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Extending of continuous real functions ¹

by

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1. Introduction

Let X and E be topological spaces. Let Y be a superspace of X . We say that X is E -embedded in Y provided that every continuous function $f : X \rightarrow E$ admits a continuous extension $f^* : Y \rightarrow E$. We shall be concerned with R -embedding, where R is the space of the reals. All spaces will be assumed to be Hausdorff completely regular. For some spaces X , the fact that X is R -embedded in Y can be decided by examining the extension of only one function $f : X \rightarrow R$ (this property of X is formulated precisely in the next section). This paper contains some partial results concerning the characterization of such spaces.

We shall now formulate a few statements of purely technical character.

A function $f : X \rightarrow R$ is said to be *absolutely extendable* provided that for every superspace Y of X , f admits a continuous extension $f^* : Y \rightarrow R$. We say that f has *vanishing oscillation outside compact subsets of X* provided that for every $\varepsilon > 0$ there exists a compact subset C of X such that

$$\omega(f, X \setminus C) = \sup \{|f(p) - f(q)| : p, q \in X \setminus C\} < \varepsilon.$$

1.1 PROPOSITION. *A function $f : X \rightarrow R$ is absolutely extendable if and only if f can be continuously extended over every compactification of X .*

1.2 PROPOSITION. *A function $f : X \rightarrow R$ has vanishing oscillation outside compact subsets of X if and only if f is the limit of a uniformly convergent sequence f_1, f_2, \dots of functions on X each of which is constant outside a compact subset of X .*

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PROOF. The sufficiency of the condition is obvious. To prove the necessity consider the functions $\alpha_{sn} : R \rightarrow R$, $s \in R$, $n = 1, 2, \dots$, defined by

$$\begin{aligned}\alpha_{sn}(t) &= s && \text{for } |t-s| \leq \frac{1}{n}; \\ \alpha_{sn}(t) &= t - \frac{1}{n} && \text{for } t \geq s + \frac{1}{n}; \\ \alpha_{sn}(t) &= t + \frac{1}{n} && \text{for } t \leq s - \frac{1}{n}.\end{aligned}$$

Each α_{sn} is continuous and $|\alpha_{sn}(t) - t| \leq 1/n$ for every $t \in R$.

Assume that $f : X \rightarrow R$ has vanishing oscillation outside compact subsets of X . For every positive integer n find a compact set $C_n \subset X$ such that $\omega(f, X \setminus C_n) \leq 1/n$. Let s be a value of f on $X \setminus C_n$ (if $X \setminus C_n$ is empty, then there is nothing to prove) and let f_n be the composition $\alpha_{sn} \circ f$. f_n is constant outside C_n and the sequence f_1, f_2, \dots is uniformly convergent to f on X .

1.3 PROPOSITION. *A function $f : X \rightarrow R$ is absolutely extendable if and only if f has vanishing oscillation outside compact subsets of X .*

PROOF. The necessity of the condition is obvious; the sufficiency follows from 1.1 and 1.2.

2. The property (P_1) . R -compact spaces

2.1 DEFINITION. We say that a space X has the property (P_1) provided that there exists a continuous function $f : X \rightarrow R$ such that for every superspace Y of X , X is R -embedded in Y if and only if the function f admits the continuous extension $f^* : Y \rightarrow R$.

A function f with the above property will be called (for a lack of a better term) a *proper function* on X .

In this section we shall give a characterization of R -compact spaces having property (P_1) . A space is R -compact iff it is homeomorphic to a closed subspace of some topological power R^m of R . Intuitively speaking, an R -compact space is a space which is either compact or admits a large number of continuous unbounded functions (precisely: X is R -compact iff for every $p_0 \in \beta X \setminus X$ there is a continuous function $f : X \rightarrow R$ which is unbounded on every neighbourhood of p_0). In the next section we shall state some partial results concerning the property (P_1) in arbitrary space.

A function $f : X \rightarrow R$ is said to be *bounded only on compact*

subsets of X provided that for every subset A of X , if f is bounded on A , then \bar{A} (= the closure of A in X) is compact.

2.2 PROPOSITION. *A space X admits a continuous function $f : X \rightarrow R$ that is bounded only on compact subsets of X if and only if X is locally compact and Lindelöf.*

Proof is obvious².

2.3 PROPOSITION. *Let $f : X \rightarrow R$ be a continuous function which is bounded only on compact subsets of X . Let Y be a superspace of X . X is R -embedded in Y if and only if the function f admits a continuous extension $f^* : Y \rightarrow R$.*

PROOF. The necessity of the condition is obvious; we shall prove the sufficiency. Let f^* be the continuous extension of f with $f^* : Y \rightarrow R$. Let g be an arbitrary continuous function with $g : X \rightarrow R$. It is clear that if a continuous function $\alpha : R \rightarrow R$ tends sufficiently fast to $+\infty$ as $|t| \rightarrow +\infty$, then the function $g(p)/\alpha(f(p))$ has vanishing oscillation outside compact subsets of X . (It suffices to take a continuous function $\alpha : R \rightarrow R$ such that

$$\alpha(t) \geq n \cdot \sup \{|g(p)| : p \in C_{n+1}\} + 1$$

for $|t| \geq n$, where $C_n = \{p \in X : |f(p)| \leq n\}$.) By 1.3, $g(p)/\alpha(f(p))$ is absolutely extendable. Let g^* be a continuous extension of $g(p)/\alpha(f(p))$ with $g^* : Y \rightarrow R$. It is clear that $g^*(p) \cdot \alpha(f^*(p))$ is a continuous extension of g over Y .

2.4 COROLLARY. *A locally compact Lindelöf subspace X of Y is R -embedded in Y if and only if there exists a continuous function $g : Y \rightarrow R$ such that $g|_X$ is bounded only on compact subsets of X .*

2.5 THEOREM. *Let X be an R -compact space. X has property (P_1) if and only if X is locally compact and Lindelöf. Furthermore, a continuous function $f : X \rightarrow R$ is a proper function if and only if f is bounded only on compact subsets of X .*

PROOF. Assume that $f : X \rightarrow R$ is a continuous function and assume that there is a set A such that f is bounded on A and \bar{A}^X is not compact. Then $\bar{A}^{\beta X} \setminus X \neq \emptyset$; let $p_0 \in \bar{A}^{\beta X} \setminus X$. f can be

² Recall that for a locally compact space X the following conditions are equivalent.

- (a) X is Lindelöf;
- (b) X is σ -compact (i.e., X is the union of countably many compact subsets);
- (c) $X = \bigcup_n C_n$, where C_n are compact and $C_n \subset \text{Int } C_{n+1}$;
- (d) the ideal point ∞ in the one-point compactification $\iota X = X \cup \{\infty\}$ of X satisfies the first axiom of countability.

extended over $X \cup \{p_0\}$; in fact, one can modify f outside a neighborhood of p_0 so that it becomes bounded on the whole of X . But X is not R -embedded in $X \cup \{p_0\}$. Thus f is not a proper function on X . The rest of the theorem follows now from Propositions 2.2 and 2.3.

3. Proper functions for arbitrary spaces

From the results of the previous section it is easy to obtain a characterization of proper functions for arbitrary spaces. We have to recall a few known facts and definitions.

An *extension* of X is any superspace εX of X such that X is dense in εX . The *canonical map* of extension $\varepsilon_1 X$ into an extension $\varepsilon_2 X$ is a continuous function $\varphi : \varepsilon_1 X \rightarrow \varepsilon_2 X$ which is the identity on X . The canonical map (if it exists) is unique. We write $\varepsilon_1 X = \text{ext } \varepsilon_2 X$ provided that there exists a canonical map of $\varepsilon_1 X$ onto $\varepsilon_2 X$ which is a homeomorphism. (For further information see [4], Chi I.) βX admits a canonical map onto any compactification of X .

The *Q -closure* of a subset P of a space X is the set of all points $q \in X$ such that for every continuous function $f : X \rightarrow R$, if $f(p) > 0$ for every $p \in P$, then $f(q) > 0$. P is said to be *Q -closed* in X provided that it is equal to its Q -closure in X . The Q -closure of any subset of an R -compact space is again R -compact. If $c_1 X$ and $c_2 X$ are compactifications of X , A is a Q -closed subset of $c_2 X$ containing X , and φ is a canonical map of $c_1 X$ onto $c_2 X$, then φ maps the Q -closure of X in $c_1 X$ into A .

$\beta_R X$ is an R -compact extension of X such that X is R -embedded in $\beta_R X$. ($\beta_R X$ is also called the *Nachbin completion* of X .) $\beta_R X$ coincides with the Q -closure of X in βX . X is R -compact iff $X = \beta_R X$; equivalently, X is R -compact iff X is Q -closed in βX .

Every continuous function $f : X \rightarrow R$ can be continuously extended over βX if we allow $\pm \infty$ to be values of this extension. We shall denote this extension by f^β (note that f^β is a function into the two-point compactification of R). It is easy to see that f is bounded only on compact subsets of X iff $f^\beta(p) = \pm \infty$ for every $p \in \beta X \setminus X$.

3.1 THEOREM. *A continuous function $f : X \rightarrow R$ is proper if and only if it satisfies the following two conditions:*

- (a) f^β is one-to-one on $\beta_R X \setminus X$;
- (b) $f^\beta(p) = \pm \infty$ for every $p \in \beta X \setminus \beta_R X$.

PROOF. Only the sufficiency requires a proof. Assume that Y is a superspace of X such that f admits a continuous extension $g : Y \rightarrow R$. We can assume that Y is R -compact (if not, replace Y by $\beta_R Y$). Let \tilde{X} be the Q -closure of X in Y ; \tilde{X} is R -compact, hence \tilde{X} is Q -closed in $\beta\tilde{X}$. Let $g_0 = g|_{\tilde{X}}$ (= the restriction of g to \tilde{X}); let φ be the canonical map of βX onto $\beta\tilde{X}$. The quality

$$(1) \quad f^\beta(p) = g_0^\beta(\varphi(p))$$

holds for every $p \in X$; hence, by continuity, (1) holds for every $p \in \beta X$. Since g_0^β is finite on \tilde{X} , we infer from (1) and (b) that $\varphi^{-1}[\tilde{X}] \subset \beta_R X$. Since \tilde{X} is Q -closed in $\beta\tilde{X}$, the reverse inclusion also holds. Consequently, $\beta_R X = \varphi^{-1}[\tilde{X}]$. It follows that $\varphi_0 = \varphi|_{\beta_R X}$ is a closed map (the restriction of a closed map to a full counter-image is again closed). From (1) and (a) we infer that φ_0 is one-to-one. Thus φ_0 is a homeomorphism; consequently, $\beta_R X =_{\text{ext}} \tilde{X}$. Hence X is R -embedded in \tilde{X} . But from (1) and (b) we infer that $g_0^\beta(q) = \pm\infty$ for every $q \in \beta\tilde{X} \setminus \tilde{X}$; hence g_0 is bounded only on compact subsets of \tilde{X} . Since \tilde{X} is R -compact, we infer from Theorem 2.5 that g_0 is a proper function for \tilde{X} . Consequently, \tilde{X} is R -embedded in Y . Thus X is R -embedded in Y . The theorem is shown.

We did not find any interesting characterization of arbitrary spaces with property (P_1) . The following partial results follow directly from Theorem 3.1.

3.2 PROPOSITION. *If X has (P_1) , then $\beta_R X$ is locally compact and Lindelöf.*

Recall that a space X is called *extremal* (in the sense of Fréchet, see [1]) provided that every continuous function $f : X \rightarrow R$ is bounded. Such spaces are also called *pseudocompact* or *quasicompact*. X is extremal iff $\beta_R X =_{\text{ext}} \beta X$.

3.3. THEOREM. *Let X be an extremal space. A continuous function $f : X \rightarrow R$ is proper if and only if f^β is one-to-one on $\beta X \setminus X$.*

3.4. THEOREM. *Let X be an extremal locally compact space. X has (P_1) if and only if the Čech outgrowth of X , $\beta X \setminus X$, is homeomorphic to a subspace of the closed interval $I = [0, 1]$.*

3.5. THEOREM. *Every space with a countable Čech outgrowth has (P_1) .*

PROOF. If $\text{card}(\beta X \setminus X) < 2^c$, then X is extremal. On the other hand, for every countable subset of an arbitrary space there exists

a continuous real function on the space which is one-to-one on this subset (see [4], Theorem 1).

On the basis of Theorem 3.5 it is easy to give examples showing that an extremal space with property (P_1) need not to be locally compact and its Čech outgrowth need not to be homeomorphic to a subspace of I .

We say that a space X has property (P_1^*) provided that there exists continuous function $f : X \rightarrow I$ (I is the closed interval $[0, 1]$) such that for every superspace Y of X , X is I -embedded in Y iff f admits a continuous extension $g : Y \rightarrow I$. A function f with this property will be called a $*$ -proper function. It can be shown that a continuous function $f : X \rightarrow I$ is $*$ -proper iff f^β is one-to-one on $\beta X \setminus X$. It follows that a space X does not have property (P_1^*) unless X is extremal. But for such spaces properties (P_1) and (P_1^*) coincide.

We conclude with two questions.

It follows from Theorem 3.4 that for a locally compact extremal space X property (P_1) depends only on the topological type of the Čech outgrowth of X . Is this true for arbitrary extremal spaces?

Does $\beta X \setminus X$ being homeomorphic to a subspace of I imply that X has (P_1) ?

4. Generalizations of properties (P_1) and (P_1^*)

The purpose of this section is to state some questions concerning generalizations of properties (P_1) and (P_1^*) to higher cardinalities.

Let m be an arbitrary cardinal; we shall say that a space X has property (P_m) provided that there exists a class \mathfrak{F} of continuous real-valued function on X such that $\text{card } \mathfrak{F} \leq m$ and for every superspace Y of X , X is R -embedded in Y iff each function in \mathfrak{F} can be extended to a continuous real-valued function on Y . Such a class \mathfrak{F} will be called a proper class on X . Property (P_m^*) and $*$ -proper classes are defined in an analogous way. It is clear that (P_m) implies (P_n) and (P_m^*) implies (P_n^*) for $n > m$; furthermore, every space has property (P_m) (as well as (P_m^*)) for a sufficiently large m . (Properties (P_0) and (P_0^*) are equivalent; each of them asserts that X is R -embedded in each of its superspaces; such spaces coincide with those having exactly one compactification.) It can be easily shown that \mathfrak{F} is a $*$ -proper class on X iff the continuous extensions of functions in \mathfrak{F} over βX separate points of $\beta X \setminus X$ (compare with Theorem 3.1 and the remarks at the end

of § 3). This, in turn, implies that X cannot have property $(P_{\aleph_0}^*)$ unless X is extremal; consequently, for $m \leq \aleph_0$, (P_m^*) implies (P_m) . Clearly, there are non-extremal spaces having property $(P_{2^{\aleph_0}}^*)$; I do not know if it can be shown without the continuum hypothesis that 2^{\aleph_0} is the first such cardinal. It can also be shown that a locally compact extremal space has (P_m) iff its Čech outgrowth is homeomorphic to a subspace of the Tihonov cube I^m (compare with 3.4). It follows that for locally compact extremal spaces properties (P_m) and (P_n) are not equivalent for any two distinct cardinals m and n .³ On the other hand, for R -compact spaces, properties $(P_1), (P_2), \dots, (P_n), \dots, n < \aleph_0$, are equivalent; this can be demonstrated by showing that $\mathfrak{F} = \{f_1, \dots, f_n\}$ is a proper class for X iff $f = \max\{|f_1|, \dots, |f_n|\}$ is a proper function for X .

Property (P_m) for R -compact spaces is somewhat related to the concept of R -defect introduced in [3]. (An R -non-extendable class for X is a class \mathfrak{F} of continuous real-valued functions on X such that for every extension εX of X with $\varepsilon X \neq X$, at least one of the functions in \mathfrak{F} does not admit a continuous real-valued extension over εX . The R -defect of X [in symbols: $\text{def}_R X$] is the smallest cardinal m such that X admits an R -non-extendable class of cardinality m . For further information see [3] and [5].) It can be easily shown that

4.1. *If an R -compact space has property (P_m) , then $\text{def}_R X \leq m$. In fact, a proper class on X is an R -non-extendable class for X .*

From 2.5 and from 5.9 in [5] we infer that for $m = 1$ the above implication can be reversed.

4.2. *Let X be R -compact. X has (P_1) if and only if $\text{def}_R X \leq 1$. The converse of 4.1 fails for infinite m . We have the following.*

4.3. *Let m be an infinite cardinal of the form $m = 2^n$ and let X be a space with weight $X \leq m$. X has (P_m) if and only if $\text{card } C(X, R) \leq m$.*

PROOF. The “if” part is obvious. To prove the converse it suffices to show that for every class \mathfrak{F} of continuous real-valued functions on X with $\text{card } \mathfrak{F} \leq m$ there is a superspace Y of X such that Y has only m continuous real-valued functions and each function in

³ In [2] Glicksberg proves that if $X \times Y$ is extremal, then $\beta(X \times Y) =_{\text{ext}} \beta X \times \beta Y$. On the other hand, if X is compact and Y is extremal, then $X \times Y$ is also extremal. Consequently, for every compact space X we have $\beta X^* \setminus X^* =_{\text{top}} X$, where $X^* = X \times S(\Omega)$ and $S(\Omega)$ is the space of all ordinals $< \Omega$.

\mathfrak{F} admits a continuous extension over Y . We can assume that \mathfrak{F} is an R -separating class for X . The parametric map h of X corresponding to F (see Theorem 2.1 in [5]) is a homeomorphism of X into R^m . It suffices to take as Y a superspace of X that is homeomorphic to R^m by an extension of the homeomorphism h . R^m has only m continuous real-valued functions; indeed, R^m has a dense subset of cardinality \aleph .

It follows from 4.2 that if $m = 2^n$ is infinite, then the discrete space X_m of cardinality m does not have (P_m) . On the other hand, $\text{def}_R X_m \leq m$ for "almost all" infinite cardinals; in particular, for all $m = 2^n$, where \aleph is Ulam non-measurable.

Consequently, the converse of 4.1 fails for all such cardinals. I do not know if 4.3 holds for infinite cardinals that are not of the form 2^n as well as if 4.2 fails for such cardinals. It appears that the answer to this question depends upon the assumed rules of exponentiation of cardinals. In particular, I do not know if 4.2 holds for the cardinal \aleph_0 . Let Q be the space of irrational numbers; we have $\text{def}_R Q = \aleph_0$; does Q have (P_{\aleph_0}) ? (Note that $\text{def}_R P > \aleph_0$, where P is the space of rational numbers; therefore P does not have (P_{\aleph_0}) .) We have $\text{def}_R R^{\aleph_0} = \aleph_0$; does R^{\aleph_0} have (P_{\aleph_0}) ?

One can discuss the above problems in a more general context. The property analogous to (P_m) but referring to functions with values in a space E will be denoted by $P_m(E)$; in the formulation of this property all spaces are assumed to be E -completely regular. F. Marin has pointed out to us that 4.1 holds true in this general context: if and E -compact space X has property $P_m(E)$, then $\text{def}_E X \leq m$. The study of property $P_m(E)$ for E -extremal spaces is more difficult. (An E -extremal space is an E -completely regular space X with the property: for every continuous function $f: X \rightarrow E$, $f[X]$ is compact.)

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