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1. Introduction and basic definitions

It has become an adopted technique in general topology to introduce more and more types of spaces of less and less interest. In this paper we submit ourselves precisely to this kind of criticism. We claim that—historically—the development of the notion of a topological space and its ramifications, e.g. Hausdorff spaces, has been—to some extent—arbitrary and we shall begin an investigation of this phenomenon. Let us make this clear by means of an example.

The union of an expanding sequence of finite discrete spaces can be topologized in several ways; firstly, by taking the open subsets of members of the sequence as an open base, secondly by taking the closed subsets of its members as a closed base. In the first case we obtain a discrete space, in the second case the so-called cofinite topology (proper closed subsets are finite; all subsets are compact). The discrete space is the strongest possible topology on a set; the cofinite topology is the weakest possible T_1-topology on this set. There seems to be no reason to favor one topology over the other, although from a "constructive point of view" the second approach seems to be the better one.

More generally, to practically every space of importance there corresponds—roughly speaking by interchanging compact and closed sets—an "antispace" which, conversely, determines the given space. This phenomenon of antispaces has been briefly discussed (without proofs) in [3]. In doing so it turns out to be necessary—and we think worthwhile for its own sake—to investigate again more closely than before, the notion of compactness and then—force majeure—the notion of a topological space. Indeed for the formalism we need to develop, it is essential to introduce these notions in a context slightly more general than usual.

Definition of a compact set and the operators ρ^n.

Let X be a set and $\mathfrak{F}$ a family of subsets of X. If $A \subset X$, then

A will be said to be *compact relative to* $\mathfrak{I}$, or equivalently $A \in \rho\mathfrak{I}$, provided that for every $\mathscr{L} \subset \mathfrak{I}$, for which $\mathscr{L} \cup \{A\}$ has the finite intersection property—(f.i.p)—the intersection of the collection $\mathscr{L} \cup \{A\}$ is non-empty. Observe that if $\mathfrak{I}$ is considered to be a subbase for the closed sets of a space (or in particular the family of all closed subsets of a space) then the collection of sets which are compact relative to $\mathfrak{I}$ coincides with the family of compact subsets of the space. $\rho\mathfrak{I}$ denotes the family of compact sets of X, relative to $\mathfrak{I}$. For $n > 1$, the family $\rho^n\mathfrak{I}$ is defined inductively

$$\rho^n\mathfrak{I} = \rho(\rho^{n-1}\mathfrak{I}).$$

Thus ρ is an operator—called the *compactness operator*—

$$\rho : 2^{2^X} \to 2^{2^X}.$$

The elements of $\rho^2\mathfrak{I}$ and $\rho^3\mathfrak{I}$ are called *squarecompact* and *cube-compact* respectively. One should not forget that—although the family of squarecompact sets has remarkable properties—the notion "squarecompact" is a "relative" one (just as is closed, for example) whereas compactness is an "absolute" notion.

Definition of a minusspace

The fact that we deal primarily with closed rather than open sets will be reflected in our notation. For example, if $\mathfrak{B}$ is the family of *all closed* subsets of a topological space on X, then the topological space will be denoted by $(X, \mathfrak{B})$. In fact, we find it useful to use the notation $(X, \mathfrak{B})$ to designate something slightly more general, namely that $\mathfrak{B} \subset 2^X$, $\mathfrak{B}$ covers X, and $\mathfrak{B}$ is closed under the formation of finite unions and arbitrary intersections[1]. Under these conditions we call $(X, \mathfrak{B})$ a minusspace. Thus a minus-space $(X, \mathfrak{B})$ is a topological space if and only if $X \in \mathfrak{B}$. Throughout the paper "space" will mean minusspace in contradistinction to topological space. Definitions of usual topological notions can be extended to (minus)spaces; e.g. B is closed in $(X, \mathfrak{B})$ provided that $B \in \mathfrak{B}$, whereas U is open iff $X - U \in \mathfrak{B}$, and A is dense iff $U \cap A \neq \emptyset$ for each nonempty open set U. Note that most spaces usually studied—e.g. non-degenerate Hausdorff spaces—contain at least one pair of disjoint non-empty open sets and so are automatically topological. The class of (minus)spaces which are not

[1] In this connection, the intersection of the empty collection has no meaning; specifically we do *not* use the convention that $\cap\, \emptyset = X$.

topological is characterized by the property that for them every open set is connected. For this reason, they are called *superconnected*. They appear to be "as important" as topological spaces and are briefly discussed in section 5.

Definition of the operator γ

If $\mathfrak{S}$ is a collection of subsets of X, we let $\gamma\,\mathfrak{S}$ denote all arbitrary intersections of finite unions of members of $\mathfrak{S}$. Thus γ is also an operator

$$\gamma : 2^{2^X} \to 2^{2^X}.$$

Since the statement that $(X, \mathfrak{B})$ is a (minus)space is precisely the statement that $\mathfrak{B}$ covers X and $\mathfrak{B} = \gamma\mathfrak{B}$, γ is called the *space-generating operator.*

We can multiply operators like γ and ρ^n by taking the composites of the corresponding functions. E.g. the relation $\rho\gamma\,\mathfrak{S} = \rho\,\mathfrak{S}$ or briefly $\rho\gamma = \rho$ is just a brief notation for Alexander's subbase theorem, as the reader should make clear to himself.

The first fact to be learned about ρ^2 has been the relation $\gamma\rho^2 = \rho^2$, that is: arbitrary intersections of finite unions of square compact sets are again squarecompact (cf. [3], [4] and section 2). So the squarecompact subsets of any space determine the closed subsets of a (minus)space. The second fact (discovered by the third author) establishes the formula $\rho^4 = \rho^2$ (cf [4]), so at most the operators ρ, ρ^2 (squarecompact), and ρ^3 (cubecompact) can be different. Moreover, if one starts with a topological space in which compact sets are closed (e.g. a Hausdorff space), then already $\rho = \rho^3$.

In general, the second and third sections are devoted to the systematic study of the relations between the elements of the semigroup generated by γ and ρ, and their topological realizations. In order to do this, we first derive other formulas, like $\rho \subset \rho^3$ (that is $\rho\,\mathfrak{S} \subset \rho^3\,\mathfrak{S}$, for all $\mathfrak{S}$). This leads to a lattice which is studied in section 4 together with its topological realizations.

Much effort has been spent on these sections and the corresponding existence problems treated in section 6. (The more sophisticated examples in this section have been found by the second author.)

The techniques involved represent a main justification for this paper. We mention another paper [10], in which these techniques are applied in order to prove the relation

$$\rho\,\mathfrak{S} = \rho\gamma\,(\mathfrak{S} \cup \rho^2\,\mathfrak{S})$$

that is: if X is compact relative to $\mathfrak{S}$ it is also compact relative to $\gamma(\mathfrak{S} \cup \rho^2\mathfrak{S})$, thus strengthening Alexander's subbase theorem substantially. Unsolved and hardly seriously attacked remains the question as to whether the notion of compactness might be axiomatized by these methods (see section 7).

We now return to the topic which first motivated our research concerning compactness.

Two minusspaces are said to be *antispaces* (of each other) and thus constitute an antipair, provided that either can be obtained by interchanging the collections of closed subsets and compact subsets of the other.

So each antispace of an antipair determines the other uniquely. Antispaces abound in topology: metrizable and locally compact Hausdorff spaces are antispaces. Each Hausdorff space $(X, \mathfrak{B})$ is such that $(X, \rho\mathfrak{B})$ is an antispace; moreover, if $(X, \mathfrak{B})$ is an arbitrary space, $(X, \rho^2\mathfrak{B})$ is an antispace. In particular a topological space is an antispace iff the closed sets coincide with the square-compact sets. (For further details, see the first part of section 5.) Moreover, we observe that the importance of the category of topological antispaces is supported by a paper of Steenrod [9], in which he looks for a convenient category of spaces in algebraic topology. He shows that the class $\mathscr{K}$ of compactly generated spaces, i.e. Hausdorff k-spaces, is such a category. However, one has to redefine the notions of topological product and of subspace (actually these appear as the k-expansions of the ordinary product and subspace, cf. Arhangel'skiĭ [1]). In the context of this paper these new definitions become natural. Indeed $\mathscr{K}$ is a family of topological antispaces and Steenrod's definitions of product and subspace correspond—by means of a categorical isomorphism—to the usual definitions if taken in the anticlass $\mathscr{K}^*$. More fundamentally, one might wonder why topological spaces—in this context—are introduced at all. Indeed, start with a set X, and a family of subsets $\mathfrak{S}$. Now define the space $(X, \rho^2\mathfrak{S})$, in which the closed sets are defined as the squarecompact sets relative to $\mathfrak{S}$. $(X, \rho^2\mathfrak{S})$ is an antispace and the additional Hausdorff axiom (or even some weaker axiom) gives us Steenrod's category which is a convenient class of topological antispaces, defined in a simple and natural way.

Section 7 contains some unsolved problems and we also mention here an elegant formula derived by P. Bacon. A forthcoming thesis by the fourth author contains more results on antispaces and the compactness operator (published University of Amsterdam, 1968).

2. Fundamental relations

In this section several fundamental relations concerning the operators ρ and γ are established. Among them are the following theorems:

THEOREM I. *The square compact subsets of any space form a minus topology, i.e., arbitrary intersections of finite unions of square compact sets are square compact.*

THEOREM II. *A subset of a space is square compact if and only if it is square compact with respect to the space generated by the square compact sets.* Thus any set X together with the square compact subsets of any topology on X forms an antispace (cf. Theorem V).

These theorems are restated below as relations (8) and (10).

Note that the relations (1) and (2) are immediate consequences of the definitions of γ and ρ. Also, (3) is merely a restatement of the well-known *Alexander's subbase theorem*; i.e., the sets which are compact relative to any subbase for a space are precisely the sets which are compact relative to the space itself. A strengthening of this relation and related results appear in [10].

Throughout the paper we will assume that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, and $\mathfrak{I}$ are subsets of 2^X. Furthermore we define the "cap" operator $\wedge$ as follows

$$\mathfrak{C} \wedge \mathfrak{D} = \{C \cap D \mid C \in \mathfrak{C}, D \in \mathfrak{D}\}$$

$$\underset{1}{\wedge}\, \mathfrak{A} = \mathfrak{A}$$

$$\underset{n}{\wedge}\, \mathfrak{A} = (\underset{n-1}{\wedge}\, \mathfrak{A}) \wedge \mathfrak{A} \text{ for } n > 1$$

$$\wedge\, \mathfrak{A} = \cup \{\underset{n}{\wedge}\, \mathfrak{A} \mid n = 1, 2, 3, \cdots\}.$$

(1) $\boxed{\gamma^2 = \gamma}$.

(2) If $\mathfrak{E} \subset \mathfrak{I}$, then $\rho\mathfrak{I} \subset \rho\mathfrak{E}$.

(3) $\boxed{\rho\gamma = \rho}$.

(4) If $\emptyset \neq \mathfrak{C} \subset \gamma\mathfrak{I} \cap \rho\mathfrak{I}$ and if $\mathfrak{C}$ has f.i.p., then $\cap\, \mathfrak{C} \neq \emptyset$.

(5) $\gamma\mathfrak{I} \wedge \rho^n\mathfrak{I} \subset \rho^n\mathfrak{I}$, $n \geqq 1$.

(6) $\rho^n\mathfrak{I} \wedge \rho^{n+1}\mathfrak{I} = \rho^n\mathfrak{I} \cap \rho^{n+1}\mathfrak{I}$, $n \geqq 1$.

(7) If $\mathfrak{I} \wedge \rho\mathfrak{I} \subset \mathfrak{I}$, then $\mathfrak{I} \subset \rho^2\mathfrak{I}$ and $\gamma\rho\mathfrak{I} = \rho\mathfrak{I}$.

(8) $\boxed{\gamma\rho^2 = \rho^2}$.

(9) $\boxed{\rho\mathfrak{F} \subset \rho^3\mathfrak{F}}$. In fact $\rho\mathfrak{F} \subset \rho\mathfrak{F} \wedge \rho\mathfrak{F} \subset \underset{3}{\wedge}\rho\mathfrak{F} \subset \cdots \subset \underset{n}{\wedge}\rho\mathfrak{F} \subset \cdots$
$\subset \wedge\,\rho\mathfrak{F} \subset \gamma\rho\mathfrak{F} \subset \rho\mathfrak{F} \wedge \rho^3\mathfrak{F} \subset \rho^3\mathfrak{F}$.

(10) $\boxed{\rho^2 = \rho^4}$.

Proof of (4). Let $C \in \mathfrak{C}$; by (3), $C \in \rho(\gamma\mathfrak{F})$. Thus since $\mathfrak{C} \subset \gamma\mathfrak{F}$ and $\mathfrak{C} \cup \{C\}$ has f.i.p., $\cap\,\mathfrak{C} = (\cap\,\mathfrak{C}) \cap C \neq \emptyset$.

Proof of (5). If $n = 1$, this is merely a restatement of the well-known fact that the intersection of a closed set and a compact set is again compact. To complete the proof by induction, assume that $\gamma\mathfrak{F} \wedge \rho^n\mathfrak{F} \subset \rho^n\mathfrak{F}$ and that $F \in \gamma\mathfrak{F}$, $C \in \rho^{n+1}\mathfrak{F}$ and $\emptyset \neq \mathfrak{C} \subset \rho^n\mathfrak{F}$ such that $\mathfrak{C} \cup \{F \cap C\}$ has f.i.p. Clearly $\mathfrak{C}' = \{F\} \wedge \mathfrak{C} \subset \gamma\mathfrak{F} \wedge \rho^n\mathfrak{F} \subset \rho^n\mathfrak{F}$ and $\mathfrak{C}' \cup \{C\}$ has f.i.p. Thus by the definition of compactness, $(\cap\,\mathfrak{C}) \cap (F \cap C) = (\cap\,\mathfrak{C}') \cap C \neq \emptyset$; and so $\gamma\mathfrak{F} \wedge \rho^{n+1}\mathfrak{F} \subset \rho^{n+1}\mathfrak{F}$.

Proof of (6). By the inductive definition of ρ^n, we need only prove (6) for $n = 1$. Clearly $\rho\mathfrak{F} \cap \rho^2\mathfrak{F} \subset \rho\mathfrak{F} \wedge \rho^2\mathfrak{F}$, and by applying the statement of (5) with $n = 1$ to $\rho\mathfrak{F}$ we have

$$\rho\mathfrak{F} \wedge \rho^2\mathfrak{F} \subset \gamma\rho\mathfrak{F} \wedge \rho^2\mathfrak{F} \subset \rho^2\mathfrak{F}.$$

Thus it remains to be shown that $\rho\mathfrak{F} \wedge \rho^2\mathfrak{F} \subset \rho\mathfrak{F}$. Let $C \in \rho\mathfrak{F}$, $D \in \rho^2\mathfrak{F}$ and $\emptyset \neq \mathfrak{A} \subset \mathfrak{F}$ be such that $\mathfrak{A} \cup \{C \cap D\}$ has f.i.p. Now $\mathfrak{A}' = \mathfrak{A} \wedge \{C\} \subset \gamma\mathfrak{F} \wedge \rho\mathfrak{F}$; so that by (5), $\mathfrak{A}' \subset \rho\mathfrak{F}$. Also $\mathfrak{A}' \cup \{D\}$ has f.i.p. so that by the definition of ρ,

$$(\cap\,\mathfrak{A}) \cap (C \cap D) = (\cap\,\mathfrak{A}') \cap D \neq \emptyset.$$

Therefore $C \cap D \in \rho\mathfrak{F}$.

Proof of (7). Suppose that $F \in \mathfrak{F}$ and $\emptyset \neq \mathfrak{C} \subset \rho\mathfrak{F}$ such that $\mathfrak{C} \cup \{F\}$ has f.i.p. By hypothesis $\{F\} \wedge \mathfrak{C} \subset \mathfrak{F}$, and by (5) $\{F\} \wedge \mathfrak{C} \subset \rho\mathfrak{F}$. Thus $\{F\} \wedge \mathfrak{C} \subset \mathfrak{F} \cap \rho\mathfrak{F}$, so that by (4)

$$\{F\} \wedge \mathfrak{C} = (\cap\,\mathfrak{C}) \cap F \neq \emptyset.$$

Hence $\mathfrak{F} \subset \rho^2\mathfrak{F}$. Clearly $\rho\mathfrak{F} \subset \gamma\rho\mathfrak{F}$. To prove the reverse inclusion we need only show that $\rho\mathfrak{F}$ is closed under the formation of arbitrary intersections. Suppose that $\emptyset \neq \mathfrak{C} \subset \rho\mathfrak{F}$ and $\emptyset \neq \mathfrak{D} \subset \mathfrak{F}$ such that $\mathfrak{D} \cup (\cap\,\mathfrak{C})$ has f.i.p. By hypothesis and (5),

$$\mathfrak{D} \wedge \mathfrak{C} \subset \mathfrak{F} \wedge \rho\mathfrak{F} \subset \mathfrak{F} \cap \rho\mathfrak{F}.$$

Thus by (4) $(\cap\,\mathfrak{D}) \cap (\cap\,\mathfrak{C}) = \cap\,(\mathfrak{D} \wedge \mathfrak{C}) \neq \emptyset$; so $\cap\,\mathfrak{C} \in \rho\mathfrak{F}$. Consequently $\rho\mathfrak{F} = \gamma\rho\mathfrak{F}$.

PROOF OF (8). By (6), for any $\mathfrak{F}$, $\rho\mathfrak{F} \wedge \rho^2\mathfrak{F} = \rho\mathfrak{F} \cap \rho^2\mathfrak{F} \subset \rho\mathfrak{F}$. Substituting $\rho\mathfrak{F}$ for $\mathfrak{F}$ in (7), we obtain $\gamma\rho^2\mathfrak{F} = \rho^2\mathfrak{F}$.

PROOF OF (9). All the inclusions except the last two are obvious from the definitions, and the last inclusion is an immediate consequence of (5). To show that $\gamma\rho\mathfrak{F} \subset \rho\mathfrak{F} \wedge \rho^3\mathfrak{F}$ assume that $C \in \gamma\rho\mathfrak{F}$ then $C = C_1 \cap (\cap \mathfrak{C})$, where $C_1 \in \rho\mathfrak{F}$ and $\emptyset \neq \mathfrak{C} \subset \rho\mathfrak{F}$. If $\emptyset \neq \mathfrak{D} \subset \rho^2\mathfrak{F}$ such that $\mathfrak{D} \cup \{\cap \mathfrak{C}\}$ has f.i.p., then by (6),

$$\mathfrak{D} \wedge \mathfrak{C} \subset \rho\mathfrak{F} \cap \rho^2\mathfrak{F}.$$

$\mathfrak{D} \wedge \mathfrak{C}$ has f.i.p., so that by (4), $\cap (\mathfrak{D} \wedge \mathfrak{C}) = (\cap \mathfrak{D}) \cap (\cap \mathfrak{C}) \neq \emptyset$. Thus $\cap \mathfrak{C} \in \rho^3\mathfrak{F}$ and $C \in \rho\mathfrak{F} \wedge \rho^3\mathfrak{F}$.

REMARK. None of the inclusions of (9) is reversible. Examples 5 and 8 show that the first and the last two inclusions may be strict. The example in [5] shows that all the others may simultaneously be strict.

PROOF OF (10). By (9), $\rho\mathfrak{F} \subset \rho^3\mathfrak{F}$ for all $\mathfrak{F} \subset 2^X$. Thus by (2) $\rho^4\mathfrak{F} \subset \rho^2\mathfrak{F}$. Also applying (9) to $\rho\mathfrak{F}$, we have $\rho(\rho\mathfrak{F}) \subset \rho^3(\rho\mathfrak{F})$, or $\rho^2\mathfrak{F} \subset \rho^4\mathfrak{F}$. Hence $\rho^2 = \rho^4$.

3. The semigroup generated by ρ and γ

The relations (1), (3), (8) and (10) of § 2 when taken by themselves determine a five element non-commutative semigroup with the following multiplication table:

Table 1

	γ	ρ	ρ^2	ρ^3	$\gamma\rho$
γ	γ	$\gamma\rho$	ρ^2	ρ^3	$\gamma\rho$
ρ	ρ	ρ^2	ρ^3	ρ^2	ρ^2
ρ^2	ρ^2	ρ^3	ρ^2	ρ^3	ρ^3
ρ^3	ρ^3	ρ^2	ρ^3	ρ^2	ρ^2
$\gamma\rho$	$\gamma\rho$	ρ^2	ρ^3	ρ^2	ρ^2

One may wonder whether or not this semigroup might lead to an axiomatic characterization of the notion of compactness; e.g., if $\rho : 2^{2^X} \to 2^{2^X}$ is an operator such that the semigroup generated by ρ and the space generating operator γ (with binary operation functional composition) is a semigroup of five distinct[2] elements

[2] Example 8 shows that for the compactness operator the five elements are distinct.

having the multiplication of table 1, then what additional conditions must be imposed to guarantee that ρ be the compactness operator?

§ 7 contains other questions concerning the characterization of ρ.

Next, in order to obtain a classification of all topological spaces, we determine all of the possible congruences of the semigroup with the multiplication of Table 1. In each case, the equalities listed are the only non-trivial equalities existing among the semigroup elements.

(E_1) $\gamma = \rho = \rho^2 = \rho^3 = \gamma\rho$.

(E_2) $\rho = \rho^2 = \rho^3 = \gamma\rho$.

(E_3) $\gamma = \rho^2$; $\rho = \rho^3 = \gamma\rho$.

(E_4) $\rho = \rho^3 = \gamma\rho$.

(E_5) $\rho^3 = \gamma\rho$.

(E_6) $\rho = \gamma\rho$.

(E_7) no equalities.

(E_8) $\rho^2 = \rho^3$; $\rho = \gamma\rho$.

(E_9) $\rho^2 = \rho^3 = \gamma\rho$.

(E_{10}) $\rho^2 = \rho^3$.

If we return to the concrete semigroup where ρ and γ are the compactness and space generating operators, then we can show that the congruences E_8, E_9, and E_{10} do not occur. First we establish two more propositions.

(11) $\rho \cap \rho^2 = \rho^2 \cap \rho^3$.

(12) $(\rho^3 \subset \rho^2 \cup \rho) \Rightarrow (\rho = \rho^3)$.

Our notational convention will be such that (11) means that for every $\mathfrak{F} \subset 2^X$, $\rho\mathfrak{F} \cap \rho^2\mathfrak{F} = \rho^2\mathfrak{F} \cap \rho^3\mathfrak{F}$; and (12) means that if $\mathfrak{F} \subset 2^X$ and $\rho^3\mathfrak{F} \subset \rho^2\mathfrak{F} \cup \rho\mathfrak{F}$, then $\rho\mathfrak{F} = \rho^3\mathfrak{F}$.

Proof of (11). Clearly since by (9), $\rho \subset \rho^3$, we have

$$\rho \cap \rho^2 \subset \rho^3 \cap \rho^2 \subset \rho^2.$$

Now if $C \in \rho^3\mathfrak{F} \cap \rho^2\mathfrak{F}$ for some $\mathfrak{F} \subset 2^X$ and if $\emptyset \neq \mathfrak{D} \subset \mathfrak{F}$ such that $\mathfrak{D} \cup \{C\}$ has f.i.p., then by (5) $\mathfrak{D} \wedge \{C\} \subset \rho^2\mathfrak{F} \cap \rho^3\mathfrak{F}$ and $\mathfrak{D} \wedge \{C\}$ has f.i.p. Thus by (4) $(\cap \mathfrak{D}) \cap C \neq \emptyset$. Hence $\rho^3 \cap \rho^2 \subset \rho$; so that (11) is established.

Proof of (12). Suppose that $\rho^3 \mathfrak{I} \subset \rho^2 \mathfrak{I} \cup \rho \mathfrak{I}$. If $A \in \rho^3 \mathfrak{I}$, then $A \in \rho \mathfrak{I}$ or $A \in \rho^2 \mathfrak{I}$. If $A \in \rho^2 \mathfrak{I}$, then $A \in \rho^2 \mathfrak{I} \cap \rho^3 \mathfrak{I}$ which by (11) is $\rho \mathfrak{I} \cap \rho^2 \mathfrak{I}$. Hence in either case $A \in \rho \mathfrak{I}$, so that $\rho^3 \mathfrak{I} \subset \rho \mathfrak{I}$. This combined with (9) completes the proof.

The above analysis leads to the following classification of all topological spaces according to compactness criteria.

Theorem 3. *Every space belongs to precisely one of the classes E_1—E_7. (The class E_6 may be empty.)*

Proof. It is clear that the classes E_1—E_{10} are disjoint and that each space must belong to one of them since they exhaust the possibilities for congruences. By (12) it is evident that the classes E_8, E_9, and E_{10} are empty since if $\rho^3 = \rho^2$, then $\rho = \rho^3 = \rho^2$. Thus any space for which $\rho^3 = \rho^2$ holds is an E_1 space or an E_2 space.

Regarding the relative "strengths" of the E-type classifications, we have the following lattice

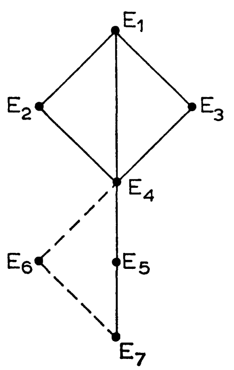


Fig. 1.

Dotted lines are shown from E_6 since we doubt that spaces satisfying this condition exist.

Following Smythe and Wilkins [7] we call the E_1 spaces "maximal compact". These are precisely those spaces for which the operator ρ is the identity. Clearly every compact Hausdorff space belongs to this class. Conversely each such space is compact T_1 but is not necessarily Hausdorff; cf. example 4. A space belongs to the class E_2 provided that it is not maximal compact, but one application of the operator ρ yields a maximal compact space. Thus these spaces may be called E_1-generative. Example 10 is such a space. The E_3 spaces are precisely the antispaces which are not maximal compact. Examples 1 and 2 or indeed all Hausdorff k-spaces (cf. § 5) are members of this class. E_4 spaces are those which are not E_3 but which are E_3-generative in the above sense. Example 6 is an E_4 space. A space is E_5 if an application of ρ does not give a space, but an application of $\gamma\rho$ yields an E_3 antispace. Example 5 is such a space. A space is E_7 if an application of ρ does not yield a space and an application of $\gamma\rho$ does not yield an antispace. Example 8 is an E_7 space. Thus we have the following:

PROPOSITION 1. *There exist spaces for which no two of the semigroup elements*: γ, ρ, ρ^2, ρ^3, *and* $\gamma\rho$ *are the same.*

4. The smallest lattice containing ρ and closed under the ρ operator

We have seen in § 3 that for the concrete semigroup under investigation, the relations among set-theoretically induced intersections and containments of the semigroup elements is of importance (since for one thing they determine the non-existence of certain congruence classes of the abstract semigroup). In this section we determine the smallest lattice (with order induced by containment) that contains the element ρ and is closed under the application of the ρ operator. To do this we need three more relations.

(13) $\rho(\rho^2 \cup \rho^3) = \rho(\rho^2 \cup \rho) = \rho^2 \cap \rho^3$.

(14) $\rho(\rho^2 \cap \rho) \supset \rho^3 \cup \rho^2$.

(15) $\rho^2(\rho^2 \cap \rho) = \rho^2 \cap \rho$.

PROOF OF (13). Since by (9) $\rho^3 \supset \rho$, it follows immediately from (2) that

$$\rho(\rho^2 \cup \rho^3) \subset \rho(\rho^2 \cup \rho) \subset \rho\rho^2 \cap \rho\rho = \rho^3 \cap \rho^2.$$

Therefore it remains to be shown that $\rho^2 \cap \rho^3$ is contained in $\rho(\rho^2 \cup \rho^3)$. Let $\mathfrak{S} \subset 2^X$, $A \in \rho^2\mathfrak{S} \cap \rho^3\mathfrak{S}$, and $\emptyset \neq \mathfrak{D} \subset \rho^2\mathfrak{S} \cup \rho^3\mathfrak{S}$ such that $\mathfrak{D} \cup \{A\}$ has f.i.p. By (8) $\mathfrak{D}' = \{A\} \wedge \mathfrak{D}$ is a subfamily of $\rho^2\mathfrak{S} \wedge \rho^3\mathfrak{S}$, which by (6) is the same as $\rho^2\mathfrak{S} \cap \rho^3\mathfrak{S}$. Clearly $\mathfrak{D}'$ has f.i.p., so from (4) it follows that $\cap\, \mathfrak{D}' = (\cap\, \mathfrak{D}) \cap A \neq \emptyset$. Thus $A \in \rho(\rho^2\mathfrak{S} \cup \rho^3\mathfrak{S})$.

The proof of (14) is immediate from (2).

PROOF OF (15). By (11) and (13) $\rho^2 \cap \rho = \rho^2 \cap \rho^3 = \rho(\rho^2 \cup \rho^3)$. But by (2) and (14) $\rho(\rho^2 \cup \rho^3) \supset \rho^2(\rho^2 \cap \rho)$. To show the reverse containment, note that by (2) $\rho(\rho^2 \cup \rho^3) \subset \rho^3 \cup \rho^4$. By (10)

$\rho^4 = \rho^2$, so that by another application of (2) we have

$$\rho^2(\rho^2 \cup \rho^3) \supset \rho(\rho^3 \cup \rho^2).$$

However, as before, by (13) and (11) $\rho(\rho^3 \cup \rho^2) = \rho^2 \cap \rho$.

The relations (9), (11), and (14) determine the following lattice:

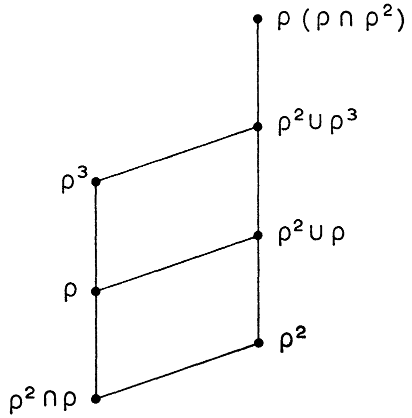


Fig. 2.

By (10), (13), and (15) the lattice is clearly closed under applications of the operator ρ.

Remark. Note that the lattice considerations above show that every $\mathfrak{J} \subset 2^X$ determines in general two pairs of anti-spaces $[(X, \rho^2\mathfrak{J}); (X, \rho^3\mathfrak{J})]$ and $[(X, \rho\mathfrak{J} \cap \rho^2\mathfrak{J}); (X, \rho(\rho\mathfrak{J} \cap \rho^2\mathfrak{J}))]$. Of course it may be true that these pairs are the same. This happens for precisely those spaces for which $\rho\mathfrak{J}$ and $\rho^2\mathfrak{J}$ are comparable, i.e., $\rho\mathfrak{J} \subset \rho^2\mathfrak{J}$ or $\rho^2\mathfrak{J} \subset \rho\mathfrak{J}$; cf. (16) below. We will show later (theorem 13) that $(X, \rho(\rho\mathfrak{J} \cap \rho^2\mathfrak{J}))$ is not only always an anti-space but it is also always a topological space and the other member of the antipair, $(X, \rho\mathfrak{J} \cap \rho^2\mathfrak{J})$, is always compact.

After the next proposition, we will be ready to establish all of the possible congruence relations (i.e., collapsings) of the lattice.

(16) The following are equivalent:

(a) $\rho(\rho^2 \cap \rho^3) \subset \rho^2 \cup \rho^3$,

(b) ρ^2 and ρ^3 are comparable,

(c) ρ and ρ^2 are comparable.

Proof. (a) $\rightarrow$ (b).

If (a) $\not\rightarrow$ (b), then there exists some $\mathfrak{J} \subset 2^X$ such that

$$\rho(\rho^2\mathfrak{J} \cap \rho^3\mathfrak{J}) \subset \rho^2\mathfrak{J} \cup \rho^3\mathfrak{J},$$

but $\rho^2\mathfrak{I} \not\subset \rho^3\mathfrak{I}$ and $\rho^3\mathfrak{I} \not\subset \rho^2\mathfrak{I}$. Let $A \in \rho^2\mathfrak{I}-\rho^3\mathfrak{I}$ and $B \in \rho^3\mathfrak{I}-\rho^2\mathfrak{I}$. By the definition of ρ there exists $\mathfrak{D}'$ and $\mathfrak{D}''$ such that $\emptyset \neq \mathfrak{D}' \subset \rho\mathfrak{I}$, $\emptyset \neq \mathfrak{D}'' \subset \rho^2\mathfrak{I}$, $\mathfrak{D}' \cup \{B\}$ has f.i.p. but $(\cap\, \mathfrak{D}') \cap B = \emptyset$, and $\mathfrak{D}'' \cup \{A\}$ has f.i.p. but $(\cap\, \mathfrak{D}'') \cap A = \emptyset$.

By (9), $\cap\, \mathfrak{D}' \in \gamma\rho\mathfrak{I} \subset \rho^3\mathfrak{I}$ so that $(\cap\, \mathfrak{D}') \cap A \in \rho^3\mathfrak{I} \wedge \rho^2\mathfrak{I}$. However, by (6), $\rho^3\mathfrak{I} \wedge \rho^2\mathfrak{I} = \rho^3\mathfrak{I} \cap \rho^2\mathfrak{I} \subset \rho^3\mathfrak{I}$. Thus

$$(\cap\, \mathfrak{D}') \cap A \subset \rho^3\mathfrak{I},$$

so that $(\cap\, \mathfrak{D}' \cap A) \cap (\cap\, \mathfrak{D}'') = \emptyset$ implies that there exists some finite collection $\mathfrak{A}'' \subset \mathfrak{D}''$ such that $(\cap\, \mathfrak{D}' \cap A) \cap (\cap\, \mathfrak{A}'') = \emptyset$.

Likewise by (8), $\cap\, \mathfrak{D}'' \subset \gamma\rho^2\mathfrak{I} = \rho^2\mathfrak{I}$, so that

$$(\cap\, \mathfrak{D}'') \cap B \in \rho^2\mathfrak{I} \wedge \rho^3\mathfrak{I} = \rho^2\mathfrak{I} \cap \rho^3\mathfrak{I} \subset \rho^2\mathfrak{I}.$$

Thus $(\cap\, \mathfrak{D}'' \cap B) \cap (\cap\, \mathfrak{D}') = \emptyset$ implies that there exists some finite collection $\mathfrak{A}' \subset \mathfrak{D}'$ such that $(\cap\, \mathfrak{D}'' \cap B) \cap (\cap\, \mathfrak{A}') = \emptyset$.

Let $E = (\cap\, \mathfrak{A}'' \cap A) \cup (\cap\, \mathfrak{A}' \cap B)$. Clearly by (8), $\cap\, \mathfrak{A}'' \cap A \in \rho^2\mathfrak{I}$ and by (8) and (9) $\cap\, \mathfrak{A}' \cap B \in \rho^3\mathfrak{I}$. Thus since $\rho^2 \subset \rho(\rho \cap \rho^2)$ and $\rho^3 \subset \rho(\rho \cap \rho^2)$ it can easily be shown that $E \in \rho(\rho\mathfrak{I} \cap \rho^2\mathfrak{I})$ which by (a) is contained in $\rho^2\mathfrak{I} \cup \rho^3\mathfrak{I}$. However $\mathfrak{D}' \cup \{E\}$ has f.i.p. and $\cap\, \mathfrak{D}' \cap E = \emptyset$, so that $E \notin \rho^2\mathfrak{I}$. Likewise $\mathfrak{D}'' \cup \{E\}$ has f.i.p. and $\cap\, \mathfrak{D}'' \cap E = \emptyset$, so that $E \notin \rho^3\mathfrak{I}$, which is a contradiction.

(b) $\rightarrow$ (c).

If $\rho^3 \subset \rho^2$, we have by (9) that $\rho \subset \rho^2$. On the other hand $\rho^2 \subset \rho^3$ implies that $\rho^2 = \rho^2 \cap \rho^3$, so that by (11) $\rho^2 = \rho^2 \cap \rho$ which shows that $\rho^2 \subset \rho$. Thus in either case ρ and ρ^2 are comparable.

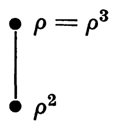
(c) $\rightarrow$ (a).

If $\rho^2 \subset \rho$ then by (9), $\rho^2 \subset \rho^3$ so that $\rho(\rho^2 \cap \rho^3) = \rho(\rho^2) = \rho^3 = \rho^2 \cup \rho^3$. If $\rho \subset \rho^2$ then by (2) $\rho^3 \subset \rho^2$ so that $\rho(\rho^2 \cap \rho^3) = \rho(\rho^3) = \rho^4$. But by (10) $\rho^4 = \rho^2 = \rho^2 \cup \rho^3$.

We now outline all of the possible ways in which the lattice under consideration can collapse and give an example for each case. Each class consists of those spaces for which only the given defining relation (and those which follow from it) hold.

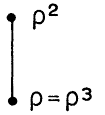
(H_1) $\rho = \rho^2$; the lattice consists of only one element; examples 3, 4, and 10.

(H_2) $\rho = \rho^2 \cup \rho^3$



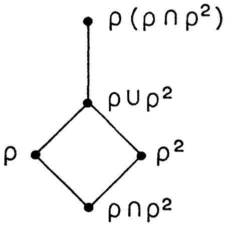
; $(X, \rho\mathfrak{I})$ of example 1.

(H_3) $\rho = \rho \cap \rho^2$



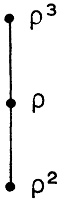
; $(X, \mathfrak{I})$ of example 1, or example 2.

(H_4) $\rho = \rho^3$



; example 7.

(H_5) $\rho^2 = \rho \cap \rho^2$



; example 8.

(H_6) no collapse; example 11.

THEOREM 4. *Every space belongs to precisely one of the classes H_1—H_6.*

PROOF. It is clear that the classes H_1—H_6 are disjoint. We must show that the cases H_1—H_6 exhaust all possibilities for spaces. Clearly if ρ and ρ^2 are not comparable, H_4 and H_6 are the only possibilities. Suppose that $\rho^2 \subset \rho$. Then by (9) $\rho^2 \subset \rho \subset \rho^3$, so that $\rho^2 \cap \rho = \rho^2$, $\rho^2 \cup \rho = \rho$, $\rho^2 \cup \rho^3 = \rho^3$ and by (16)

$$\rho(\rho \cap \rho^2) = \rho^2 \cup \rho^3 = \rho^3.$$

Thus the only possibilities are H_1, H_2, and H_5. If $\rho \subset \rho^2$, we have by (2) that $\rho^3 \subset \rho^2$, so that by (12), $\rho = \rho^3 \subset \rho^2$. Hence using (16), the right side of the lattice collapses to ρ^2 and the left side collapses to ρ, so that the only possibilities are H_1 and H_3.

It is clear: that the class H_1 is the union of classes E_1 and E_2; that the union of classes H_2, H_3, and H_4 is the union of classes E_3 and E_4; and that the union of classes H_5 and H_6 is the union of classes E_5, E_6, and E_7.

5. Antispaces

Recall that two (minus)spaces over X are said to be antispaces (of each other), and thus constitute an antipair, provided that either can be obtained by interchanging the collections of closed subsets and compact subsets of the other.

In this section we will develop some of the antispace theory and provide detailed proofs of the results in [3]. Note that by the definition, all antispaces will be T_1, and for any antipair, the topologies when restricted to closed compact subsets are identical.

First we consider a particular class of spaces which are important from an antispace standpoint.

DEFINITION. A Hausdorff space is said to be a k-space (cf. [6, p. 230]) (or a *compactly generated space* (cf. [8, p. 5])) provided that a subset is closed if and only if its intersection with every closed compact subset is closed; i.e., the space is Hausdorff and the collection of closed sets, $\mathfrak{B}$, satifies (k_1) below.

Along with (k_1) we have stated other variations of the k-axiom, all of which are equivalent for Hausdorff spaces or even for those spaces in which all compact sets are closed.

(k_1) $\quad A \in \mathfrak{B} \Leftrightarrow \{A\} \wedge (\mathfrak{B} \cap \rho\mathfrak{B}) \subset \mathfrak{B}$

(k_2) $\quad A \in \mathfrak{B} \Leftrightarrow \{A\} \wedge (\mathfrak{B} \cap \rho\mathfrak{B}) \subset \rho\mathfrak{B}$

(k_3) $\quad A \in \mathfrak{B} \Leftrightarrow \{A\} \wedge \rho\mathfrak{B} \subset \mathfrak{B}$

(k_4) $\quad A \in \mathfrak{B} \Leftrightarrow \{A\} \wedge \rho\mathfrak{B} \subset \rho\mathfrak{B}.$

A space $(X, \mathfrak{B})$ is called a k_i-space provided that $\mathfrak{B}$ satisfies k_i, $i = 1, 2, 3, 4$. In general these axioms have relative strengths indicated by:

$$\begin{tikzcd} & k_2 \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ k_4 \arrow[dr, Rightarrow] & & k_1 \arrow[dl, Rightarrow] \\ & k_3 & \end{tikzcd}$$

The proofs of these implications, as well as the fact that they cannot be reversed in general, but are reversed for Hausdorff spaces, are straightforward; cf. [2]. The k_2-spaces are precisely the c-spaces of [3] and a space $(X, \mathfrak{B})$ is k_4 provided that every $\mathfrak{D} \subset 2^X$ satisfying $\rho\mathfrak{D} = \rho\mathfrak{B}$, is such that $\mathfrak{D} \subset \mathfrak{B}$; cf. [10]. The connection between antispaces and the k_2, k_3, and k_4 axioms will be demonstrated in the next four theorems.

THEOREM 5. *For a space* $(X, \mathfrak{B})$, *the following are equivalent:*

(i) $(X, \mathfrak{B})$ *is an antispace*;

(ii) $\mathfrak{B} = \rho^2\mathfrak{B}$, *i.e. the closed sets are precisely the square compact sets.*

(iii) $(X, \mathfrak{B})$ *is a* k_3*-space and* $\rho\mathfrak{B} \wedge \rho^2\mathfrak{B} \subset \mathfrak{B}$, *i.e. every intersection of a compact set with a square compact set is closed.*

Proof. (i) $\leftrightarrow$ (ii).

Clearly by definition $(X, \mathfrak{B})$ is an antispace if and only if there exists a space $(X, \mathfrak{B}^*)$ such that $\mathfrak{B} = \rho\mathfrak{B}^*$ and $\mathfrak{B}^* = \rho\mathfrak{B}$, and this is true provided that $\mathfrak{B} = \rho^2\mathfrak{B}$.

(ii) $\to$ (iii).

Clearly if $\mathfrak{B} = \rho^2\mathfrak{B}$, we have by (6) that

$$\rho\mathfrak{B} \wedge \rho^2\mathfrak{B} = \rho\mathfrak{B} \cap \rho^2\mathfrak{B} \subset \rho^2\mathfrak{B} = \mathfrak{B}.$$

Thus the second condition of (iii) is satisfied. To show that $(X, \mathfrak{B})$ is a k_3-space, first let $A \in \mathfrak{B}$; then by the above,

$$\{A\} \wedge \rho\mathfrak{B} \subset \mathfrak{B} \wedge \rho\mathfrak{B} = \rho^2\mathfrak{B} \wedge \rho\mathfrak{B} \subset \mathfrak{B}.$$

Conversely suppose that $A \subset X$ and that $\{A\} \wedge \rho\mathfrak{B} \subset \mathfrak{B} = \rho^2\mathfrak{B}$. If $\emptyset \neq \mathfrak{C} \subset \rho\mathfrak{B}$ such that $\mathfrak{C} \cup \{A\}$ has f.i.p., then

$$\mathfrak{C}' = \{A\} \wedge \mathfrak{C} \wedge \{C_0\} \subset \rho^2\mathfrak{B} \wedge \rho\mathfrak{B} \subset \mathfrak{B}$$

where C_0 is any member of $\mathfrak{C}$. Thus, since $\mathfrak{C}' \cup \{C_0\}$ has f.i.p., we have by the definition of ρ, $(\cap\, \mathfrak{C}') \cap C_0 \neq \emptyset$. But

$$(\cap\, \mathfrak{C}') \cap C_0 = (\cap\, \mathfrak{C}) \cap A;$$

thus $A \in \rho^2\mathfrak{B} = \mathfrak{B}$.

(iii) $\to$ (ii).

Let $(X, \mathfrak{B})$ be k_3 and $\rho\mathfrak{B} \wedge \rho^2\mathfrak{B} \subset \mathfrak{B}$. If $A \in \mathfrak{B}$ and $\emptyset \neq \mathfrak{C} \subset \rho\mathfrak{B}$ such that $\mathfrak{C} \cup \{A\}$ has f.i.p., then by the k_3 axiom

$$\mathfrak{C}' = \{A\} \wedge \mathfrak{C} \subset \{A\} \wedge \rho\mathfrak{B} \subset \mathfrak{B},$$

and for any $C_0 \in \mathfrak{C}$, $\mathfrak{C}' \cup \{C_0\}$ has f.i.p.; so that

$$(\cap\, \mathfrak{C}') \cap C_0 = (\cap\, \mathfrak{C}) \cap A \neq \emptyset.$$

Hence $\mathfrak{B} \subset \rho^2\mathfrak{B}$.

If $A \in \rho^2\mathfrak{B}$, then $\{A\} \wedge \rho\mathfrak{B} \subset \rho^2\mathfrak{B} \wedge \rho\mathfrak{B} \subset \mathfrak{B}$, so that by the k_3 axiom $A \in \mathfrak{B}$. Thus $\rho^2\mathfrak{B} \subset \mathfrak{B}$, so $\mathfrak{B} = \rho^2\mathfrak{B}$.

Corollary. *For every* $\mathfrak{F} \subset 2^X$, $(X, \rho^2\mathfrak{F})$ *is an antispace.*

Proof. Immediate from (10).

For *topological* spaces, i.e. those spaces for which the underlying set is closed, we obtain the following theorem and thus have a characterization for antispaces in which we need not mention square compact sets.

THEOREM 6. *For a topological space* $(X, \mathfrak{B})$, *the following are equivalent*:

(i) $(X, \mathfrak{B})$ *is an antispace*;

(ii) $\mathfrak{B} = \rho^2\mathfrak{B}$;

(iii) $(X, \mathfrak{B})$ *is a* k_3*-space and* $\rho\mathfrak{B} \wedge \rho^2\mathfrak{B} \subset \mathfrak{B}$;

(iv) $(X, \mathfrak{B})$ *is a* k_2*-space*;

(v) $(X, \mathfrak{B})$ *is a* k_4*-space and the intersection of compact sets is compact* ($\gamma\rho\mathfrak{B} = \rho\mathfrak{B}$);

(vi) $(X, \mathfrak{B})$ *is a* k_4*-space and the intersection of any pair of compact sets is compact* ($\rho\mathfrak{B} \wedge \rho\mathfrak{B} = \rho\mathfrak{B}$);

(vii) *Every set which is compact or square compact is closed* ($\rho\mathfrak{B} \cup \rho^2\mathfrak{B} \subset \mathfrak{B}$).

PROOF. (i), (ii) and (iii) have been shown to be equivalent under the more general hypothesis of Theorem 5. We will show that (ii) $\to$ (iv) $\to$ (v) $\to$ (vi) $\to$ (vii) $\to$ (ii).

(ii) $\to$ (iv).

Suppose that $\mathfrak{B} = \rho^2\mathfrak{B}$. If $\{A\} \wedge (\mathfrak{B} \cap \rho\mathfrak{B}) \subset \rho\mathfrak{B}$ and if $\mathfrak{C} \subset \rho\mathfrak{B}$ such that $\mathfrak{C} \cup \{A\}$ has f.i.p., then since the space is topological, $X \in \mathfrak{B} = \rho^2\mathfrak{B}$. Thus $\mathfrak{C}' = \{A\} \wedge \mathfrak{C} = \{A\} \wedge \{X\} \wedge \mathfrak{C} \subset \{A\} \wedge \rho^2\mathfrak{B} \wedge \rho\mathfrak{B}$. Thus by (6) $\mathfrak{C}' \subset \{A\} \wedge (\rho^2\mathfrak{B} \cap \rho\mathfrak{B}) = \{A\} \wedge (\mathfrak{B} \cap \rho\mathfrak{B}) \subset \rho\mathfrak{B}$. By hypothesis, $X \in \rho^2\mathfrak{B}$ and $\mathfrak{C}' \cup \{X\}$ has f.i.p. Thus

$$(\cap\, \mathfrak{C}') \cap X = (\cap\, \mathfrak{C}) \cap A \neq \emptyset;$$

so that $A \in \rho^2\mathfrak{B} = \mathfrak{B}$. Conversely if $A \in \mathfrak{B}$ then by (5) $\{A\} \wedge (\mathfrak{B} \cap \rho\mathfrak{B}) \subset \rho\mathfrak{B} \wedge \mathfrak{B} \subset \rho\mathfrak{B}$. Consequently $(X, \mathfrak{B})$ is a k_2-space.

(iv) $\to$ (v).

Suppose that $(X, \mathfrak{B})$ is k_2. If $A \in \mathfrak{B}$, then by (5), $\{A\} \wedge \rho\mathfrak{B} \subset \rho\mathfrak{B}$. Conversely $\{A\} \wedge \rho\mathfrak{B} \subset \rho\mathfrak{B}$ clearly implies that $\{A\} \wedge (\mathfrak{B} \cap \rho\mathfrak{B}) \subset \rho\mathfrak{B}$. Thus every k_2-space is k_4. If $C \in \rho\mathfrak{B}$, then $\{C\} \wedge (\mathfrak{B} \cap \rho\mathfrak{B})$ is clearly contained in $\rho\mathfrak{B} \wedge \mathfrak{B}$, which by (5) is contained in $\rho\mathfrak{B}$. Thus by the k_2 axiom, $C \in \mathfrak{B}$. Hence $\rho\mathfrak{B} \subset \mathfrak{B}$, so that by (4) $\gamma\rho\mathfrak{B} = \rho\mathfrak{B}$.

(v) $\to$ (vi).

Trivial.

(vi) $\to$ (vii).

Suppose that $(X, \mathfrak{B})$ is k_4 and $\rho\mathfrak{B} \wedge \rho\mathfrak{B} = \rho\mathfrak{B}$. If $C \in \rho\mathfrak{B}$, then $\{C\} \wedge \rho\mathfrak{B} \subset \rho\mathfrak{B} \wedge \rho\mathfrak{B} = \rho\mathfrak{B}$; so that by the k_4 axiom, $C \in \mathfrak{B}$. Thus $\rho\mathfrak{B} \subset \mathfrak{B}$. If $A \in \rho^2\mathfrak{B}$, then by (6),

$$\{A\} \wedge \rho\mathfrak{B} \subset \rho^2\mathfrak{B} \wedge \rho\mathfrak{B} = \rho^2\mathfrak{B} \cap \rho\mathfrak{B} \subset \rho\mathfrak{B},$$

so that by k_4, $A \in \mathfrak{B}$. Hence $\rho^2\mathfrak{B} \subset \mathfrak{B}$.

(vii) $\rightarrow$ (ii).

Suppose that $\rho\mathfrak{B} \cup \rho^2\mathfrak{B} \subset \mathfrak{B}$. Since $\mathfrak{B} = \gamma\mathfrak{B}$ and $\rho\mathfrak{B} \subset \mathfrak{B}$, $\mathfrak{B} \wedge \rho\mathfrak{B} \subset \mathfrak{B}$. Thus, by (7) $\mathfrak{B} \subset \rho^2\mathfrak{B}$, so that $\rho^2\mathfrak{B} = \mathfrak{B}$.

COROLLARY. *Every Hausdorff k-space is an antispace.*

We shall now see that if we restrict our attention to certain important classes of spaces, we obtain an equivalence between the Hausdorff property and the property of being an antispace.

DEFINITION. A space $(X, \mathfrak{B})$ is said to be *locally compact* if for every $p \in X$ and $B \in \mathfrak{B}$ such that $p \notin B$, there exists some $G \in \mathfrak{B}$ and some $C \in \rho\mathfrak{B}$ such that $p \in X-G \subset C \subset X-B$.

THEOREM 7. *For topological spaces $(X, \mathfrak{B})$ which are locally compact or satisfy the first axiom of countability, the following are equivalent:*

(i) $(X, \mathfrak{B})$ *is an antispace;*

(ii) $(X, \mathfrak{B})$ *is a k_2-space;*

(iii) $\rho\mathfrak{B} \subset \mathfrak{B}$; *i.e. every compact subset is closed;*

(iv) $(X, \mathfrak{B})$ *is Hausdorff.*

PROOF. Clearly by theorem 6, (i) is equivalent to (ii) and (ii) implies (iii).

(iii) $\rightarrow$ (iv).

Let a and b be distinct points of X. For the case that $(X, \mathfrak{B})$ is locally compact, by definition and the fact that any space satisfying (iii) is T_1, there exist G_a, $G_b \in \mathfrak{B}$ and C_a, $C_b \in \rho\mathfrak{B}$ such that $a \in X-G_b \subset C_b \subset X-\{b\}$ and $b \in X-G_a \subset C_a \subset X-\{a\}$. Then by (iii) C_a and C_b are closed, so that $(X-G_b)-C_a$ and $(X-G_a)-C_b$ are disjoint open neighborhoods of a and b respectively. For the case that $(X, \mathfrak{B})$ satisfies the first axiom of countability, let $\{V_n\}$ be a countable system of neighborhoods for a, and $\{W_n\}$ a countable system of neighborhoods for b. If for each n, $V_n \cap W_n \neq \emptyset$, choose $c_n \in V_n \cap W_n$. Clearly $\{a\} \cup \{c_n \mid n = 1, 2, 3 \cdots\}$ is compact, so by (iii) it is closed, which is impossible since every neighborhood of b meets it.

(iv) $\rightarrow$ (ii).

It is well known that every locally compact Hausdorff space and every first countable Hausdorff space is a k-space (cf. [6,

p. 231]) and as has been mentioned, for Hausdorff spaces all four k axioms are equivalent.

THEOREM 8. *If* $(X, \mathfrak{B}) = (X, \gamma\rho\mathfrak{E})$ *for some* $\mathfrak{E}$, *then the following are equivalent*:

(i) $(X, \mathfrak{B})$ *is an antispace*;

(ii) $(X, \mathfrak{B})$ *is a* k_3*-space.*

PROOF. By Theorem 5, (i) implies (ii). Conversely suppose that $(X, \mathfrak{B})$ is a k_3-space and $\mathfrak{B} = \gamma\rho\mathfrak{E}$. Then $\rho\mathfrak{B} \wedge \rho^2\mathfrak{B} = \rho\gamma\rho\mathfrak{E} \wedge \rho^2\gamma\rho\mathfrak{E}$. By (3), (6), and (11), this becomes:

$$\rho\mathfrak{B} \wedge \rho^2\mathfrak{B} = \rho^2\mathfrak{E} \wedge \rho^3\mathfrak{E} = \rho^2\mathfrak{E} \cap \rho^3\mathfrak{E} = \rho\mathfrak{E} \cap \rho^2\mathfrak{E} \subset \rho\mathfrak{E} \subset \gamma\rho\mathfrak{E} = \mathfrak{B}.$$

Therefore by theorem 5, $(X, \mathfrak{B})$ is an antispace.

The next three theorems provide some means of comparing the notions of compactness and square compactness.

THEOREM 9. *For a space* $(X, \mathfrak{B})$, *the following are equivalent*:

(i) $(X, \mathfrak{B})$ *is compact, i.e.* $X \in \rho\mathfrak{B}$;

(ii) $\mathfrak{B} \subset \rho\mathfrak{B}$;

(iii) $\mathfrak{B} \cup \rho^2\mathfrak{B} \subset \rho\mathfrak{B}$.

PROOF.

(i) $\to$ (ii).

Clearly if $X \in \rho\mathfrak{B}$, we have, using (5),

$$\mathfrak{B} = \mathfrak{B} \wedge \{X\} \subset \mathfrak{B} \wedge \rho\mathfrak{B} \subset \rho\mathfrak{B}.$$

(ii) $\to$ (iii).

If $\mathfrak{B} \subset \rho\mathfrak{B}$, we have, by (2),

$$\rho^2\mathfrak{B} \subset \rho\mathfrak{B}.$$

(iii) $\to$ (i).

Suppose that $\mathfrak{B} \cup \rho^2\mathfrak{B} \subset \rho\mathfrak{B}$. Then if $\emptyset \neq \mathfrak{D} \subset \mathfrak{B}$ and has f.i.p., we have by (4) that $\cap\, \mathfrak{D} \neq \emptyset$; thus $X \in \rho\mathfrak{B}$.

THEOREM 10. *For a space* $(X, \mathfrak{B})$, *the following are equivalent*:

(i) $(X, \mathfrak{B})$ *is square compact, i. e.* $X \in \rho^2\mathfrak{B}$;

(ii) $\rho\mathfrak{B} \subset \rho^2\mathfrak{B}$;

(iii) $\rho\mathfrak{B} = \rho^3\mathfrak{B}$ *and* $\mathfrak{B} \cup \rho\mathfrak{B} \subset \rho^2\mathfrak{B}$;

(iv) $\mathfrak{B} \cup \rho\mathfrak{B} \subset \rho^2\mathfrak{B}$.

PROOF.

(i) $\to$ (ii).

If $X \in \rho^2\mathfrak{B}$, then $\rho\mathfrak{B} = \rho\mathfrak{B} \wedge \{X\} \subset \rho\mathfrak{B} \wedge \rho^2\mathfrak{B} = \rho\mathfrak{B} \cap \rho^2\mathfrak{B} \subset \rho^2\mathfrak{B}$.

(ii) $\to$ (iii).

If $\rho\mathfrak{B} \subset \rho^2\mathfrak{B}$, then by (2),

$\rho^3\mathfrak{B} \subset \rho^2\mathfrak{B}$; so that by (12),

$\rho\mathfrak{B} = \rho^3\mathfrak{B}$.

Let $B \in \mathfrak{B}$ and $\emptyset \neq \mathfrak{C} \subset \rho\mathfrak{B}$ such that $\mathfrak{C} \cup \{B\}$ has f.i.p. Then by (5), $\mathfrak{C}' = \{B\} \wedge \mathfrak{C} \subset \mathfrak{B} \wedge \rho^2\mathfrak{B} \subset \rho^2\mathfrak{B}$. If $C_0 \in \mathfrak{C}$, then $C_0 \in \rho^3\mathfrak{B}$ and $\mathfrak{C}' \cup \{C_0\}$ has f.i.p. so that

$$(\cap\, \mathfrak{C}') \cap C_0 = (\cap\, \mathfrak{C}) \cap B = \emptyset.$$

Consequently $B \in \rho^2\mathfrak{B}$, and we have $\mathfrak{B} \cup \rho\mathfrak{B} \subset \rho^2\mathfrak{B}$.

(iii) $\to$ (iv). Trivial.

(iv) $\to$ (i).

If $\mathfrak{B} \cup \rho\mathfrak{B} \subset \rho^2\mathfrak{B}$ and if $\emptyset \neq \mathfrak{D} \subset \rho\mathfrak{B}$ has f.i.p. then by (4), $\cap\, \mathfrak{D} \neq \emptyset$; thus $X \in \rho^2\mathfrak{B}$.

COROLLARY. *Every Hausdorff space is square compact.*

PROOF. $(X, \mathfrak{B})$ Hausdorff $\Rightarrow \rho\mathfrak{B} \subset \mathfrak{B} \Rightarrow \rho\mathfrak{B} \subset \rho^2\mathfrak{B}$.

N.B. The above corollary is also immediate from (4).

THEOREM 11. *For a space* $(X, \mathfrak{B})$, *the following are equivalent:*

(i) $(X, \mathfrak{B})$ *is compact and square compact;*

(ii) $\mathfrak{B} \subset \rho\mathfrak{B} = \rho^2\mathfrak{B} = \cdots = \rho^n\mathfrak{B} \cdots$

PROOF. Immediate from theorems 9 and 10.

Next we seek a classification of the pairs of antispaces.

DEFINITION. A space $(X, \mathfrak{B})$ is called *superconnected* provided that X itself is not closed, i.e. there do not exist $B_1, B_2 \in \mathfrak{B}$ such that $X = B_1 \cup B_2$.

The following lemma is easily verified. Part (iv) explains our terminology.

LEMMA. For a space $(X, \mathfrak{B})$, *the following are equivalent:*

(i) $(X, \mathfrak{B})$ *is superconnected;*

(ii) $(X, \mathfrak{B})$ *is not topological;*

(iii) *every open set is dense;*

(iv) *every open set is connected*[3].

[3] We have thus adopted the convention that the empty set is not connected.

THEOREM 12. *Every pair of antispaces satisfies precisely one of the following conditions*:

(A) *The spaces are identical, maximal compact*[4], *and topological.*

(B) *One space is a k_2 non-compact topological space, whereas the other is a compact superconnected space.*

(C) *Each space is superconnected and non-compact.*

PROOF. Clearly no pair of spaces can satisfy more than one of the conditions. To show that they include all possibilities, let $(X, \mathfrak{B})$ and $(X, \mathfrak{B}^*)$ be a pair of antispaces. If neither is compact, then $X \notin \rho\mathfrak{B} \cup \rho\mathfrak{B}^* = \mathfrak{B}^* \cup \mathfrak{B}$; hence neither space is topological, so that by the lemma both are superconnected, and condition (C) is satisfied.

Thus we may suppose that one space, say $(X, \mathfrak{B})$, is compact. Hence $X \in \rho\mathfrak{B} = \mathfrak{B}^*$, so that $(X, \mathfrak{B}^*)$ is topological and so by theorem 6 is a k_2-space. If $(X, \mathfrak{B}^*)$ is also compact, we have by theorem 9 that $\rho\mathfrak{B}^* \subset \mathfrak{B}^* = \rho\mathfrak{B} \subset \mathfrak{B} = \rho\mathfrak{B}^*$. Thus ρ is the identity and the spaces are maximal compact and identical (condition (A)). If $(X, \mathfrak{B}^*)$ is not compact, then $X \neq B_1 \cup B_2$ where B_1, $B_2 \in \rho\mathfrak{B}^* = \mathfrak{B}$. Thus $(X, \mathfrak{B})$ is superconnected, which completes the proof.

Note that by (iii) of the lemma, superconnected spaces are extremely non-Hausdorff in the sense that no pair of open sets is disjoint. However by theorem 12 (B) and theorem 7 we see that (for example) every non-compact metrizable space is completely determined by some superconnected space — namely its antispace.

Examples 3 and 4 are antispaces satisfying (A), whereas examples 1 and 2 satisfy (B), and example 7 satisfies (C). We now introduce a natural method of combining a collection of spaces.

DEFINITION. If $(X_i, \mathfrak{B}_i)$, $i \in I$ is a disjoint collection of spaces, then their *ultra-union* is the space $(X, \mathfrak{B})$ where $X = \cup \{X_i \mid i \in I\}$ and $\mathfrak{B} = \gamma (\cup \{\mathfrak{B}_i \mid i \in I\})$. The following is easily established.

LEMMA. *Every ultra-union of superconnected spaces is superconnected and every ultra-union of an infinite collection of spaces is superconnected.*

Thus since the ultra-union topology restricted to any of the original sets yields the original space, we have the following:

[4] A space is maximal compact iff it is compact and there exists no strictly finer (stronger) compact topology on the underlying set; cf. [7].

Proposition 2. *Every space can be embedded in a superconnected space.*

Moreover we have the following result for antispaces.

Proposition 3. *If* $(X_i, \mathfrak{B}_i)$, $i \in I$ *is a disjoint collection of compact antispaces of type* (B) *then their ultra-union* $(X, \mathfrak{B})$ *and the topological union* $(X, \mathfrak{C})$ *of the spaces* $(X_i, \rho\mathfrak{B}_i)$, $i \in I$ *form an antipair of type* (B).

Proof. Clearly $(X, \mathfrak{C})$ is k_2 and topological since it is the disjoint union of k_2-topological spaces. Thus, by theorem 6, $(X, \mathfrak{C})$ is an antispace. Now if $C \in \rho\mathfrak{C}$, then there exists some finite collection $J \subset I$ such that $C \subset \cup \{X_i \mid i \in J\}$ and $C \cap X_i \in \rho(\rho\mathfrak{B}_i) = \mathfrak{B}_i$ for each i. Thus

$$C = \cup \{(C \cap X_i) \mid i \in J\} \in \gamma(\cup\{\mathfrak{B}_i \mid i \in I\}) = \mathfrak{B}.$$

Conversely, if $B \in \mathfrak{B}$, then there exists a finite collection $J \subset I$ such that $B = \cup \{B_i \mid i \in J\}$ where $B_i \in \mathfrak{B}_i = \rho(\rho\mathfrak{B}_i) \subset \rho\mathfrak{C}$. Hence $B \in \rho\mathfrak{C}$, so that $\mathfrak{B} = \rho\mathfrak{C}$. But since $(X, \mathfrak{C})$ is an antispace, $(X, \mathfrak{B})$ must be an antispace and by the lemma $(X, \mathfrak{B})$ is superconnected. Thus $(X, \mathfrak{B})$ and $(X, \mathfrak{C})$ form an antipair of type (B).

We now will show that given any collection of subsets of some set, one can obtain by application of the operator ρ not only an antispace (cf. Corollary to theorem 5), but also a k_2 antispace which is topological and which is paired with a compact space.

Theorem 13. *For every*

$$\mathfrak{F} \subset 2^X,\ (X, \rho\mathfrak{F} \cap \rho^2\mathfrak{F}) \text{ and } \big(X, \rho(\rho\mathfrak{F} \cap \rho^2\mathfrak{F})\big)$$

form a pair of antispaces, where $(X, \rho\mathfrak{F} \cap \rho^2\mathfrak{F})$ *is compact and* $\big(X, \rho(\rho\mathfrak{F} \cap \rho^2\mathfrak{F})\big)$ *is* k_2 *and topological.*

Proof. By (15) and theorem 5,

$$(X, \rho^2\mathfrak{F} \cap \rho\mathfrak{F}) \text{ and } \big(X, \rho(\rho^2\mathfrak{F} \cap \rho\mathfrak{F})\big)$$

form a pair of antispaces. By (14),

$$\rho\mathfrak{F} \cap \rho^2\mathfrak{F} \subset \rho^2\mathfrak{F} \cup \rho^3\mathfrak{F} \subset \rho(\rho^2\mathfrak{F} \cap \rho\mathfrak{F}),$$

so that by theorem 9 $(X, \rho\mathfrak{F} \cap \rho^2\mathfrak{F})$ is compact. Thus by theorem 12 $\big(X, \rho(\rho\mathfrak{F} \cap \rho^2\mathfrak{F})\big)$ is topological, whence by theorem 6, it is a k_2 space.

Having investigated the conditions for which a given $\mathfrak{F} \subset 2^X$ is the collection of closed sets of an antispace, we now list several conditions which are sufficient for one application of the operator

ρ to obtain an antispace, i.e., *conditions which are sufficient to guarantee that* $\rho\mathfrak{J} = \rho^3\mathfrak{J}$.

(a) $\rho^3\mathfrak{J} \subset \rho\mathfrak{J} \cup \rho^2\mathfrak{J}$;

PROOF. Relation (12).

(b) $\mathfrak{J} \wedge \rho^3\mathfrak{J} \subset \rho^2\mathfrak{J}$;

PROOF. If $A \in \rho^3\mathfrak{J}$ and $\mathfrak{C} \subset \mathfrak{J}$ is such that $\mathfrak{C} \cup \{A\}$ has f.i.p., then $\mathfrak{C}' = \{A\} \wedge \mathfrak{C} \subset \rho^2\mathfrak{J}$ and $\mathfrak{C}' \cup \{A\}$ has f.i.p., thus

$$(\cap \mathfrak{C}') \cap A \neq \emptyset.$$

(c) $X \in \rho^2\mathfrak{J}$;

PROOF. Theorem 10.

(d) $\rho\mathfrak{J} \subset \rho^2\mathfrak{J}$;

PROOF. Theorem 10.

(e) $\rho\mathfrak{J} \subset \mathfrak{J} \cup \rho^2\mathfrak{J}$;

PROOF. $\rho(\mathfrak{J} \cup \rho^2\mathfrak{J}) = \rho\mathfrak{J}$ (cf. [10]). Thus by (2) $\rho\mathfrak{J} \subset \rho^2\mathfrak{J}$ which is (d).

(f) $\mathfrak{J} \subset \rho^2\mathfrak{J}$;

PROOF. Relation (2).

(g) $\mathfrak{J} \wedge \rho\mathfrak{J} \subset \rho^2\mathfrak{J}$.

PROOF. This condition implies condition (b).

(h) $\mathfrak{J} \wedge \rho\mathfrak{J} \subset \mathfrak{J}$.

PROOF. By (7), this implies condition (f).

PROBLEM. Does the equality $\rho\mathfrak{J} = \gamma\rho\mathfrak{J}$ also imply $\rho\mathfrak{J} = \rho^3\mathfrak{J}$?

6. Examples

EXAMPLE 1. Let $\mathfrak{B} = 2^X$; then $\rho\mathfrak{B} = \{A \subset X \mid A \text{ is finite}\}$ and $\rho^2\mathfrak{B} = \mathfrak{B}$. Thus $(X, \mathfrak{B})$ is the discrete space and $(X, \rho\mathfrak{B})$ is the co-finite (or Zariski) space.

EXAMPLE 2. Let $(X, \mathfrak{B})$ be the real line with the usual topology. Then $\rho\mathfrak{B} \subset \mathfrak{B} = \rho^2\mathfrak{J}$; thus (theorem 5) $(X, \mathfrak{B})$ has a compact antispace with a properly weaker topology.

EXAMPLE 3. Let $(X, \mathfrak{B})$ be any compact Hausdorff space. Then $\mathfrak{B} = \rho\mathfrak{B} = \rho^2\mathfrak{B}$.

Example 4. Let $(X, \mathfrak{B})$ be the one-point compactification[5] of the rationals (with the usual topology). Then $\mathfrak{B} = \rho\mathfrak{B} = \rho^2\mathfrak{B}$, but $(X, \mathfrak{B})$ is not Hausdorff.

Example 5. Let X be infinite and let a and b be two distinguished elements of X. Let $\mathfrak{B} = \{A \subset X \mid \text{if } A \text{ is infinite, then } \{a, b\} \subset A\}$. Then

$$\rho\mathfrak{B} = \{C \subset X \mid \text{if } C \text{ is infinite, then } \{a, b\} \cap C \neq \emptyset\};$$
$$\rho^2\mathfrak{B} = \{E \subset X \mid E \text{ is finite}\}; \text{ and}$$
$$\gamma\rho\mathfrak{B} = \rho^3\mathfrak{B} = 2^X.$$

Example 6. Let $(X, \mathfrak{B})$ be the integers together with one point from their Stone-Čech compactification. Then

$$\rho\mathfrak{B} = \gamma\rho\mathfrak{B} = \rho^3\mathfrak{B} = \{C \subset X \mid C \text{ is finite}\}; \text{ and}$$
$$\rho^2\mathfrak{B} = 2^X.$$

Example 7. Let X be the set of real numbers with their natural order, let $(X, \mathfrak{S})$ be the space with the usual order topology, and let $\mathfrak{B} = \{A \subset X | A \text{ has a lower bound and } A \in \mathfrak{S}\}$. Then

$$\rho\mathfrak{B} = \rho^3\mathfrak{B} = \{C \subset X | X \text{ has an upper bound and } C \in \mathfrak{S}\}$$
$$\rho^2\mathfrak{B} = \mathfrak{B}.$$

Example 8. Let W be the space of all ordinal numbers less than the first uncountable ordinal number ω_1, with the natural order topology. Let $X = W \times \{0, 1\}$ and let $\pi : X \to W$ be the projection. Let

$$\mathfrak{B} = \{A \subset X \mid \text{if } \alpha \in (\pi(A) - \{a\})^-, \text{ then } \{(\alpha, 0), (\alpha, 1)\} \subset A\}.$$

Then

$$\rho\mathfrak{B} = \{C \subset X | |C| \leqq \aleph_0 \text{ and if } \alpha \in (\pi(A) - \{a\})^-, \text{ then}$$
$$\{(\alpha, 0), (\alpha, 1)\} \cap A \neq \emptyset\};$$
$$\gamma\rho\mathfrak{B} = \{D \subset X | |D| \leqq \aleph_0\};$$
$$\rho^2\mathfrak{B} = \{E \subset X | |E| < \aleph_0\};$$
$$\rho^3\mathfrak{B} = 2^X.$$

Note in particular that $\rho\mathfrak{B} \wedge \rho^3\mathfrak{B} = \gamma\rho\mathfrak{B} \neq \rho^3\mathfrak{B}$.

Example 9. Let X be the set of example 8 and let $Y = X \cup \{\omega_1\}$. Let A be a member of $\mathfrak{B}$ provided that (a) and (b) below are satisfied:

[5] A set U is an open neighborhood for the adjoined point if the complement of U is closed and compact in the original space.

(a) $\alpha \in (\pi(A-\{\omega_1, (\alpha, 0), (\alpha, 1)\}))^-$ implies that $\{(\alpha, 0), (\alpha, 1)\} \subset A$

(b) if A is uncountable, then $\omega_1 \in A$.

Then $\rho\mathfrak{B}$ consists of the collection of sets A satisfying (b) and (c).

(c) $\alpha \in (\pi(A-\{\omega_1, (\alpha, 0)\ (\alpha, 1)\}))^-$ implies that $\{(\alpha, 0), (\alpha, 1)\} \cap A \neq \emptyset$.

$\gamma\rho\mathfrak{B}$ consists of those sets A satisfying (b).
$\rho^2\mathfrak{B}$ consists of all finite subsets of Y.
$\rho^3\mathfrak{B} = 2^Y$.

Consequently

$$\rho^2\mathfrak{B} \subsetneqq \mathfrak{B} = \gamma\mathfrak{B} \subsetneqq \rho\mathfrak{B} \subsetneqq \gamma\rho\mathfrak{B} \subsetneqq \gamma\rho\mathfrak{B} \wedge \rho^3\mathfrak{F} = \rho^3\mathfrak{F}.$$

EXAMPLE 10. Let W be the space of all ordinal numbers less than the first uncountable ordinal number ω_1 with the natural order-topology. Let N be the discrete space of all natural numbers, Y be the space $W \times N$ with the product topology. Let a, b be two different points, which are not contained in Y. We set $X = Y \cup \{a, b\}$ and define:

A subset F of X belongs to $\mathfrak{B}$ iff F fulfills at least one of the following three conditions:

(1) F is a compact subset of Y,

(2) There exist $\alpha \in W$, $n \in N$ with $F = \{(\beta, n) : \beta \geqq \alpha\} \cup \{a\}$,

(3) There exist $\alpha \in W$, $\beta \in W$, $n \in N$ with $F = \{(\gamma, m) : \alpha < \gamma \leqq \beta, n < m\} \cup \{b\}$.

Then the following statements are true (where π_N and π_W are the projections from Y onto N and W, respectively):

1) A subset G of X belongs to $\rho\mathfrak{B}$ iff G fulfills the following three conditions:
   1. $G \cap Y$ is closed in Y;
   2. $|\pi_N(A \cap Y)| \geqq \aleph_0 \Rightarrow b \in A$,
   3. $|\pi_W(A \cap Y)| \geqq \aleph_1 \Rightarrow a \in A$.

2) $(X, \gamma\mathfrak{B})$ is a k_3-space but $\gamma\mathfrak{B} \neq \rho^2\mathfrak{B}$.

3) $\rho\mathfrak{B} = \rho^2\mathfrak{B}$ but $(X, \rho\mathfrak{B})$ is not Hausdorff.

4) There is no set $\mathfrak{D} \subset 2^X$ such that any of the following is true:
   $\rho\mathfrak{D} = \mathfrak{B}$;
   $\rho\mathfrak{D} = \gamma\mathfrak{B}$;
   $\gamma\rho\mathfrak{D} = \gamma\mathfrak{B}$.

(This follows from (12) and the fact that $\rho\mathfrak{B}$ equals $\rho^2\mathfrak{B}$ but is different from $\mathfrak{B}$ and $\gamma\mathfrak{B}$.)

Example 11. Let $(X, \mathfrak{B})$ be the disjoint topological union of the spaces of example 1 and example 5. Then $\rho\mathfrak{B}$ is not equal to $\rho^3\mathfrak{B}$, and $\rho^2\mathfrak{B}$ is comparable neither with $\rho\mathfrak{B}$ nor with $\rho^3\mathfrak{B}$. From this it follows that no two elements of the general lattice (fig. 2) are equal.

7. Problems

The main problem which remains unsolved is the axiomatization of the notion of compactness in terms of an operator, i.e.

1. Given a function $\rho : 2^{2^X} \to 2^{2^X}$, what is the "best" collection of relations (possibly concerning powers of ρ, its relationship with the space generating operator γ, and/or lattice relations) that are necessary and sufficient for ρ to be the compactness operator?

We have proved, for instance, that the relations (1)—(16) are necessary. To what extent are they sufficient? In this context, or purely for its own sake, one might consider extending the lattice considerations of § 4 to encompass the operator γ, and so include the results of § 3; or of extending either section to include the operation $\wedge$. Possibly the axiomatization questions might also be looked into from a wider context, e.g. from the standpoint of category theory.

The following result of P. Bacon might be useful.

$$\rho(\mathfrak{F} \cup \mathfrak{G}) = \rho(\mathfrak{F} \wedge \mathfrak{G}) \cap \rho\mathfrak{F} \cap \rho\mathfrak{G}.$$

A related problem is that of describing ρ^{-1} (where ρ is again the compactness operator), i.e. given $\mathfrak{F} \subset 2^X$;

2. (a) Characterize $\{\mathfrak{D} \subset 2^X \mid \rho\mathfrak{D} = \mathfrak{F}\}$,
or even less:

(b) Determine whether or not $\{\mathfrak{D} \subset 2^X \mid \rho\mathfrak{D} = \mathfrak{F}\}$ is empty. Similarly,

(c) Determine whether or not $\{\mathfrak{D} \subset 2^X \mid \gamma\rho\mathfrak{D} = \mathfrak{F}\}$ is empty.

Note that 2(b) and 2(c) are actually different questions since the collection of all countable subsets of an uncountable space can be $\gamma\rho\mathfrak{D}$ for some $\mathfrak{D}$ (example 8) but cannot be $\rho\mathfrak{D}$ for any $\mathfrak{D}$.

3. (a) Do E_6 spaces exist?
The following are related problems:
(b) If $\gamma\rho\mathfrak{F} = \rho\mathfrak{F}$, then does $\rho\mathfrak{F} = \rho^3\mathfrak{F}$?
(c) If $\gamma\rho\mathfrak{F} \cup \rho^2\mathfrak{F} \subset \rho\mathfrak{F}$, then must $\mathfrak{F} \subset \rho\mathfrak{F}$?

We have called a space $(X, \mathfrak{B})$ square compact provided that $X \in \rho^2\mathfrak{B}$. A subset $A \subset X$ has been called a square compact *subset* of $(X, \mathfrak{B})$ provided that $A \in \rho^2\mathfrak{B}$. Clearly every square compact *subset* is a square compact *subspace* (with the relative topology). It should be pointed out, however, that the converse is not true (every non-closed subset of the real line is a square compact subspace but not a square compact subset.) However, for closed subsets of any space, the two notions coincide. Thus it follows that the property of being a square compact space is "closed hereditary."

4. (a) For what other operations is the property "square compact space" an invariant?
(b) Can a theory similar to that for compact spaces be developed for square compact spaces (cube compact spaces)?

In this paper we have investigated the operator ρ. One can easily see that the square compactness operator ρ^2 has "nicer" properties than ρ (it is idempotent and commutes with the space generating function γ). Thus we should state for ρ^2 the questions corresponding to 1 and 2 above, which were stated for ρ; namely

5. (a) Characterize ρ^2
(b) Describe $(\rho^2)^{-1}$.

Antispaces are easily defined by the ρ operator ($\mathfrak{B} = \rho^2\mathfrak{B}$). However, for *topological* spaces we have a relatively simple characterization without using the ρ operator (especially without using ρ^2), i.e. a topological space is an antispace provided that every set which has a compact intersection with each closed compact set, is closed.

6. Is there a simple characterization of antispaces in general without using the ρ operator?

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