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by

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1. Introduction

If G is a group and r is a positive integer, G is said to have *finite rank* r if each finitely generated subgroup of G can be generated by r or fewer elements and if r is the least such integer. Here we consider the effect of imposing finiteness of rank on groups which have some degree of solubility in a sense which will now be made precise.

If $\mathfrak{X}$ is a class of groups, let

$$\dot{P}\mathfrak{X}$$

denote the class of all groups which have an ascending series with each factor in $\mathfrak{X}$ and let

$$L\mathfrak{X}$$

denote the class of locally-$\mathfrak{X}$ groups, i.e., groups such that each finite subset lies in a subgroup belonging to the class $\mathfrak{X}$. $\dot{P}$ and L are closure operations on the class of all classes of groups. A class $\mathfrak{X}$ is said to be $\dot{P}$-closed if $\mathfrak{X} = \dot{P}\mathfrak{X}$ and L-closed if $\mathfrak{X} = L\mathfrak{X}$. Let us denote by

$$\overline{\mathfrak{X}}$$

the intersection of all the classes of groups which contain $\mathfrak{X}$ and are both $\dot{P}$ and L-closed: clearly $\overline{\mathfrak{X}}$ is just the smallest $\dot{P}$ and L-closed class containing $\mathfrak{X}$. It is easy to show that $\overline{\mathfrak{X}}$ is simply the union of all the classes $(\dot{P}L)^{\alpha}\mathfrak{X}$, α = an ordinal number: these classes are defined by

$$(\dot{P}L)^{\alpha+1}\mathfrak{X} = \dot{P}L((\dot{P}L)^{\alpha}\mathfrak{X})$$

and

$$(\dot{P}L)^{\lambda}\mathfrak{X} = \bigcup_{\alpha<\lambda} (\dot{P}L)^{\alpha}\mathfrak{X}$$

for all ordinals α and all limit ordinals λ, ([5], p. 534).

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Let $\mathfrak{A}$ denote the class of abelian groups. We shall be concerned here with the class

$$\overline{\mathfrak{A}};$$

this is a class of generalized soluble groups containing for example all locally soluble groups and all SN*-groups (see [6] for terminology). Our object is to prove the following.

THEOREM. *Let G be a group belonging to $\overline{\mathfrak{A}}$, the smallest class of groups containing all abelian groups which is $\dot{P}$-closed and L-closed, and suppose that G has finite rank r. Then G is locally a soluble minimax group with minimax length bounded by a function of r only.*

By a *minimax group* we mean a group G with a *minimax series* of finite length, i.e. a series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G$$

in which each factor satisfies either Max (the maximal condition on subgroups) or Min (the minimal condition of subgroups). The length of a shortest minimax series of G is called the *minimax length* of G and is denoted by

$$m(G).$$

The theorem implies for example that every finitely generated soluble group of finite rank is a minimax group: this furnishes a partial solution to a problem raised in a previous paper ([9], p. 518).

2. Proofs

We recall the well-known fact that an abelian group has finite rank if and only if its p-component is the direct product of a boundedly finite number (r_p) of cyclic and quasicyclic subgroups for each prime p and the factor group of its torsion-subgroup is is isomorphic with an additive subgroup of a rational vector space of finite dimension (r_0). Moreover if r_0 is the least such integer, the rank of the group is precisely $r_0 + \operatorname{Max}_p r_p$. (For example see Fuchs [3] pp. 36 and 68).

Two preliminary results will be required.

LEMMA 1. *Let G be a nilpotent group. Then G is a minimax group if and only if G/G', its derived factor group, is a minimax group.*

For a proof of this see [10], Corollary 1.

LEMMA 2 (Mal'cev [7], Theorem 4). *Let G be a group with a series* *of normal subgroups*[2] *of finite length such that each factor of the series is an abelian group of finite rank in which only finitely many primary components are non-trivial. Then G has a normal subgroup of finite index whose derived subgroup is nilpotent, i.e. G is nilpotent-by abelian-by-finite.*

The proof of this lemma is a straightforward application of the Kolchin-Mal'cev theorem on the structure of soluble linear groups.

PROOF OF THE THEOREM

(a) Assume that G is a finitely generated soluble group of finite rank r. We will prove that G is a minimax group and we note first of all that in order to do this it is sufficient to show that G is nilpotent-by-abelian-by-finite. For suppose that G has this structure. Since subgroups of finite index in G are also finitely generated, we can assume that G is nilpotent-by-abelian, i.e. G has a normal nilpotent subgroup N such that G/N is abelian. By Lemma 1 we can suppose without loss of generality that N is abelian, so that G is finitely generated and metabelian and therefore satisfies the maximal condition on normal subgroups by a result of P. Hall ([4], Theorem 3). The torsion-subgroup of N satisfies the maximal condition on characteristic subgroups and also has finite rank. Hence this subgroup is finite and we may take N to be a torsion-free abelian group of rank $\leqq r$. Also

$$N = a_1^G a_2^G \cdots a_n^G$$

for a suitable finite subset $\{a_1, a_2, \cdots, a_n\}$. If follows that G is a minimax group if and only if every $A = a^G$ $(a \in N)$ is. We can therefore concentrate on A.

We identify A with an additive subgroup of an r-dimensional rational vector space V and extend the action of G from A to V in the natural way, so that G is represented by a group of linear operators on V. Choose a basis for V. We can represent each element g of G by an $r \times r$ matrix $M(g)$ with rational entries. Let the components of a with respect to the basis be $a_1, \cdots, a_r$ and let G be generated by $g_1, \cdots, g_r$. The primes occurring non-trivially in the denominator of an a_i or of an entry in an $M(g_j)$ or $M(g_j^{-1})$ form a finite set π, say. If $b \in A$ has components $b_1, \cdots, b_r$, then the denominators of the b_i's may be taken to be π-numbers. Hence A is isomorphic with a subgroup of the direct sum of r copies of Q_π, the additive group of all rational numbers whose denominators

[2] Actually it is not necessary for the terms of the series to be normal subgroups here.

are π-numbers. Since Q_π is a minimax group, so is A.

Now let G be *any* finitely generated soluble group with finite rank r. Then G has a normal series of finite length,

$$1 = G_0 \leqq G_1 \leqq \cdots \leqq G_n = G,$$

in which each G_{i+1}/G_i is either torsion-free and abelian of rank $\leqq r$ or else a direct product of abelian p-groups, each of rank $\leqq r$. Let $n > 1$ and write $A = G_1$; by induction on n G/A is a minimax group. If A is torsion-free, the hypotheses of Lemma 2 are fulfilled, so G is nilpotent-by-abelian-by-finite and the first part of this proof shows that G is a minimax group.

Suppose that A is periodic and G is not a minimax group. Then A has infinitely many non-trivial primary components and there is a normal subgroup B of G contained in A such that A/B has infinitely many non-trivial primary components and the p-component is either elementary abelian of order $\leqq p^r$ or a direct product of $\leqq r$ groups of type p^∞. Clearly we can take $B = 1$. The action of G on the p-component of A yields a representation of G as a linear group of degree r over either $GF(p)$ or the field of p-adic numbers. In either case the strong form of the Kolchin-Mal'cev Theorem ([11], Theorem 21) shows that there is an integer m depending only on r such that $R = (G^m)'$ acts unitriangularly on each primary component of A. Hence

$$[A, \underbrace{R, \cdots, R}_{r}] = 1. \tag{2}$$

Since G/A is a minimax group, it is nilpotent-by-abelian-by-finite by Lemma 2; hence for some $n > 0$ $S = (G^n)'$ is such that SA/A is nilpotent. Let $T = (G^{mn})'$; then G/G^{mn} is finite and T is nilpotent by (2), so G is nilpotent-by-abelian-by-finite. Hence G is a minimax group, which is a contradiction.

We have still to provide a bound for $m(G)$ when G is any finitely generated soluble group of rank r. Let P denote the maximal normal periodic subgroup of G and let N/P be the Fitting subgroup of G/P. Clearly P satisfies Min and by Theorem 2.11 of [8], G/N satisfies Max. Hence writing H for N/P we have

$$m(G) \leqq m(H)+2.$$

H is locally nilpotent and torsion-free and has finite rank, so by a theorem of Mal'cev, ([7], Theorem 5), H is nilpotent. Let M be a maximal normal abelian subgroup of H; then M coincides with its centralizer in H and H/M is essentially a group of automorphisms of M. Since M is torsion-free and abelian of rank $\leqq r$ and since H is nilpotent, it follows that

$$[M, \underbrace{H, \cdots, H}_{r}] = 1;$$

also H/M, being isomorphic with a group of unitriangular $r \times r$ matrices, has nilpotent class $\leqq r-1$. Hence if c is the nilpotent class of H, $c \leqq 2r-1$. By Theorem 4.22 of [8]

$$m(H) \leqq 3[\log_3(c+1)]+3.$$

By combining these inequalities we obtain

$$m(G) \leqq 3[\log_3(2r)]+5.$$

(b) Let G be a locally soluble group of finite rank r. Some information about the structure of G is necessary before we can go further. Let H be any finitely generated subgroup of G. Then H is soluble with rank $\leqq r$ and consequently H has an ascending normal series each factor of which is either torsion-free and abelian of rank $\leqq r$ or elementary abelian of order dividing p^r for some prime p. The action of H on a factor of this series gives rise to a representation of H as a linear group of degree r. Now a well-known theorem of Zassenhaus ([12]) asserts that the derived length a soluble linear group of degree r does not exceed a certain number $n = n(r)$ depending only on r. Hence $H^{(n)}$, the $(n+1)$th term of the derived series of H, centralizes every factor of the original ascending series of H. It follows that $H^{(n)}$ is a hypercentral (or ZA)-group. Since n is independent of H, $G^{(n)}$ is locally hypercentral, i.e. locally nilpotent. By results of Mal'cev and Černikov ([7], p. 12) in a locally nilpotent group of finite rank each primary component is hypercentral and satisfies Min and the torsion-factor group is nilpotent. Thus we have established the following.

Let G be a locally soluble group of finite rank. Then G has a normal subgroup T such that G/T is soluble and T is a periodic hypercentral group with each of its primary components satisfying Min.[3]

(c) It remains only to show that every $\overline{\mathfrak{A}}$-group with finite rank is locally soluble. Suppose that this is not the case and that α is the first ordinal for which groups of finite rank in the class $(\dot{P}L)^{\alpha}\mathfrak{A}$ need not be locally soluble. α cannot be a limit ordinal. Let G be a group of finite rank in the class $(\dot{P}L)^{\alpha}\mathfrak{A}$; then G has an

[3] Thus torsion-free locally soluble groups of finite rank are soluble (Čarin [2]). On the other hand locally soluble groups of finite rank are not soluble in general — see [1], p. 27.

ascending series whose factors all belong to the class $L(\dot{P}L)^{\alpha-1}\mathfrak{A}$ and by minimality of α are therefore locally soluble. We will denote this ascending series by $\{G_\beta : \beta < \gamma\}$. Suppose that G is not locally soluble and let β be the first ordinal for which G_β is not locally soluble. Again β is not a limit ordinal, so both $G_{\beta-1}$ and $G_\beta/G_{\beta-1}$ are locally soluble.

Let H be a finitely generated subgroup of G_β. Then $H/H \cap G_{\beta-1}$ is soluble and $H \cap G_{\beta-1}$ is locally soluble; consequently by (b) there is an integer n such that $H^{(n)}$ is periodic and hypercentral. Now by (a) $H/H^{(n+1)}$ is a minimax group and this implies that $H^{(n)}/H^{(n+1)}$ satisfies Min and so has only finitely many non-trivial primary components. Let $S = H^{(n)}$. Then for all but a finite number of primes p, S_p, the p-component of S, lies in S'. Since S is the direct product of its primary components, this means that $S_p = (S_p)'$. But each S_p is soluble, as a locally nilpotent p-group of finite rank, so all but a finite number of the S_p's are trivial and therefore S is soluble. However this implies that H is soluble and G_β is locally soluble, a contradiction.

In conclusion we remark that in [5] (p. 538) P. Hall has shown that even SI^*-groups need not be locally soluble, so certainly $\overline{\mathfrak{A}}$-groups need not be either.

REFERENCES

R. Baer

[1] Polyminimaxgruppen. Math Ann. 175 (1968), 1—43.

V. S. Čarin

[2] On locally soluble groups of finite rank. Mat. Sb. (N.S.) 41 (1957), 37—48.

L. Fuchs

[3] Abelian groups. Oxford: Pergamon Press (1960).

P. Hall

[4] Finiteness conditions for soluble groups. Proc. London Math. Soc. (3), 4 (1954), 419—436.

P. Hall

[5] On non-strictly simple groups. Proc. Cambridge Philos. Soc. 59 (1963), 531—553.

A. G. Kuroš

[6] The theory of groups. Second edition. New York: Chelsea (1960).

A. I. Mal'cev

[7] On certain classes of infinite soluble groups. Mat. Sbornik (N.S.) 28 (70 (1951), 567—588. Amer. Math. Soc. Translations (2), 2 (1956), 1—21.

D. J. S. Robinson

[8] On soluble minimax groups. Math. Zeit. 101 (1967), 13—40.

D. J. S. Robinson

[9] Residual properties of some classes of infinite soluble groups. Proc. London Math. Soc. (3) 18 (1968), 495—520.

D. J. S. Robinson

[10] A property of the lower central series of a group. Math. Zeit. 107 (1968), 225—231.

D. A. Suprunenko

[11] Soluble and nilpotent linear groups. Amer. Math. Soc. Translations, Math. Monographs 9 (1963).

H. Zassenhaus

[12] Beweis eines Satzes über diskrete Gruppen. Abh. Math. Sem. Univ. Hamburg, 12 (1938), 289—312.

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