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A note concerning the fine topology on function spaces

by

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1. Introduction

In the following we will verify several properties of the fine topology on the set of functions which map from a topological space to a metric space (definitions in § 2). This topology was first introduced by Whitney [9] and still sometimes carries his name. It later appears (for example) in Morlet [6], Peixoto [8], Eells-McAlpin [3], and Elworthy [4]. These authors were faced with the problem of showing that an arbitrary function could be approximated in a special sense by a certain set of nice functions, i.e., that a particular set of functions was dense in the fine topology. Here, however, we are interested in the topological structure of the fine topology itself. The main result (Corollary 5.2) is that a function space with the fine topology is totally disconnected when the domain space is an infinite dimensional manifold — a fact which appears to discourage further interest in its structure. We also answer questions concerning metrizability, completeness, and (Baire) category. The terminology of [2] is followed; all maps are taken to be continuous.

2. The fine topology

The fine topology on the set of maps from the space X to the metric space Y is generated by the "tubes"

$$T(f, \varepsilon) = \{g : \rho(f(x), g(x)) < \varepsilon(x) \text{ for all } x \in X\}$$

where ε is any map $X \to$ reals > 0 and ρ is a fixed metric on Y. These tubes actually form a basis. We call the resulting space $C^0_{\text{fine}}(X, Y)$. If X is paracompact, the topology of this space turns out to be independent of the choice of metric on Y by the following.

[1] The results of this work form a chapter of the author's doctoral dissertation directed by J. Eells at Cornell University and supported in part by NSF Grants GP5882 and GP8413.

Proposition 1. *Let X be paracompact, then the fine topology is generated by the collection of sets*

$$\mathscr{A}((C_\alpha), (V_\beta), \varphi) = \{g : g(C_\alpha) \subset V_{\varphi\alpha} \text{ for all } \alpha\}$$

where (C_α) is a locally finite closed cover of X, (V_β) is an open cover of Y, and φ is a map of index sets $(\alpha) \to (\beta)$. (It turns out that the collection $(\mathscr{A})$ also forms a basis).

Proof. By way of notation let $B(y, r)$ be the ρ-ball of radius r at y in Y.

Suppose $f \in \mathscr{A}((C_\alpha), (V_\beta), \varphi)$ and $x \in X$; then x is contained in exactly $C_1, \cdots, C_N$, and so $f(x)$ is contained in $V_{\varphi 1}, \cdots, V_{\varphi N}$ and possibly more. We define the function

$$\delta(x) = \sup \{\delta : B(f(x), \delta) \subset V_{\varphi 1} \cap \cdots \cap V_{\varphi N}\}$$

and show that it is lower semi-continuous. Suppose $x_\nu \to x$ and $\delta(x_\nu) \leqq a$. Since (C_α) is a locally finite closed cover, we can find a neighborhood V of x which meets $C_1, \cdots, C_N$ and no other C_α. Assume $(x_\nu) \subset V$; then each x_ν is contained in some subcollection of $C_1, \cdots, C_N$ and no other C_α, and therefore

$$\delta(x) - \rho(f(x), f(x_\nu)) \leqq \delta(x_\nu) \leqq a.$$

And since $\rho(f(x), f(x_\nu)) \to 0$, we get $\delta(x) \leqq a$. Finally since δ is a positive lower semi-continuous function on a paracompact space X, there is a map $\varepsilon : X \to$ reals such that $0 < \varepsilon(x) < \delta(x)$ for all $x \in X$. Then clearly we have $T(f, \varepsilon) \subset \mathscr{A}((C_\alpha), (V_\beta), \varphi)$.

Now suppose we are given $T(f, \varepsilon)$. Let $V_{f(x)} = B(f(x), \varepsilon(x)/4)$ and let $V_x = f^{-1}(V_{f(x)}) \cap \{\bar{x} : \varepsilon(\bar{x}) > \varepsilon(x)/2\}$. Then (V_x) is an open cover of X which can be refined to a locally finite closed subcover (C_x). Now suppose $g \in \mathscr{A}((C_x), (V_{f(x)}), f)$. For any $\bar{x}$ there is some C_x which contains it. Since $f(C_x) \subset V_{f(x)}$ and $g(C_x) \subset V_{f(x)}$, we clearly have $\rho(f(\bar{x}), g(\bar{x})) < \varepsilon(x)/2 < \varepsilon(\bar{x})$. Therefore we get

$$f \in \mathscr{A}((C_x), (V_{f(x)}), f) \subset T(f, \varepsilon).$$

Remarks. We can define a topology, as above, on the set of maps between two spaces X and Y which reduces to the fine topology when X is paracompact and Y is a metric space.

We should further remark that the metrics

$$\rho_\varepsilon(f, g) = \sup_X \frac{\rho(f(x) g(x))}{\varepsilon(x)}$$

(which allow the value infinity) also generate the fine topology, and therefore $C^0_{\text{fine}}(X, Y)$ is a completely regular Hausdorff space with a uniform structure given by the collection (ρ_ε).

3. Metrizability

It is clear that, if X is compact, the fine topology reduces to the topology of uniform convergence and therefore has a metric induced from that of Y. If, however, X is not compact, the fine topology is not in general metrizable.

PROPOSITION 2. *If X is paracompact and noncompact and Y contains an arc, then $C^0_{\text{fine}}(X, Y)$ is not metrizable.*

PROOF. We will show that $C^0_{\text{fine}}(X, Y)$ is not even first countable.

First consider the special case of $C^0_{\text{fine}}(X, I)$ where I is the closed unit interval. Suppose that there is a countable base $T(z, \varepsilon_n)$ at the "zero" map $z(X) = 0 \in I$. The assumptions on X imply that it is not countably compact and therefore not sequentially compact (see Kelley [5], p. 162E, p. 171V), so there is a countable sequence $(x_n) \subset X$ having no cluster points. Define the map $\varepsilon : (x_n) \to (0, 1)$ by $\varepsilon(x_n) = \frac{1}{4}\varepsilon_n(x_n)$ (we can assume that all $\varepsilon_n < 1$). Since (x_n) is a closed subset of normal space X, we can apply Tietze's theorem to get an extension $\varepsilon : X \to (0, 1)$ (see Dugundji [1], p. 149). Then clearly we have $\frac{1}{2}\varepsilon_n \in T(z, \varepsilon_n)$ and $\frac{1}{2}\varepsilon_n \notin T(z, \varepsilon)$ for all n. Therefore we cannot have $T(z, \varepsilon_n) \subset T(z, \varepsilon)$ for any n; a contradiction.

Now since Y contains an arc, there is an embedding $\varphi : I \to Y$. We show that φ induces another embedding $\Phi: C^0_{\text{fine}}(X, I) \to C^0_{\text{fine}}(X, Y)$ where $\Phi(f) = \varphi \circ f$, and therefore $C^0_{\text{fine}}(X, I)$ cannot be first countable since it contains a subspace which is not first countable. It is easy to see that Φ maps one-one onto $C^0_{\text{fine}}(X, \varphi(I))$. To show Φ is continuous, let ε be given and let $U_n = \{x : \varepsilon(x) > (1/n)\}$. Since I is compact, for each n there is a $\delta_n > 0$ such that $|f(x) - g(x)| < \delta_n$ implies $\rho(\varphi f(x), \varphi g(x)) < (1/n) < \varepsilon(x)$ for all $x \in U_n$. We can assume all the δ_n are bounded, and therefore we can define the function

$$\bar{\delta}(x) = \sup_{\{n : x \in U_n\}} \delta_n.$$

It is easily seen that $\bar{\delta}$ is lower semicontinuous, so as before there is a map $\delta : X \to$ reals such that $0 < \delta(x) < \bar{\delta}(x)$ for all $x \in X$. Hence we have $\Phi(T(f, \delta)) \subset T(\Phi(f), \varepsilon)$. The continuity of

$$\Phi^{-1} : C^0_{\text{fine}}(X, \varphi(I)) \to C^0_{\text{fine}}(X, I)$$

follows similarly.

REMARK. Note that the argument above shows that if $\varphi : Y \to Z$ is uniformly continuous, then $\Phi : C^0_{\text{fine}}(X, Y) \to C^0_{\text{fine}}(X, Z)$ is continuous.

4. Completeness and category

PROPOSITION 3. *If Y is complete, then $C^0_{\text{fine}}(X, Y)$ is complete in its uniform structure.*

PROOF. It is clear that for our purposes (g_α) is a Cauchy net if for every $\varepsilon : X \to \text{reals} > 0$ there is an A_ε such that $g_\alpha \in Y(g_{A_\varepsilon}, \varepsilon)$ for $\alpha > A_\varepsilon$ and that $(g_\alpha) \to g$ if for any ε there is an A_ε such that $g_\alpha \in T(g, \varepsilon)$ for $\alpha > A_\varepsilon$. We show that any Cauchy net (g_α) converges to some $g \in C^0_{\text{fine}}(X, Y)$. Fixing $x \in X$ we have that

$$g_\alpha(x) \in B\big(g_{A_\varepsilon}(x), \varepsilon(x)\big)$$

for $\alpha > A_\varepsilon$. Since ε and therefore $\varepsilon(x)$ is arbitrary, this says that $(g_\alpha(x))$ is a Cauchy net in Y and hence converges to some $y \in Y$; define g by $g(x) = y$. Clearly $g_\alpha(x) \in B\big(g(x), 3\varepsilon(x)\big)$ and therefore $g_\alpha \in T(g, 3\varepsilon)$ for $\alpha > A_\varepsilon$, so we have $(g_\alpha) \to g$. We need only show that g is continuous, but this follows easily since the convergence $(g_\alpha) \to g$ is stronger than uniform convergence.

PROPOSITION 4. *If X is a k-space and Y is complete, then $C^0_{\text{fine}}(X, Y)$ is a Baire space* (See Peixoto [8], p. 225). X is a k-space if U is open in X iff $U \cap C$ is open in C for all compact C in X. First countable spaces are k-spaces, see Kelley [5], p. 230.)

PROOF. Let (θ_n) be a countable collection of open sets dense in $C^0_{\text{fine}}(X, Y)$. We show that $\cap\, \theta_n$ meets any given $T(f, \varepsilon)$ and is therefore dense. Since θ_1 is dense, there is an $f_1 \in T(f, \varepsilon)$ and an ε_1 such that $\bar{T}(f_1, \varepsilon_1) \subset T(f, \varepsilon) \cap \theta_1$ and $0 < \varepsilon_1(x) < \varepsilon(x)/2$ for all $x \in X$. Since θ_2 is dense, there is an $f_2 \in T(f_1, \varepsilon_1)$ and an ε_2 such that $\bar{T}(f_2, \varepsilon_2) \subset T(f_1, \varepsilon_1)$ and $0 < \varepsilon_2(x) < \varepsilon_1(x)/2$ for all $x \in X$. Continuing this process we get maps (f_n) and (ε_n) where

$$\bar{T}(f_n, \varepsilon_n) \subset T(f_{n-1}, \varepsilon_{n-1}) \subset \cdots \subset T(f, \varepsilon) \cap \theta_1 \cap \cdots \cap \theta_n$$

and $0 < \varepsilon_n(x) < \varepsilon(x)/2$ for all $x \in X$. Since

$$\rho\big(f_n(x), f_p(x)\big) < \varepsilon_n(x) < \varepsilon(x)/2^n \text{ for } p > n,$$

we have that $(f_n(x))$ is a Cauchy sequence and therefore $(f_n(x)) \to y$; define g by $g(x) = y$. Then since $\rho\big(f_n(x), g(x)\big) \leqq \varepsilon_n(x)$ for all n and $x \in X$, we have $g \in \bar{T}(f_n, \varepsilon_n) \subset \cdots \subset T(f, \varepsilon) \cap \theta_1 \cap \cdots \cap \theta_n$ for all n, and therefore $T(f, \varepsilon)$ and $\cap\, \theta_n$ meet at g.

We need only show that g is continuous. Take a compact C in X and let $\bar{\varepsilon} = \sup_{x \in C} \varepsilon(x)$, then we get $\rho(f_n(x), g(x)) < \bar{\varepsilon}/2^n$ for all $x \in C$. So on any fixed compact subset of X the convergence $(f_n) \to g$ is uniform and hence g is continuous there. And finally, since X is a k-space, g is therefore continuous everywhere.

5. Connectedness

Lemma. *If X is a metric space and the closure of $\{x : f(x) \neq g(x)\}$ is noncompact, then f and g lie in different components of $C^0_{\text{fine}}(X, Y)$.* (See Kelley [5] p. 107 V for a special case.)

Proof. By hypothesis there is a sequence $(x_n) \subset X$, with no cluster points, on which $f(x_n) \neq g(x_n)$ for all n. Define

$$\varepsilon(x_n) = (1/n)\, \rho\big(f(x_n), g(x_n)\big) > 0;$$

then $\varepsilon : (x_n) \to (0, \infty)$ is continuous. Since (x_n) is closed we can apply Tietze's theorem to get an extension $\varepsilon : X \to (0, \infty)$.

Define $Z = \{h : \exists$ a constant C_h where $\rho\big(h(x), f(x)\big) < C_h \in (x)$ for all $x \in X$; then clearly $f \in Z$. Furthermore $g \notin Z$ since otherwise $\rho(f(x_h), g(x_h)) < (1/n)\, C_g\, \rho(f(x_n), g(x_n))$; a contradiction for large n. We now show that Z separates f and g.

First we show Z is open. Let $h \in Z$ so that we have

$$\rho\big(h(x), f(x)\big) < C_h \varepsilon(x)$$

for all $x \in X$, and let

$$\delta(x) = C_h \varepsilon(x) - \rho\big(h(x), f(x)\big) > 0.$$

Then if $k \in T(h, \delta)$ we have $\rho\big(k(x), h(x)\big) < \delta(x)$. By an easy calculation $\rho\big(k(x), f(x)\big) < C_h \varepsilon(x)$ so $k \in Z$ and therefore $T(h, \delta) \subset Z$.

And finally Z is closed. Let $p \in \bar{Z}$, then there is a $q \in Z$ such that $\rho\big(p(x), q(x)\big) < \varepsilon(x)$ for all $x \in X$. But

$$\begin{aligned} \rho\big(p(x), f(x)\big) &\leqq \rho\big(p(x), q(x)\big) + \rho\big(q(x), f(x)\big) < \varepsilon(x) + C_q \varepsilon(x) \\ &= (1 + C_q)\varepsilon(x). \end{aligned}$$

Therefore we have $p \in Z$.

Corollary 1. *$C^0_{\text{fine}}\ (R, R)$ is not locally connected at any point.*

Proof. For any f and ε-tube around it, the map $f + \varepsilon/2$ is in this tube and is never equal to f.

Remark. Notice that even though any two maps f and $g : R \to R$ are homotopic, there may not be a path between them in $C^0_{\text{fine}}(R, R)$.

COROLLARY 2. *If X is a metric space which is not locally compact at any point* (for example, X could be an infinite dimensional manifold), *then $C^0_{\text{fine}}(X, Y)$ is totally disconnected.*

PROOF. Suppose f and g are not identical; then they differ on some open set and therefore on some noncompact set of X.

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