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by

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Near-rings on certain finite groups have been considered by Clay [3], Jacobson [9], Clay and Malone [4], Maxson [12] and Heatherly [7]. It was shown in [4] that any near-ring with identity defined on a finite simple group is a field. This result was generalized in [7] by showing that the above result holds under a weaker hypothesis: the existence of a nonzero right distributive element. It is the purpose of this paper to extend the above results to near-rings with a chain condition on arbitrary simple groups. We also extend some known theorems [1] in ring theory to distributively generated near-rings.

1. Definitions

A near-ring R is a system with two binary operations, addition and multiplication such that:

(i) The elements of R form a group R^+ under addition,
(ii) The elements of R form a multiplicative semigroup,
(iii) $x(y+z) = xy+xz$, for all $x, y, z \in R$,
(iv) $0x = 0$, where 0 is the additive identity of R^+ and for all $x \in R$.

In particular, if R contains a multiplicative semigroup S whose elements generate R^+ and satisfy

(v) $(x+y)s = xs+ys$, for all $x, y \in R$ and $s \in S$, we say that R is a distributively generated (d.g.) near-ring.

The most natural example of a near-ring is given by the set R of identity preserving mappings of an additive group G (not necessarily abelian) into itself. If the mappings are added by adding images and multiplication is iteration, then the system $(R, +, \cdot)$ is a near-ring. If S is a multiplicative semigroup of

[1] Portions of this paper appear in the author's Ph.D. dissertation written under the direction of Professor J. J. Malone, Jr., at Texas A & M University.

endomorphisms of G and R' is the sub-near-ring generated by S, then R' is a d.g. near-ring. Other examples of d.g. near-rings may be found in [6].

A near-ring R that contains more than one element is said to be a division near-ring if and only if the set R' of nonzero elements is a multiplicative group. Every division ring is an example of a division near-ring. For examples of division near-ring which are not division rings, see [14].

An element r of R is right distributive if $(b+c)r = br+cr$; for all b, $c \in R$. An element $x \in R$ is anti-right distributive if $(y+z)x = zx+yx$, for all y, $z \in R$. It follows at once that an element r is right distributive if and only if $(-r)$ is anti-right distributive. In particular, any element of a d.g. near-ring is a finite sum of right and anti-right distributive elements.

A subgroup H of a near-ring R is called an R-subgroup if $HR = \{hr\colon h \in H, r \in R\} \subseteqq H$.

Division near-rings were first considered by L. E. Dickson [5]. In 1936 H. Zassenhaus [14] proved that the additive group of a finite division near-ring is abelian. Four years later, B. H. Neumann [13] extended this result to the general case. For easy reference, we state

THEOREM 1. *The additive group of a division near-ring is abelian.*

2. Descending chain condition on principal R-subgroups

The element e in the d.g. near-ring R is an identity for R if $er = re = r$ for each r in R. The element $z \neq 0$ in R is a zero divisor if there exists $w \neq 0$ in R such that either $wz = 0$ or $zw = 0$. For each x in R, $xR = \{xr\colon r \in R\}$ is an R-subgroup of R. In particular, xR will be called a principal R-subgroup of R. The following results are generalizations of those given in [1].

THEOREM 2. *Let R be a d.g. near-ring with d.c.c. on principal R-subgroups. Then R has an identity if (and only if) at least one element in R is not a zero divisor.*

PROOF. Suppose $x \neq 0$ is not a zero divisor. Since

$$xR \supseteqq x^2R \supseteqq \dots,$$

the d.c.c. assures us that there must exist a positive integer n such that $x^n R = x^{n+1} R = \cdots$. Thus $x^n x = x^{n+1}e$ for some e in R. It follows that $x^n(x-xe) = 0$. This implies that $x = xe$. From the fact that $x(ex-x) = 0$, we see that e is a two-sided identity for x. Let w be any element in R. Then $x(ew-w) = 0$ and this implies that e is a left identity for w. Since R is a d.g. near-ring, any element in R is a finite sum of right and anti-right distributive elements. Let $x = x_1+x_2+\cdots+x_n$. Then

$$(we-w)x = (we-w)x_1+(we-w)x_2+\cdots+(we-w)x_n = 0.$$

This follows since $(we-w)x_i = -wx_i+wex_i = 0$ if x_i is anti-right distributive and $(we-w)x_i = wex_i-wx_i = 0$ if x_i is right distributive. The fact that x is not a zero divisor implies that $we = w$. Hence e is a two-sided identity for R.

In 1939 C. Hopkins [8] proved that if a ring R contains a left identity or a right identity for R, then the maximum condition for left ideals in R is a consequence of the minimum condition for left ideals in R. As Baer [1] pointed out, Hopkins theorem can be improved slightly by applying the ring analogue of Theorem 2. In 1966 Beidleman [2] proved a similar theorem for distributively generated near-rings with identity whose additive groups is solvable. Thus we can also improve Beidleman's theorem slightly as follows.

Corollary 1. *Let R be a d.g. near-ring whose additive group R^+ is solvable. If R satisfies the d.c.c. on R-subgroups, then either each element is a zero divisor or R is Noetherian.*

As another application of Theorem 2 we extend another result [1, p. 634] in ring theory to d.g. near-rings.

Corollary 2. *A d.g. near-ring R is a division ring if and only if R has no zero divisors and the d.c.c. is satisfied by the principal R-subgroups in R.*

Proof. Necessity is quite clear. From Theorem 2 R has an identity e. For each $x \neq 0$ in R, there is a positive integer n such that $x^n R = x^{n+1}R$. Thus $x^n e = x^{n+1}y$ and this implies that $x^n(e-xy) = 0$. Thus $e = xy$ and each nonzero element in R has a right inverse and hence R is a division near-ring. By Theorem 1, R^+ is abelian. It now follows [6, p. 93] that R is a division ring.

Corollary 3. *A finite d.g. near-ring with no zero divisors is a field.*

Corollary 4. *Any finite integral domain is a field.*

By employing a similar argument used in Theorem 2 and Corollary 2, we have two other characterizations of division near-rings. For other characterizations of division near-rings, see [10].

COROLLARY 5. *Let R be a near-ring with a nonzero right distributive element. Then R is a division near-ring if and only if R has no zero divisors and the d.c.c. is satisfied by the principal R-subgroups in R.*

COROLLARY 6. *A finite near-ring R with a nonzero right distributive element is a division near-ring if and only if R has no zero divisors.*

REMARKS. Let G be a finite additive group with at least three elements. For each $g \neq 0$ in G, define $gx = x$ for all x in G and $0y = 0$ for all y in G. Then $(G, +, \cdot)$ is a near-ring [11]. This near-ring is not distributively generated. Thus we see that Theorem 2, Corollaries 2, 3 and 4 cannot be extended to arbitrary near-rings.

3. Near-rings on simple groups

Clay and Malone [4] have shown that a near-ring with identity on a finite simple group is a field. Heatherly [7] has extended this result by assuming only the existence of a nonzero right distributive element. Now we generalize their results to near-rings with d.c.c. on principal R-subgroups defined on arbitrary simple groups.

THEOREM 3. *Let $(R, +)$ be any simple group and $(R, +, \cdot)$ a near-ring defined on $(R, +)$ such that $(R, +, \cdot)$ satisfies the d.c.c. on principal R-subgroups and has a nonzero right distributive element r. Then either $ab = 0$ for each $a, b \in R$ or $(R, +, \cdot)$ is a field.*

PROOF. Suppose $a \neq 0$. Define $T(a) = \{x \in R : ax = 0\}$. This is a normal subgroup of $(R, +)$. If $ab \neq 0$ for some $b \neq 0$, then $T(a) = 0$. Let $L(r) = \{y \in R : yr = 0\}$. Since $L(r)$ is a normal subgroup of $(R, +)$ and since $(R, +)$ is simple it follows that $L(r) = 0$ or $L(r) = (R, +)$. In case $L(r) = (R, +)$ it follows easily that $ab = 0$ for each $a, b \in R$. Therefore suppose $L(r) = 0$. Now let c be any nonzero element in R. Then $T(c) = 0$ since $cr \neq 0$. It follows that no element is a zero divisor and thus Corollary 5 implies that $(R, +, \cdot)$ is a division near-ring. By Theorem 1, $(R, +)$ is abelian.

Let $M = \{r \in R : (x+y)r = xr+yr\}$. It is easily shown that M is a normal subgroup of $(R, +)$. Since $e \in M$, it follows that $M = R$. Thus $(R, +, \cdot)$ is a division ring. Finally let $C = \{x \in R : xy = yx$ for all $y \in R\}$. Since $(R, +)$ is abelian, we see that C is a normal subgroup of $(R, +)$. But $e \in C$, we conclude that $C = R$. This shows that $(R, +, \cdot)$ is a field.

COROLLARY 7. (Clay and Malone, Heatherly) *Any near-ring with identity defined on a finite simple group is a field.*

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