+1} \frac{l_{in}(t)}{(1-t^2)^{\frac12}}\,dt = \frac{\pi}{(n+1)}\,(1-\cos n\theta_{in}) \leqq \frac{7}{n}$$

$$(4.6)\qquad \sum_{i=2}^{n+1} \frac{(1-x^2)l_{in}^2(x)}{(1-x_i^2)} \leqq 2.$$

$$(4.7)\qquad \left|\int_{-1}^{x} \frac{u_n'(t)}{(1-t^2)^{\frac12}}\,dt\right| \leqq 6(n+1)^2$$

$$(4.8)\qquad \left|\int_{-1}^{x} \frac{p_{in}(t)}{(1-t^2)^{\frac12}}\,dt\right| \leqq 8(n+1), \qquad i = 2, 3, \ldots, n+1$$

where $p_{in}(t)$ *is defined by* (2.10).

Proof. Inequality (4.6) is due to P. Szasz [9]. (4.3) is immediate from (4.1) and (4.2). Since

$$(4.9)\qquad u_n'(x) = 4\sum_{j=1}^{n/2} j u_{2j-1}(x) \qquad (n \text{ even})$$

we get (4.7) from (4.9) and (4.3). From a result of Fejer we have

$$(4.10)\qquad l_{in}(x) = \frac{2(1-x_{in}^2)}{(n+1)} \sum_{r=0}^{n-1} u_r(x_{in})u_r(x), \qquad i = 2, 3, \ldots, n+1,$$

$$= \frac{1}{(n+1)}\,[1+2\sum_{r=1}^{n-1} \cos r\theta_{in} \cos r\theta - \cos n\theta_{in}(u_{n-2}(x) + \cos \theta_{in} u_{n-1}(x))]$$

Now using (4.3) we get (4.4) and (4.5). From (2.10) we have

$$(4.11)\qquad p_{in}(t) = \frac{1}{(n+1)}\,[\sum_{r=1}^{n-1} (4ru_{r-2}(t)+2\alpha_r T_r(t)) \cos r\theta_{in} - \alpha_0 - \cos n\theta_{in}(u_{n-2}(t)+\cos \theta_{in} u_{n-1}(t))]$$

where

$$T_r(t) = \cos r\theta \quad \text{and} \quad \int_{-1}^{x} \frac{T_r(t)dt}{(1-t^2)^{\frac12}} = \frac{\sin r\theta}{r}, \quad \cos\theta = x.$$

Further using (4.3) we get (4.8).

In the estimation of $\sum_{i=1}^{n+2} |A_{in}(x)|$ we need the following results. Let

$$(4.12)\qquad k_1(t) \equiv k_1(x_{in}, t) = \sum_{r=1}^{n-1} u_r(x_{in})u_r'(t)$$

$$(4.13)\qquad k_2(t) \equiv k_2(x_{in}, t) = \sum_{r=0}^{n-1} (e_r-1)u_r(x_{in})u_r(t)$$

$$(4.14)\qquad k_3(t) \equiv k_3(x_{in}, t) = \sum_{r=1}^{n-1} e_r(1-x_{in}^2)u_r(x_{in})u_r'(t)$$

where

$$e_r = n(n+2)-r(r+2). \tag{4.15}$$

Now we prove the following

Lemma 4.3 *For $-1 \leqq x \leqq +1$, we have*

$$\left|\int_{-1}^{x} \frac{k_1(t)}{(1-t^2)^{\frac{1}{2}}}\, dt\right| \leqq \frac{36n^2}{(1-x_{in}^2)}, \qquad i = 2, 3, \ldots, n+1 \tag{4.16}$$

$$\left|\int_{-1}^{x} \frac{k_2(t)}{(1-t^2)^{\frac{1}{2}}}\, dt\right| \leqq \frac{36n^2}{(1-x_{in}^2)}, \qquad i = 2, 3, \ldots, n+1 \tag{4.17}$$

and

$$\left|\int_{-1}^{x} \frac{k_3(t)}{(1-t^2)^{\frac{1}{2}}}\, dt\right| \leqq \frac{70(n+1)^3}{\sqrt{-x_{in}^2}}, \qquad i = 2, 3, \ldots, n+1. \tag{4.18}$$

Proof. Proof of (4.16) and (4.17) are similar to (4.8). A simple computation leads

$$4k_3(t) = \sum_{r=1}^{n-1} \beta_r(t) u_r(x_{in}) + k_4(t)$$

where

$$\begin{aligned} k_4(x_{in}, t) &= k_4(t) \\ &= -e_2 u_2'(t) - \cos{(n+1)\theta_{in}} \big(e_{n+1} u_{n+1}'(t) + e_{n-1} u_{n-1}'(t)\big) \end{aligned}$$

and

$$\beta_r(t) = 8u_r'(t) - 2(r+2)e_{r+2} T_{r+1}(t) + 4\big(5r(r+2) - n(n+2) + 8\big) u_{r-1}(t).$$

Using (4.15), (4.7) and (4.3) we have

$$\left|\int_{-1}^{x} \frac{k_4(t)}{(1-t^2)^{\frac{1}{2}}}\, dt\right| \leqq 72(n+1)^3 \tag{4.19}$$

and

$$\begin{aligned} \left|\sum_{r=1}^{n-1} u_r(x_{in}) \int_{-1}^{x} \frac{\beta_r(t)}{(1-t^2)^{\frac{1}{2}}}\, dt\right| &\leqq \frac{16(n+1)^3}{\sqrt{1-x_{in}^2}} + \frac{2(n+1)^3}{\sqrt{1-x_{in}^2}} + \frac{192(n+1)^3}{\sqrt{1-x_{in}^2}} \\ &= \frac{210(n+1)^3}{\sqrt{1-x_{in}^2}}. \end{aligned}$$

Therefore

$$\left|\int_{-1}^{x} \frac{k_3(t)}{\sqrt{1-t^2}}\, dt\right| \leqq \frac{70(n+1)^3}{\sqrt{1-x_{in}^2}}.$$

LEMMA 4.4 *For* $-1 \leqq x \leqq +1$ *we have*

$$(4.20)\quad \left|\int_{-1}^{x} \frac{q_{in}(t)}{\sqrt{1-t^2}}\,dt\right| \leqq \frac{36(n+1)}{1-x_{in}^2} + \frac{23(n+1)^2}{(1-x_{in}^2)^{\frac{1}{2}}}$$

$$i = 2, 3, \ldots, n+1$$

where $q_{in}(t)$ *is defined by* (2.18).

PROOF. It can be shown that [15]

$$(4.21)\quad q_{in}(t) = \frac{x_{in}}{6(1-x_{in}^2)}\frac{u_n'(t)}{u_n'(x_{in})} + \frac{1}{3(n+1)} \cdot [2x_{in}k_1(t) - x_{in}k_2(t) + k_3(t)].$$

Now using (4.7) and Lemma (4.3) we get (4.20).

5

Here we shall investigate the estimation of the fundamental polynomials.

LEMMA 5.1 *For* $-1 \leqq x \leqq +1$ *we have*

$$(5.1)\quad |C_{in}(x)| \leqq \frac{10(1-x_{in}^2)}{n^3}, \qquad i = 2, 3, \ldots, n+1,\ n = 4, 6, \ldots$$

$$(5.2)\quad \sum_{i=2}^{n+1} (1-x_{in}^2)^{-1}|C_{in}(x)| \leqq 10n^{-2}.$$

LEMMA 5.2 *For* $n = 4, 6, \ldots$ *and for* $-1 \leqq x \leqq +1$ *we have*

$$(5.3)\quad |B_{1n}(x)| \leqq \frac{3}{n}, \qquad |B_{n+2,n}(x)| \leqq \frac{3}{n}$$

$$(5.4)\quad |B_{in}(x)| \leqq \frac{(1-x^2)l_{in}^2(x)}{n(1-x_{in}^2)} + \frac{80}{n} \qquad i + 2, 3, \ldots, n+1$$

and

$$(5.5)\quad \sum_{i=1}^{n+2} |B_{in}(x)| \leqq 88.$$

The proof of these two lemmas follows very easily from Lemma 4.2 and (2.5)—(2.12).

LEMMA 5.3 *For* $-1 \leqq x \leqq +1$ *we have*

$$(5.6)\quad |A_{1n}(x)| \leqq 7n, \quad |A_{n+2n}(x)| \leqq 7n$$

$$(5.7)\quad |A_{in}(x)| \leqq \frac{6n(1-x^2)l_{in}^2(x)}{1-x_{in}^2} + \frac{1400}{(n+1)(1-x_{in}^2)} + \frac{119}{\sqrt{1-x_{in}^2}}$$

and

$$\sum_{i=1}^{n+2} |A_{in}(x)| \leqq 1626\, n \log n. \tag{5.8}$$

Proof. (5.6) follows easily from (2.13), (2.14) and (4.7). From (2.16)—(2.18) and (4.20) we have

$$|c_i'| \leqq \frac{20(n+1)^2}{(1-x_{in}^2)} \qquad i = 2, 3, \ldots, n+1 \tag{5.9}$$

$$|d_i'| \leqq \frac{52(n+1)}{1-x_{in}^2} + \frac{8(n+1)^2}{\sqrt{1-x_{in}^2}} \tag{5.10}$$

$$\left|\frac{(1-x^2)l_{in}(x)}{(1-x_{in}^2)}\right| \leqq 5 \qquad -1 \leqq x \leqq +1. \tag{5.11}$$

Therefore using (5.4), lemma 4.2

$$|A_{in}(x)| \leqq 6n\, \frac{(1-x^2)l_{in}^2(x)}{1-x_{in}^2} + \frac{1400}{(n+1)(1-x_{in}^2)} + \frac{119}{\sqrt{1-x_{in}^2}}$$

$$\therefore \sum_{i=2}^{n+1} |A_{in}(x)| \leqq 12n + 1400(n+1) + 119n \log n \leqq 1612n \log n.$$

Lemma 5.4 *Let* $f'(x) \in \mathrm{Lip}\, \alpha$, $0 < \alpha < 1$ *in* $[-1, +1]$. *Then there exists a sequence of polynomial* $\{\varphi_n(x)\}$ *of degree at most n with the following properties for* $-1 \leqq x \leqq +1$

$$|f(x) - \varphi_n(x)| \leqq \frac{c}{n^{1+\alpha}} \left[(\sqrt{1-x^2})^{1+\alpha} + \frac{1}{n^{1+\alpha}}\right] \tag{5.12}$$

$$|f'(x) - \varphi_n'(x)| \leqq \frac{c_1}{n^{\alpha}} \left[(\sqrt{1-x^2})^{\alpha} + \frac{1}{n^{\alpha}}\right] \tag{5.13}$$

$$|\varphi_n'''(x) \leqq \frac{n^{2-\alpha}}{(1-x^2)} \quad \text{in} \quad -1 < x < +1. \tag{5.14}$$

Proof. The existence of $\varphi_n(x)$ satisfying (5.12) and (5.13) are well known [see Timan [14]]. (5.14) follows closely on the lines of Freud G. [4].

6

Proof of Theorem 3.1 From the uniqueness theorem we have

$$\begin{aligned} R_n(x,f) - f(x) &= \sum_{i=1}^{n+2} [f(x_{in}) - \varphi_n(x_{in})] A_{in}(x) \\ &\quad + \sum_{i=1}^{n+2} [f'(x_{in}) - \varphi_n'(x_{in})] B_{in}(x) + \sum_{i=2}^{n+1} [\delta_{in} - \varphi_n'''(x_{in})] C_{in}(x) \\ &\quad + \varphi_n(x) - f(x) = I_1 + I_2 + I_3 + I_4 \qquad \text{say.} \end{aligned}$$

From (5.8) and (5.12) $I_1 = c/n^{1+\alpha}$ 1616 $n \log n = o(1)$.
From (5.13) and (5.5) we have $I_2 = o(1)$.
Using (5.2) and (3.2), and (5.14) we have immediately

$$\begin{aligned} I_3 &= \sum_{i=2}^{n+1} \left[\frac{o(n^2)}{(1-x_{in}^2)} + \frac{n^{2-\alpha}}{(1-x_{in}^2)} \right] |C_{in}(x)| \\ &= o(1) \quad \text{for} \quad 0 < \alpha < 1. \end{aligned}$$

and lastly from (5.12) $I_4 = o(1)$.

Thus $R_n(f, x) - f(x) = o(1)$ which proves the theorem.

7

Here we shall consider existence and uniqueness theorem for modified (0, 1, 3) interpolation when nodes are taken as zeros of ultraspherical polynomials. Since λ and n are fixed, we shall denote $P_n^{(\lambda)}(x) = \varphi_n(x)$. It is well known [11] that the differential equation for $\varphi_n(x)$ for all non-negative integers n's is given by

$$(7.1) \qquad (1-x^2)\varphi_n''(x) - (2\lambda+1)x\varphi_n'(x) + n(n+2\lambda)\varphi_n(x) = 0.$$

It is also known [11] that all the zeros of $\varphi_n(x)$ are real, simple and lying in $-1 < x < 1$.

We shall prove the following theorem (a special case of which is theorem 2.1 which corresponds to $\lambda = 1$).

Theorem 7.1 *Let $n = 2k$, and $\lambda \neq \pm\frac{1}{4}$,*

$$\lambda \neq \frac{m-1}{2}, \qquad m = 1, 2, \ldots, n+2,$$

then to prescribed values a_i, b_i, $(i = 1, 2, \ldots, n+2)$, c_i $(i = 2, 3, \ldots, n+1)$ there is a uniquely determined polynomial $f(x)$ of degree $\leqq 3n+3$ such that

$$(7.2) \qquad f(x_i) = a_i,\ f'(x_i) = b_i, \qquad i = 1, 2, \ldots, n+2$$

$$(7.3) \qquad f'''(x_i) = c_i \qquad i = 2, 3, \ldots, n+1.$$

Here x_i's are zeros of $(1-x^2)\varphi_n(x)$

$$(7.4) \qquad +1 = x_1 > x_2 > \ldots \qquad > x_{n+1} > x_{n+2} = -1$$

Proof. Proof of theorem 7.1 is on the lines of the paper [9] by J. Suranyi and P. Turán. We shall show that in the case

$$(7.5) \qquad \begin{aligned} f(x_i) = f'(x_i) &= 0, \qquad i = 1, 2, \ldots, n+2, \\ f'''(x_i) &= 0, \qquad i = 2, 3, \ldots, n+1 \end{aligned}$$

the only polynomial of degree $\leqq 3n+3$ is $f(x) \equiv 0$. Thus, using first part of (7.5) we have

$$(7.6) \qquad f(x) = (1-x^2)^2\varphi_n^2(x)r_{n-1}(x)$$

where $r_{n-1}(x)$ is a polynomial in x of degree $\leqq n-1$. Also, $f'''(x_i) = 0$, $i = 2, 3, \ldots, n+1$ and since the zeros of $\varphi_n(x)$ are simple, we obtain

$$(7.7) \quad (1-x^2)r'_{n-1}(x_i)+(2\lambda-3)x_i r_{n-1}(x_i) = 0 \quad i = 2, 3, \ldots, n+1.$$

Since the polynomial

$$(1-x^2)r'_{n-1}(x)+(2\lambda-3)xr_{n-1}(x)$$

is of degree $\leqq n$, and by (7.7) all its zeros are the same as those of $\varphi_n(x)$, we obtain

$$(7.8) \qquad (1-x^2)r'_{n-1}(x)+(2\lambda-3)xr_{n-1}(x) = c\varphi_n(x)$$

with numerical c.

Now we have to investigate whether or not the equation (7.8) has a polynomial solution of degree $\leqq n-1$ (n even). We try to solve the equation by

$$(7.9) \qquad r_{n-1}(x) = \sum_{i=0}^{n-1} c_i\varphi_i(x).$$

We shall be using the identities for $i \geqq 1$ [see Szego [11]].

$$(7.10) \quad (1-x^2)\varphi'_i(x) = \frac{1}{2(i+\lambda)}\{(i+2\lambda-1)(i+2\lambda)\varphi_{i-1}(x) \\ -i(i+1)\varphi_{i+1}(x)\}$$

and

$$(7.11) \quad x\varphi_i(x) = \frac{1}{2(i+\lambda)}\{(i+1)\varphi_{i+1}(x)+(i+2\lambda-1)\varphi_{i-1}(x)\}.$$

Substituting (7.9) in (7.8) we obtain

$$c\varphi_n(x) = \sum_{i=1}^{n-1} c_i(1-x^2)\varphi'_i(x)+\sum_{i=0}^{n-1} c_i(2\lambda-3)x\varphi_i(x)$$

and using (7.10) (7.11)

$$
\begin{aligned}
c\varphi_n(x) &= \sum_{i=1}^{n-1} \frac{c_i}{2(i+\lambda)} \{(i+2\lambda-1)(i+2\lambda)\varphi_{i-1}(x) - i(i+1)\varphi_{i+1}(x)\} \\
&\quad + \frac{c_0(2\lambda-3)\varphi_1(x)}{2\lambda} + \sum_{i=1}^{n-1} \frac{c_i(2\lambda-3)}{2(i+\lambda)} \\
&\qquad \cdot \{(i+1)\varphi_{i+1}(x) + (i+2\lambda-1)\varphi_{i-1}(x)\} \\
&= \frac{c_0(2\lambda-3)}{2\lambda}\varphi_1(x) + \sum_{i=1}^{n-1} \frac{c_i(i+1)(2\lambda-3-i)}{2(i+\lambda)}\varphi_{i+1}(x) \\
&\quad + \sum_{i=1}^{n-1} \frac{c_i(i+2\lambda-1)(i+4\lambda-3)}{2(i+\lambda)}\varphi_{i-1}(x) \\
&= \frac{c_0(2\lambda-3)\varphi_1(x)}{2\lambda} + \sum_{i=2}^{n} \frac{ic_{i-1}(2\lambda-i-2)\varphi_i(x)}{2(i-1+\lambda)} \\
&\quad + \sum_{i=0}^{n-2} c_{i+1} \frac{(i+2\lambda)(i+4\lambda-2)\varphi_i(x)}{2(i+1+\lambda)}.
\end{aligned}
$$

We have to compare the coefficients of $\varphi_i(x)$ in (7.12). Comparing the coefficient of $\varphi_{n-1}(x)$ we find for $n \geqq 4$

$$0 = \frac{c_{n-2}(n-1)(2\lambda-n-1)}{2}. \tag{7.13}$$

If $n \geqq 4$ and $n \neq 2\lambda-1$ then (7.13) implies

$$c_{n-2} = 0. \tag{7.14}$$

Comparing the coefficients of $\varphi_0(x)$ in (7.12) we obtain

$$0 = \frac{c_1(2\lambda)(2\lambda-1)}{\lambda+1}.$$

From the conditions imposed in λ in theorem 7.1

$$(\lambda \neq 0,\ \lambda \neq \tfrac{1}{2}\ \lambda > -\tfrac{1}{2})$$

we have

$$c_1 = 0. \tag{7.15}$$

If $n \geqq 4$ and $i = 1, 2, \ldots, n-2$, the comparison of the coefficients of $\varphi_i(x)$ in (7.12) gives

$$0 = \frac{ic_{i-1}(2\lambda-i-2)}{2(i-1+\lambda)} + \frac{(i+2\lambda)(i+4\lambda-2)}{2(i+1+\lambda)} c_{i+1}. \tag{7.16}$$

Evidently c_{i+1} can be expressed always by c_{i-1} from (7.16).

Starting from $c_1 = 0$ we have

$$c_1 = c_3 = \cdots = c_{n-1} = 0 \qquad (n \text{ even}). \tag{7.18}$$

Similarly starting from (7.14) (i.e. $c_{n-2} = 0$) and using (7.16) we have

$$(7.19) \qquad c_{n-2} = c_{n-4} = \ldots = c_2 = c_0 = 0 \qquad (n \text{ even}).$$

Here we remark that (7.19) was possible owing to the condition $\lambda \neq (m-1)/2\ m = 1, 2, n+2$. Therefore we conclude from (7.10) and (7.19) that $r_{n-1}(x) \equiv 0$. This implies that $f(x) \equiv 0$. Therefore in general equations (7.2) and (7.3) determines a unique polynomial $f(x)$ degree $\leqq 3n+3$. This proves the theorem.

This research has been supported by the University of Alberta Post-Doctoral Fellowship, Dept. of Mathematics, Edmonton (1966-67) and N.R.C. grant M.C.A. 26 (1964).

REFERENCES

J. Balázs, and P. Turán

[1] Notes on interpolation II, Acta. Math. Acad. Sci. Hung. Vol. 8 (1957) pp. 201—215.

J. Balázs and P. Turán

[2] Notes on interpolation III, ibid. 9 (1958) pp. 195—214.

J. Balázs and P. Turán

[3] Notes on interpolation IV, ibid. 9, (1958) pp. 243—258.

G. Freud

[4] Bemerkung über die Konvergenz eines Interpolations fahrens von P. Turán, ibid. 9 (1958) pp. 337—341.

O. Kis

[5] On trigonometric interpolation (Russian) (0, 2) case, ibid II (1960) pp. 256—276.

O. Kis

[6] Remarks on interpolation (Russian), ibid. (1960) 49—64.

R. B. Saxena and A. Sharma

[7] Convergence of interpolatory polynomials (0, 1, 3) interpolation, ibid. (1959) pp. 157—175.

R. B. Saxena

[8] Convergence of interpolatory polynomials (0, 1, 2, 4) interpolation. Tran. of the Amer. Math. Society, Vol. 95, No. 2 (1960) pp. 361—385.

J. Suranyi and P. Turán

[9] Notes on interpolation I on some interpolatory properties of ultraspherical polynomials, Acta Math. Acad. Sci. Hung. 6 (1955) pp. 67—79.

P. Szasz

[10] On quasi-Hermite-Fejer interpolation, ibid. (1959) pp. 413—439.

G. Szegö

[11] Orthogonal polynomials, Amer. Math. Soc. Coll. Pub. Vol. 23 (1959) 2nd edition.

A. SHARMA and A. K. VARMA

[12] Trigonometric interpolation, Duke Math. Jour. 1965, pp. 341—357.

A. SHARMA and A. K. VARMA

[13] Some interpolatory properties of Tchebycheff polynomials (0, 2) case modified. Pub. Math. Deb. Hungary 1961, pp. 336—349.

A. TIMAN

[14] Theory of Approx. Pergamon Press.

A. K. VARMA

[15] On some interpolatory properties of Tchebycheff polynomials. Duke Math. Jour. 1961 Vol. 28, No. 3, pp. 449—462.

A. K. VARMA

[16] A problem of P. Turán. Canadian Math. Bulletin, Vol. 10, no. 4, 1967, pp. 531—557.

(Oblatum 7-3-1967)

Department of Mathematics
University of Florida
Gainesville, Florida, U.S.A.