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by

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1. Introduction

Suppose that $f(z) = z+\sum_2^\infty a_k z^k$ is analytic for $z \in E$ ($|z| < 1$). For $0 \leqq \alpha < 1$, let $\bar{S}_\alpha$ denote the class of functions $f(z)$ which satisfy $|zf'(z)/f(z)-1| \leqq 1-\alpha$ for $z \in E$. These functions form a subclass of the class of functions starlike of order α introduced by Robertson [2]. The function $f(z)$ is said to be convex for $|z| < r < 1$ if it satisfies the condition $\text{Re}\,\{zf''(z)/f'(z)+1\} \geqq 0$, $|z| < r$. This condition expresses analytically the fact that $f(z)$ maps the disk $|z| < r$ onto a convex domain [1, problem 5, page 224].

In a recent paper [3] R. Singh studies the class $\bar{S}_0$. He showed that each function in $\bar{S}_0$ is convex for $|z| < (\sqrt{13}-3)/2 \sim \cdot 303$. This is not the best possible result, however, and in the present paper we give the exact radius of convexity for this class. We also give some extensions of the other results in [3] to the class $\bar{S}_\alpha$.

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Those results in this section which are stated without proof can be obtained by minor obvious changes in the proofs of the corresponding theorems in [3].

THEOREM 1. *$f(z) \in \bar{S}_\alpha$ if, and only if,*

$$f(z) = z \exp\left\{(1-\alpha)\int_0^z \phi(t)dt\right\}, \tag{1}$$

where ϕ is regular and bounded by 1 in E.

COROLLARY. *If $f(z) \in \bar{S}_\alpha$, then*

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq 1-(1-\alpha)|z|,$$

and

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \leqq \text{arc sin } (1-\alpha)|z|.$$

PROOF. Both assertions follow from the fact that

$$\frac{zf'(z)}{f(z)} = 1+(1-\alpha)z\phi(z),$$

and so $zf'(z)/f(z)$ maps $|z| \leqq r$ into the circle with center 1, radius $(1-\alpha)r$.

THEOREM 2. *If* $f(z) \in \bar{S}_\alpha$, *then*

$$|z|e^{-(1-\alpha)|z|} \leqq |f(z)| \leqq |z|e^{(1-\alpha)|z|},$$

and

$$[1-(1-\alpha)|z|]e^{-(1-\alpha)|z|} \leqq |f'(z)| \leqq [1+(1-\alpha)|z|]e^{(1-\alpha)|z|}.$$

COROLLARY. *If* $f \in \bar{S}_\alpha$, *then each point in the disk* $|w| < e^{-(1-\alpha)}$ *is the image under* $f(z)$ *of some point in* E.

THEOREM 3. *If* $f(z) \in \bar{S}_\alpha$, *then* $|a_n| \leqq (1-\alpha)/(n-1)$, *and equality is attained for* $f(z) = z \exp \{(1-\alpha)z^{n-1}/(n-1)\}$.

THEOREM 4. *If* $f(z) \in \bar{S}_0$, *then* $f(z)$ *is convex for* $|z| < (3-\sqrt{5})/2$.

PROOF. From (1) with $\alpha = 0$ we obtain

$$f'(z) = (1+z\phi(z)) \exp \left\{\int_0^z \phi(t)dt\right\},$$

and a calculation yields

$$\frac{zf''(z)}{f'(z)}+1 = 1+z\phi(z)+ \frac{z(z\phi'(z)+\phi(z))}{1+z\phi(z)}.$$

Now, $|\phi'(z)| \leqq (1-|\phi(z)|^2)/(1-|z|^2)$ [1, p. 168], so

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\} &\geqq 1-|z\phi(z)|- \frac{|z|(|z|\,|\phi'(z)|+|\phi(z)|)}{1-|z\phi(z)|}\\ &\geqq 1-|z|- \frac{|z|(|z|+|\phi(z)|)}{1-|z|^2}\\ &\geqq 1-|z|- \frac{|z|}{1-|z|}\\ &= \frac{1-3|z|+|z|^2}{1-|z|}.\end{aligned}$$

The last expression is positive for $|z| < (3-\sqrt{5})/2$, so f is convex in that disk.

If $f(z) = ze^z$, then $\operatorname{Re}\{zf''(z)/f'(z)+1\} = 0$ for $z = -(3-\sqrt{5})/2$, so this function is not convex in any disk of radius r for $r > (3-\sqrt{5})/2 \sim \cdot 382$.

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(Oblatum 2–4–68)