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by

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The primary purpose of this paper is to apply the following two theorems to the study of certain subsets of F-manifolds. These theorems can be found, for example, in [1].

THEOREM 0.1. *Each F-manifold can be embedded as an open subset of Hilbert space.*

THEOREM 0.2. *Two F-manifolds are homeomorphic if and only if they have the same homotopy type.*

The major results in this paper are an engulfing theorem for certain subsets of an F-manifold (Theorem 2.1) and an annulus theorem for disjoint, bicollared spheres in an F-manifold (Theorem 3.2).

H will be used to denote separable Hilbert space, and M will denote an arbitrary F-manifold. By an F-manifold is meant a separable metric space which is a manifold modeled on a separable infinite-dimensional Fréchet space.

$B_r(x)$ will be the ball in H of radius r centered at x; $S_r(x) = Bd\, B_r(x)$; $B_r = B_r(\theta)$; and $S_r = S_r(\theta)$, where θ is the zero element of H.

By an *open H-cell* (or open cell) in a space X is meant an open subset of X which is the homeomorphic image of H. A *closed H-cell* (or cell) is a subset C of the space X such that there is a homeomorphism from the pair (B_1, S_1) in H onto the pair $(C, Bd\, C)$ in X. A closed subset K of X-Int C is a *collar* of C if there exists a homeomorphism h from the pair (B_2, B_1) in H onto the pair $(K \cup C, C)$ in X such that $h(S_2) = Bd\,(K \cup C)$.

1. When F-manifolds are homeomorphic to H

The monotone union property for an F-manifold M is the property that the union of an increasing sequence of copies of M, which are open in the union, must be homeomorphic to M.

The following theorem gives several necessary and sufficient conditions for an F-manifold M to be Hilbert space.

THEOREM 1.1. *The following are equivalent for an F-manifold M.*
(1) *M is homeomorphic to H.*
(2) *M is contractible.*
(3) *M has trivial homotopy groups.*
(4) *M is an AR.*
(5) *M has the monotone union property.*

PROOF. (1) is equivalent to (3) follows from Theorem 0.2. That (2), (3), and (4) are equivalent can be found in [4]. (1) implies (5) follows from the Monotone Union Theorem for Hilbert Space found in [2]. Finally to show that (5) implies (1), by Theorem 0.1, let h be an embedding of M as an open subset of H. Choose $\delta > 0$ and $x \in h(M)$ so that $B_\delta(x) \subset h(M)$. Let g be a homeomorphism of H onto itself such that $g[B_\delta(x)] = B_1$. For each $n = 1, 2, \cdots$, let f_n be a homeomorphism from H onto Int B_{n+1} such that $f_n(B_1) = B_n$, and let $M_n = f_n gh(M)$. Then $H = \bigcup_{n=1}^{\infty} M_n$ is a monotone union of open copies of M, so that M is homeomorphic to H.

2. An engulfing theorem for F-manifolds

LEMMA 2.1. *If U and V are open cells in H such that $U \cap V$ is a cell, then $U \cup V$ is a cell.*

PROOF. U, V, and $U \cap V$ are AR's, so that $U \cup V$ is an AR. Then by Theorem 1.1, $U \cup V$ is a cell.

Lemma 2.1 can be generalized slightly using Van Kampen's Theorem.

LEMMA 2.2. *Let U be a connected open subset of H. Then every two points in U can be joined by a piecewise linear arc lying in U.*

PROOF. Let $x, y \in U$. Since U is connected and open in H, it will be arcwise connected. Because an arc between x and y is compact, there exists $B_{\delta_1}(x_1), \cdots, B_{\delta_n}(x_n)$ such that each $B_{\delta_i}(x_i) \subset U$, $B_{\delta_i}(x_i) \cap B_{\delta_{i+1}}(x_{i+1}) \neq \emptyset$ for $i = 1, \cdots, n-1$, and $x_1 = x$ and $x_n = y$. Then $[x_i : x_{i+1}] \subset B_{\delta_i}(x_i) \cup B_{\delta_{i+1}}(x_{i+1})$ for $i = 1, \cdots, n-1$, so that $\bigcup_{i=1}^{n-1} [x_i : x_{i+1}]$ is a piecewise linear arc joining x and y, where $[x_i : x_{i+1}]$ is the line segment from x_i to x_{i+1}.

LEMMA 2.3. *Let U be a connected open subset of H containing B_1, let $x \in U - B_1$, and let $y \in S_1$. Then there is a piecewise linear arc joining x and y lying in $(U - B_1) \cup y$.*

PROOF. Let $\delta > 0$ be such that $B_\delta(y) \subset U$. Let z be the point such that Ray $[\theta : y] \cap (B_\delta(y) - \text{Int } B_1) = [y : z]$, where Ray $[\theta : y]$ is the ray eminating from θ and passing through y. By Lemma 2.2, there exists a piecewise linear arc α from x to z lying in $U - B_1$. Let w be the point of $\alpha \cap [y : z]$ such that the portion of α lying between w and x, call it β, intersects $[y : z]$ only at w. Then $[y : w] \cup \beta$ is the desired piecewise linear arc.

LEMMA 2.4. *Let U be a connected open subset of H containing B_1, and let $x \in U - B_1$. Then there exists an open cell contained in U and containing* Int $B_1 \cup x$.

PROOF. Let $y \in S_1$. Then by Lemma 2.3 there exists a piecewise linear arc α joining x and y and lying in $(U - B_1) \cup y$. Let $\alpha_1, \cdots, \alpha_n$ be the linear pieces of α, starting at y and going to x. Let $\delta_1 > 0$ be such that $N_{\delta_1}(\alpha_1) \cap (\bigcup_{i=3}^n \alpha_i) = \emptyset$, where $N_{\delta_1}(\alpha_1)$ is the open δ_1 neighborhood of α_1. Then for each $i = 2, \cdots, n$, inductively define δ_i so that

$$N_{\delta_i}(\alpha_i) \cap ([\bigcup_{j=1}^{i-2} N_{\delta_j}(\alpha_j)] \cup [\bigcup_{j=i+2}^{n} \alpha_j] \cup \text{Int } B_1) = \emptyset.$$

Each $N_{\delta_i}(\alpha_i)$ is an open cell, and $N_{\delta_1}(\alpha_1) \cap \text{Int } B_1$ and $N_{\delta_i}(\alpha_i) \cap N_{\delta_{i+1}}(\alpha_{i+1})$ are convex and are hence open cells. Then by repeated applications of Lemma 2.1, $[\bigcup_{i=1}^n N_{\delta_i}(\alpha_i)] \cup \text{Int } B_1$ is an open cell containing Int $B_1 \cup x$ and contained in U.

THEOREM 2.1. *Let X be a subset of a connected F-manifold M, which is contained is some collared cell in M, and let U be an open subset of M. Then there exists a homeomorphism h of M onto itself and a collared cell C such that $X \subset h(U)$ and $h|(M - C) =$ identity.*

PROOF. By Theorem 0.1, consider M as an open subset of H. By hypothesis, X is contained in a collared cell C' which is contained in M. Let $x \in U$ and $x \in \text{Int } C'$. From an application of Lemma 2.4, $x \cup y$ is contained in a collared cell C'' in H such that $C \cup M$. Then let f be a homeomorphism of H onto itself so that $f(C'') = B_1$. Define g to be a homeomorphism of H onto itself so that $gf(y) = f(x)$ and $g|(H - B_1) =$ identity. Then define the homeomorphism φ of M onto itself by $\varphi(x) = f^{-1}gf(x)$ if $x \in C'$, and $\varphi(x) = x$ otherwise. Now $\varphi(C')$ is a collared cell in M such that $U \cap \text{Int } \varphi(C) \neq \emptyset$. If K is a collar of $\varphi(C')$ in M, then $K \cup \varphi(C')$ is homeomorphic to H. Since the image of $\varphi(C')$ under this homeomorphism is collared in H, the theorem can be established using the Engulfing Theorem for Hilbert Space found in [3].

3. Bicollared spheres in *F*-manifolds

A closed subset A of a space X is an *annulus* if there exists a homeomorphism h from B_2—Int B_1 onto A such that

$$Bd\,A = h(S_1 \cup S_2).$$

A *bicollared sphere* in X is a closed set S such that there exists a homeomorphism g from B_3—Int B_1 onto an annulus in X such that $g(S_2) = S$.

THEOREM 3.1. *A closed complementary domain of a bicollared sphere in an F-manifold M is a closed H-cell if and only if it satisfies one of conditions* (1) *through* (5) *in Theorem* 1.1.

The proof of Theorem 3.1 is similar to the proof of the following theorem.

THEOREM 3.2. *The closed region between two disjoint, bicollared spheres in an F-manifold M is an annulus if and only if it satisfies one of conditions* (1) *through* (5) *in Theorem* 1.1.

PROOF. The annular region plus the collars of the spheres can be seen to be a contractible F-manifold, which is therefore homeomorphic to H by Theorem 1.1. The images of the spheres in H under this homeomorphism are tame since they are bicollared (see [5]), so that the region between them is an annulus by Corollary 1 of [3].

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