COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 21, nº 4 (1969), p. 417-430
<http://www.numdam.org/item?id=CM_1969__21_4_417_0>

COMPOSITIO MATHEMATICA, Vol. 21, Fasc. 4, 1969, pag. 417-430
Wolters-Noordhoff Publishing
Printed in the Netherlands

A remark on the vanishing of certain cohomology groups

by

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For a q-pseudoconvex space, cohomology with values in a coherent sheaf is finite dimensional above dimension q. Analogous results are valid for q-pseudoconcave spaces.

In this note we show that in some favorable instances these finite dimensional groups are actually zero. We prove that if $\mathscr{F}$ is a given coherent sheaf on the space X and F is a holomorphic line bundle having a certain strong type of convexity, then $H^r(X, \mathscr{F} \otimes \Omega(F^k)) = 0$ for large values of k, and for r in the expected range given by the convexity or concavity of X.

The idea of the proof is that used for the same purpose in [1] and this note can be considered as a straightforward application of the method of [1] and there explicitly carried out for the case when the base space X is a compact manifold.

These theorems are of the same nature as the vanishing theorem of Serre ([3] n. 74 théorème 1).

1. Preliminaries

a). Let X be a (reduced) complex space and let $\pi : F \to X$ be a holomorphic line bundle over X.

Let us consider a hermitian metric on the fibres of F. If F is trivializable on the open sets of the covering $U = \{U_i\}$ of X, $F_{|U_i} \cong U_i \times \boldsymbol{C}$ and if $g_{ij} : U_i \cap U_j \to \boldsymbol{C}$ are the corresponding transition functions for F, then the hermitian metric is locally given by C^∞ functions $h_i : U_i \to \boldsymbol{R}$, $h_i > 0$, such that

$$h_i(z) = |g_{ji}(z)|^2 h_j(z).$$

If $v \in \pi^{-1}(U_i)$ has base coordinate $\pi(v) = z$ and fibre coordinate $\xi_i \in \boldsymbol{C}$ then

$$||v||^2 = h_i(z)|\xi_i|^2 = \chi(z, \xi)$$

is the length of v in the metric we have considered.

Let F^* be the dual bundle of F. On the same covering U, F^* is represented by the transition functions $\{g_{ij}^{-1}\}$. Given the hermitian metric $\{h_i\}$ on the fibres of F, a hermitian metric on the fibres of F^* is given by the collection of local functions $\{h_i^{-1}\}$. If $v^* \in \pi^{-1}(U_i)$ has base coordinate z and fibre coordinate $\eta_i \in \boldsymbol{C}$ then

$$||v^*||^2 = h_i^{-1}(z)|\eta_i|^2 = \chi_*(z, \eta).$$

We will call the bundle F *metrically pseudoconvex* if a hermitian metric on the fibres of F can be chosen so that the function

$$\chi(z, \xi) = ||v||^2$$

is strongly pseudoconvex outside of the 0-section of F.[1]

If X is a manifold this is certainly the case if the exterior form

$$\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log h_i(z) \in c^1_{\boldsymbol{R}}(F)$$

corresponds to a hermitian positive definite quadratic form $\partial\bar{\partial}\log h_i(z)$.

We will call the line bundle F *metrically pseudoconcave* if the dual bundle F^* is metrically pseudoconvex.

Example. Consider the line bundle F associated with the $\boldsymbol{C}^*$-principal bundle (the Hopf bundle)

$$\boldsymbol{C}^{n+1} - \{0\} \to \boldsymbol{P}_n(\boldsymbol{C})$$

which serves as definition for the projective space $\boldsymbol{P}_n(\boldsymbol{C})$. On the covering $U_i = \{z = z_0, \cdots, z_n) \in \boldsymbol{P}_n(\boldsymbol{C}) : z_i \neq 0\}$ of $\boldsymbol{P}_n(\boldsymbol{C})$ the transition functions are

$$g_{ij} = \frac{z_i}{z_j} \quad \text{in} \quad U_i \cap U_j$$

so that F is isomorphic to the monoidal transform of $\boldsymbol{C}^{n+1}$ with center 0:

$$F = \{(z, t) \in \boldsymbol{C}^{n+1} \times \boldsymbol{P}_n(\boldsymbol{C}) : z_i t_j = z_j t_i\}.$$

Clearly F is a metrically pseudoconvex bundle since we can take the euclidean metric on $\boldsymbol{C}^{n+1}$ and lift it to F to get a metric on its fibres.

[1] I.e. for each point $z_0 \in X$ we can choose a neighborhood U of z_0 such that (i) $F_{|U} \cong U \times \boldsymbol{C}$, (ii) U is realized as an analytic subset of some open set $G \subset \boldsymbol{C}^N$ for some N, (iii) there exists a C^∞ function $\hat{h}(z) > 0$ on G such that $\hat{\chi} = \hat{h}(z)|\xi|^2$, $\xi \in \boldsymbol{C}$, is a C^∞ strongly Levi-convex function for $\xi \neq 0$ and $\chi = \hat{\chi}_{|U\times\boldsymbol{C}}$.

If on the other hand we identify $\boldsymbol{C}^{n+1}$ with the open set U_{n+1} of $\boldsymbol{P}_{n+1}(\boldsymbol{C})$ so that $\mathbf{0} = (0, \cdots, 0, 1)$ and

$$\boldsymbol{P}_n(\boldsymbol{C}) = \{z \in \boldsymbol{P}_{n+1}(\boldsymbol{C}) : z_{n+1} = 0\}$$

then the dual bundle of F is isomorphic to

$$F^* = \boldsymbol{P}_{n+1}(\boldsymbol{C}) - \{\mathbf{0}\}$$

the projection being the projection from $\mathbf{0}$ to the hyperplane $\boldsymbol{P}_n(\boldsymbol{C})$.

2. Filtration of cohomology

a). Let A be an analytic subset of a domain of holomorphy U of $\boldsymbol{C}^N$.

LEMMA. *Every holomorphic function f on $A \times \boldsymbol{C}$ admits a power series expansion*

$$f = \sum_0^\infty {}_s\, c_s(z)\xi^s$$

uniformly convergent on compact sets, where the coefficients $c_s(z)$ are holomorphic on A. Such an expansion is unique.

PROOF. Consider $A \times \boldsymbol{C}$ as an analytic subset of $U \times \boldsymbol{C}$. Since $U \times \boldsymbol{C}$ is a domain of holomorphy, there exists a holomorphic function $\hat{f}$ on $U \times \boldsymbol{C}$ such that $\hat{f}_{|A\times \boldsymbol{C}} = f$. For $\hat{f}$ we have a power series expansion (uniformly convergent on compact sets)

$$\hat{f} = \sum_0^\infty {}_s\, \hat{c}_s(z)\xi^s \qquad z \in U, \quad \xi \in \boldsymbol{C}$$

with holomorphic $\hat{c}_s(z)$. By restriction to $A \times \boldsymbol{C}$ we get an expansion for f.

To prove unicity assume that $f = \sum_0^\infty {}_s\, c'_s(z)\xi^s$ so that for every point $a \in A$

$$\sum_0^\infty {}_s\, \{c_s(a) - c'_s(a)\}\xi^s \equiv 0.$$

This implies that for every $s \in \boldsymbol{N}$, $c_s(a) = c'_s(a)$. Thus the conclusion follows.

Let $\hat{A} = A \times \boldsymbol{C}$ and define

$$\Gamma(\hat{A}, O_{\hat{A}})_k = \{f \in \Gamma(\hat{A}, O_{\hat{A}}) : f = \sum_0^\infty {}_s\, c_s(z)\xi^s,$$
$$c_0(z) = \cdots = c_{k-1}(z) = 0\}$$

so that we get a filtration of $\Gamma(\hat{A}, O_{\hat{A}})$

$$\Gamma(\hat{A}, O_{\hat{A}}) = \Gamma(\hat{A}, O_{\hat{A}})_0 \supset \Gamma(\hat{A}, O_{\hat{A}})_1 \supset \cdots.$$

Note that

$$\mathscr{I} = \Gamma(\hat{A}, O_{\hat{A}})_1$$

is the ideal of holomorphic functions on $\hat{A}$ vanishing on $A \times \{0\}$, and that the filtration considered is the $\mathscr{I}$-adic filtration

$$\Gamma(\hat{A}, O_{\hat{A}})_k = \mathscr{I}^k.$$

We have a natural isomorphism

$$j_k : \Gamma(A, O_A) \simeq \Gamma(\hat{A}, O_{\hat{A}})_k / \Gamma(\hat{A}, O_{\hat{A}})_{k+1}$$

given by

$$c(z) \to c(z)\xi^k.$$

Let $\mathscr{F}$ be a coherent sheaf on A and set

$$\hat{\mathscr{F}} = \pi^* \mathscr{F} \otimes_{O_A} O_{\hat{A}}$$

where $\pi : \hat{A} \to A$ is the natural projection.

We can consider $\Gamma(\hat{A}, \hat{\mathscr{F}})$ as a $\Gamma(\hat{A}, O_{\hat{A}})$-module and thus we can consider on $\Gamma(\hat{A}, \hat{\mathscr{F}})$ the induced filtration

$$\Gamma(\hat{A}, \hat{\mathscr{F}})_k = \mathscr{I}^k \Gamma(\hat{A}, \hat{\mathscr{F}}).$$

By the nature of the sheaf $\hat{\mathscr{F}}$ obtained by inverse image by π we get natural isomorphisms

$$j_k : \Gamma(A, \mathscr{F}) \simeq \Gamma(\hat{A}, \hat{\mathscr{F}})_k / \Gamma(\hat{A}, \hat{\mathscr{F}})_{k+1}.$$

Let us assume that we have on A a finite presentation of $\mathscr{F}$

$$O_A^l \xrightarrow{\tau} \mathscr{F} \to 0$$

so that $\Gamma(A, O_A)^l \to \Gamma(A, \mathscr{F})$ is surjective (since A is Stein).

Let $s_1, \cdots, s_l$ be the generators of $\Gamma(A, \mathscr{F})$, the images by τ of the sections $(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$ of $\Gamma(A, O_A)^l$. By tensoring over O_A by $O_{\hat{A}}$ we get then the exact sequence

$$O_{\hat{A}}^l \xrightarrow{\hat{\tau}} \hat{\mathscr{F}} \to 0$$

and correspondingly a surjective map $\Gamma(\hat{A}, O_{\hat{A}})^l \to \Gamma(\hat{A}, \hat{\mathscr{F}})$. If $\hat{s} \in \Gamma(\hat{A}, \hat{\mathscr{F}})$ then

$$\hat{s}(z, \xi) = \sum_{1}^{l}{}_i\, \alpha_i(z, \xi) s_i(z).$$

One has thus

$$\Gamma(\hat{A}, \hat{\mathscr{F}})_k = \{s \in \Gamma(\hat{A}, \hat{\mathscr{F}}) : s = \sum_{1}^{l}{}_i \alpha_i s_i, \alpha_i \in \Gamma(\hat{A}, O_{\hat{A}})_k, 1 \leqq i \leqq l\}.$$

b). Let $U = \{U_i\}_{i \in I}$ be a covering of the space X by open sets U_i which are holomorphically complete and such that $F_{|U_i}$ is trivial for each $i \in I$.

Let $\hat{U}_i = \pi^{-1}(U_i) \cong U_i \times \boldsymbol{C}$. Then $\hat{U} = \{\hat{U}_i\}_{i \in I}$ is a Stein covering of the bundle space F.

Let $\mathscr{F}$ be a coherent sheaf on X and let $\hat{\mathscr{F}} = \pi^* \mathscr{F} \otimes_{O_X} O_F$ be the inverse image sheaf on F.

We consider the cochain groups

$$C^r(\hat{U}, \hat{\mathscr{F}}) = \prod_{(i_0 \cdots i_r)} \Gamma(\hat{U}_{i_0} \cap \cdots \cap \hat{U}_{i_r}, \hat{\mathscr{F}}).$$

On each of the factors we have defined a filtration. This induces a filtration

$$C^r(\hat{U}, \hat{\mathscr{F}}) = C^r(\hat{U}, \mathscr{F})_0 \supset C^r(\hat{U}, \hat{\mathscr{F}})_1 \supset \cdots$$

of the cochain group which is compatible with coboundary operator δ.

If $\mathscr{F} = O$ we get a split exact sequence for every $k \geqq 1$

$$0 \to C^r(\hat{U}, O)_{k-1} \to C^r(\hat{U}, O)_k \underset{\sigma}{\overset{\alpha}{\rightleftarrows}} C^r(U, \Omega(F^{*k})) \to 0$$

the map α being defined by

$$f_{i_0 \cdots i_r} = \sum_{s=k}^{\infty} c_{i_0 \cdots i_r; s}(z) \xi_{i_r}^s \to c_{i_0 \cdots i_r; k}(z).$$

The splitting σ is obviously given as a left inverse of α by

$$c_{i_0 \cdots i_r}(z) \to c_{i_0 \cdots i_r}(z) \xi_{i_r}^k.$$

For a general sheaf $\mathscr{F}$ on X we thus have analogously split exact sequences

$$0 \to C^r(\hat{U}, \hat{\mathscr{F}})_{k-1} \to C^r(\hat{U}, \hat{\mathscr{F}})_k \underset{\sigma}{\rightleftarrows} C^r(U, \mathscr{F} \otimes \Omega(F^{*k})) \to 0.$$

The homomorphisms here considered are homomorphisms of the cochain complexes.

The filtration on the cochain complexes induce a filtration on the cohomology groups

$$H^r(F, \hat{\mathscr{F}}) = H^r(F, \hat{\mathscr{F}})_0 \supset H^r(F, \hat{\mathscr{F}})_1 \supset \cdots$$

where $H^r(F, \hat{\mathscr{F}})_k$ is the image of the cocycles of $C^r(\hat{U}, \hat{\mathscr{F}})_k$. From the above remark it follows that $H^r(F, \hat{\mathscr{F}})_k$ is also the r-th cohomology group of the cochain complex $\{C^*(\hat{U}, \hat{\mathscr{F}})_k, \delta\}$.

We thus have split exact sequences

$$0 \to H^r(F, \hat{\mathscr{F}})_{k-1} \to H^r(F, \hat{\mathscr{F}})_k \underset{\sigma_*}{\overset{\alpha_*}{\rightleftarrows}} H^r(X, \mathscr{F} \otimes \Omega(F^{*k})) \to 0.$$

In particular we get the following conclusions:

(i) for any coherent sheaf $\mathscr{F}$ on X the graded group associated to the filtration of $H^r(F, \hat{\mathscr{F}})$ is

$$GH^r(F, \hat{\mathscr{F}}) \cong \coprod_{k=0}^{\infty} H^r(X, \mathscr{F} \otimes \Omega(F^{*k}))$$

(ii) we have a natural injection

$$\coprod_{k=0}^{\infty} H^r(X, \mathscr{F} \otimes \Omega(F^{*k})) \to H^r(F, \hat{\mathscr{F}})$$

so that

$$\dim_C GH^r(F, \hat{\mathscr{F}}) \leqq \dim_C H^r(F, \hat{\mathscr{F}}).$$

3. Compact base space (cf. [1])

If X is compact then

(i) *if F is metrically pseudoconcave, given any coherent sheaf $\mathscr{F}$ on X there exists an integer $k_0 = k_0 (\mathscr{F}, F)$ such that*

$$H^r(X, \mathscr{F} \otimes \Omega(F^k)) = 0$$

for $k \geqq k_0$ and any $r > 0$

(ii) *if F is metrically pseudoconvex, given any coherent sheaf $\mathscr{F}$ on X there exists an integer $k_0 = k_0(\mathscr{F}, F)$ such that*

$$H^r(X, \mathscr{F} \otimes \Omega(F^k)) = 0$$

for $k \geqq k_0$ and $r <$ prof $(\mathscr{F})$.

In case (i) the bundle space of F^* is a strongly 0-pseudoconvex space. Thus by [1]

$$\dim_C H^r(F^*, \hat{\mathscr{F}}) < +\infty$$

for $r > 0$. From the end of the previous section we derive the conclusion.

In case (ii) the bundle space F^* is a strongly 0-pseudoconcave space. Moreover we have

$$\text{prof}\,(\hat{\mathscr{F}}) = \text{prof}\,(\mathscr{F})+1.$$

The conclusion then follows from the finiteness theorem [1]:

$$\dim_{\boldsymbol{C}} H^r(F^*, \hat{\mathscr{F}}) < +\infty$$

for $r < \text{prof}\,(\hat{\mathscr{F}})-1$.

4. Pseudoconvex base space

a). For a C^∞ function on a complex manifold we adopt the notion of "strongly q-pseudoconvex" as given in [1] or [2].

As usual this notion is carried to complex spaces as follows.

A function $\phi : X \to \boldsymbol{R}$ will be called *strongly q-pseudoconvex* at a point $x_0 \in X$ if we can find

(i) an embedding τ of a neighborhood V of $x_0 \in X$ into an analytic set of some open neighborhood U of the origin of the Zariski tangent space T_{x_0} to X at x_0

(ii) a strongly q-pseudoconvex function $\hat{\phi} : U \to \boldsymbol{R}$ such that

(a$_1$) $\phi_{|V} = \hat{\phi} \circ \tau$

(a$_2$) $\overline{\{x \in V : \phi(x) < c\}} = \{x \in V : \phi(x) \leqq c\}$ for every $c \in \boldsymbol{R}$.

A complex space X is called *strongly q-pseudoconvex* if we can find a compact set $K \subset X$ and a C^∞ function $\phi : X \to \boldsymbol{R}$ such that

(i) for every $c < \sup_X \phi$ the sets

$$B_c = \{x \in X : \phi(x) < c\}$$

are relatively compact

(ii) on $X-K$, ϕ is strongly q-pseudoconvex.

For a strongly q-pseudoconvex space one has the following theorem of finiteness for cohomology (cf. [1]): for any coherent sheaf $\mathscr{F}$ on X

$$\dim_{\boldsymbol{C}} H^r(X, \mathscr{F}) < +\infty$$

if $r > q$.

b). Proposition 1. *The bundle space of a metrically pseudoconvex line bundle F over a strongly q-pseudoconvex space X is a strongly q-pseudoconvex space.*

Proof. With the notations introduced we consider on F the function

$$\Theta = \pi^* \phi + \mu(\chi)$$

where μ is a C^∞ function defined for $0 \leqq t < +\infty$ such that

$$\mu(t) \geqq 0, \quad \mu'(t) > 0, \quad \mu''(t) \geqq 0$$

(i.e. positive, increasing, convex).

Without loss of generality we may assume $\phi \geqq 0$ and $\sup_X \phi = +\infty$.

For every $c \in \boldsymbol{R}$ we have

$$B_c = \{v \in F : \Theta(v) < c\}$$
$$\subset \pi^{-1}\{x \in X : \phi(x) < c\} \cap \{v \in F : \chi(v) < \mu^{-1}(c)\}.$$

Since $\lim_{t\to+\infty}\mu(t) = +\infty$ and since π restricted on the set $\{v \in F : \chi(v) \leqq \text{const}\}$ is a proper map, it follows that the sets B_c are relatively compact.

Let U be an open set in X. If U is sufficiently small we may assume that

(i) $F_{|U} \cong U \times \boldsymbol{C}$

(ii) U is an analytic subset of some open set $G \subset \boldsymbol{C}^N$ and there exists a C^∞ strongly q-pseudoconvex function $\hat{\phi} : G \to \boldsymbol{R}$ such that $\phi = \hat{\phi}_{|U}$

(iii) $\chi_{|U\times\boldsymbol{C}} = h(z)|\xi|^2$ and exists a C^∞ function $\hat{h}$ on G such that $\hat{\chi} = \hat{h}(z)|\xi|^2$ is strongly 0-pseudoconvex on $G \times \boldsymbol{C}$ ($z \in G$, $\xi \in \boldsymbol{C}$, $\xi \neq 0$).

Let $G' \Subset G$; there exists a constant $c = c(G')$ such that if $\mu' > c$ the function

$$\hat{\Theta} = \hat{\phi} + \mu(\hat{\chi})$$

is strongly 0-pseudoconvex on the set

$$\{(z, \xi) \in G' \times \boldsymbol{C} : \hat{\chi} \geqq 1\}.$$

This is straightforward verification.

Making use of this remark we may select μ in such a way that on

$$\pi^{-1}(K) \cap \{v \in F : \chi(v) \geqq 1\}$$

the function Θ be strongly 0-pseudoconvex.

On each point of F outside of the compact set

$$C = \{v \in F : \chi(v) \leqq 1\} \cap \pi^{-1}(K)$$

the function Θ is strongly q-pseudoconvex. This follows from the fact that everywhere on F (including the 0-section) the Levi form $\mathscr{L}(\hat{\chi})$ has one positive eigenvalue in the direction of the fibre.

Finally the verification that $\hat{\Theta}$ has the property (a_2) stated in a) follows from the fact that $\hat{\chi}$ is an open map on each fibre.

Arguing now as we did for the case where X is compact and using the theorem of finiteness quoted before for the bundle space of F, we get the following

THEOREM 1. *Let X be a strongly q-pseudoconvex space. Let F be a metrically pseudoconcave line bundle on X and let $\mathscr{F}$ be a coherent sheaf on X. There exists an integer $k_0 = k(\mathscr{F}, F)$ such that*

$$H^r(X, \mathscr{F} \otimes \Omega(F^k)) = 0$$

for $k \geqq k_0$ and $r > q$.

5. Pseudoconcave base space

a). A complex space X is called *strongly q-pseudoconcave* if we can find a compact set $K \subset X$ and a C^∞ function $\phi : X \to \boldsymbol{R}$ such that

(i) for any $c > \inf_X \phi$ the sets

$$B_c = \{x \in X : \phi(x) > c\}$$

are relatively compact in X,

(ii) for each point $x_0 \in X - K$ we can find a positive constant $k(x_0)$ such that $e^{k\phi}$ is strongly q-pseudoconvex for $k \geqq k(x_0)$ in a neighborhood of x_0.

As in [1] one establishes for strongly q-pseudoconcave spaces the following theorem of finiteness for cohomology: for any coherent sheaf $\mathscr{F}$ on X one has

$$\dim_{\boldsymbol{C}} H^r(X, \mathscr{F}) < +\infty$$

for $r < \text{prof}\,(\mathscr{F}) - q - 1$.

b). PROPOSITION 2. *The bundle space of a metrically pseudoconcave line bundle F over a strongly q-pseudoconcave space X is strongly $(q+1)$-pseudoconcave space.*

PROOF. By the hypothesis we can choose on F^* a metric on the fibres such that the function

$$||v^*||^2 = \chi_*(z, \eta) = h(z)|\eta|^2$$

is 0-pseudoconvex outside of the 0-section.

On F we can consider the function

$$||v||^{-2} = h(z)|\xi|^{-2}$$

where ξ is the fibre coordinate on F.

Let $\mu(t)$ be a C^∞ function defined for $0 < t < +\infty$ such that

$\mu(t) > 0$, $\mu(t) = t$ if $0 < t < \frac{1}{2}$, $\mu(t) = \frac{1}{2}$ if $t \geqq 1$, $\mu'(t) \geqq 0$, and set $g = \mu(\|v\|^{-2})$.

On the bundle space F we then define the function

$$\Theta = \frac{\pi^*(\phi) \cdot g}{e^{\pi^*(\phi)+g}}.$$

It is no loss of generality to assume that $\inf_X \phi = 0$ and $\phi \leqq \frac{1}{2}$ on X.

First we remark that $e^{\pi^*(\phi)+g} > \sup \{\pi^*(\phi), g\}$ so that

$$\Theta < \frac{\pi^*(\phi) \cdot g}{\sup \{\pi^*(\phi), g\}} = \inf \{\pi^*(\phi), g\}.$$

Therefore the sets

$$\{v \in F : \Theta(v) > c\}$$

are relatively compact for $c > 0$.

For U sufficiently small on X we may make the assumptions (i), (ii), (iii) given in the proof of proposition 1. We thus have to compute the pseudoconvexity of the function

$$\hat{\Theta} = \frac{\hat{\phi}\hat{g}}{e^{\hat{\phi}+\hat{g}}}$$

where $\hat{g} = \mu(\hat{h}(z)|\xi|^{-2})$.

Writing $\hat{\Theta} = \exp \log \hat{\Theta}$ we get for the Levi form the expression

$$\mathscr{L}(\hat{\Theta}) = \hat{\Theta} \left\{ |\partial \log \hat{\Theta}|^2 + \left(\frac{1}{\hat{\phi}} - 1\right) \partial\bar{\partial}\hat{\phi} + \left(\frac{1}{\hat{g}} - 1\right) \partial\bar{\partial}\hat{g} - \frac{1}{\hat{\phi}^2} |\partial\hat{\phi}|^2 - \frac{1}{\hat{g}^2} |\partial\hat{g}|^2 \right\}.$$

Let $v = (z, \xi) \in \pi^{-1}(U)$ and restrict the Levi form to the directions at v for which

$$\partial\hat{\phi} = \partial\hat{g} = 0.$$

If ξ is sufficiently large, then the eigenvalues of $((1/\hat{g}) - 1)\, \partial\bar{\partial}\hat{g}$ will be prevalent and the Levi form will be positive definite. If on the other hand $U \cap K = \emptyset$, the Levi form of $\hat{g}$ on $\partial\hat{g} = 0$ is positive so that we do keep the $n-q-1$ positive eigenvalues coming from $\hat{\phi}$.

In conclusion, we can find a compact set $C \in \pi^{-1}(K)$ on F such that for any point $v_0 \notin C$ for large values of $k \geqq k(v_0)$ the function $e^{k\Theta}$ is strongly $(q+1)$-pseudoconvex.

Arguing now as in the compact case we obtain the following

THEOREM 2. *Let X be a strongly q-pseudoconcave space and let F be a metrically pseudoconvex line bundle on X. Let $\mathscr{F}$ be a coherent sheaf on X. There exists an integer $k_0 = k_0(\mathscr{F}, F)$ such that*

$$H^r(X, \mathscr{F} \otimes \Omega(F^k)) = 0$$

for $k \geqq k_0$ and $r <$ prof $(\mathscr{F})-q-1$.

6. An application

a). By a *q-corona* we mean a complex space X endowed with a C^∞ and positive function $\phi : X \to \boldsymbol{R}$ having the following properties

(i) for every $\varepsilon > 0$, $c > 0$ the sets

$$X_{\varepsilon, c} = \{x \in X : \varepsilon < \phi(x) < c\}$$

are relatively compact

(ii) ϕ is a strongly q-pseudoconvex function outside of some compact subset $K \subset X$.

For a q-corona one has the following theorem of finiteness for cohomology

THEOREM 3. *Let X be a q-corona and let $\mathscr{F}$ be a coherent sheaf on X. Then*

$$\dim_{\boldsymbol{C}} H^r(X, \mathscr{F}) < +\infty$$

if $q < r <$ prof $(\mathscr{F})-q-1$.

The proof of this theorem is obtained by the procedure of [1] combining the arguments for the pseudoconvex and pseudoconcave case.

Precisely one establishes (cf. [1] propositions 16 and 17) that given ε, c, with $\varepsilon < \inf_K \phi$ and $c > \sup_K \phi$ we can find $0 < \varepsilon' < \varepsilon$, $c' > c$ such that the restriction map

$$H^r(X_{\varepsilon', c'}, \mathscr{F}) \to H^r(X_{\varepsilon, c}, \mathscr{F})$$

is surjective if $q < r <$ prof $(\mathscr{F})-q-1$.

This implies the finiteness of $\dim_{\boldsymbol{C}} H^r(X_{\varepsilon, c}, \mathscr{F})$ for

$$q < r < \text{prof}\,(\mathscr{F})-q-1.$$

Then one establishes the "Runge theorem" and obtains the complete statement of the theorem (cf. theorems 12 and 13 of [1]).

Can we establish a vanishing theorem on a q-corona for the same ranges of r for which we have the theorem of finiteness? To this question we give a partial answer in what follows.

b). We consider the following situation: Z is a compact complex space, Y_1 and Y_2 are two open subsets of Z such that

(i) $Z = Y_1 \cup Y_2$
(ii) Y_1 is q-complete and Y_2 is q-pseudoconcave
(iii) $X = Y_1 \cap Y_2$ is a q-corona.

THEOREM 4. *Let X, Y_1, Y_2, Z be as above, let $\mathscr{F}$ be a coherent sheaf on Z, let F be a metrically pseudoconvex line bundle on Z. Then there exists an integer $k_0 = k_0(\mathscr{F}, F)$ such that*

$$H^r(X, \mathscr{F} \otimes \Omega(F^k)) = 0$$

for $k \geqq k_0$ and $q < r <$ $\operatorname{prof}_Z(\mathscr{F})-q-1$.

PROOF. From the Mayer-Vietoris sequence we get the exact cohomology sequence

$$\begin{aligned} \cdots &\to H^r(Z, \mathscr{F} \otimes \Omega(F^k)) \\ &\to H^r(Y_1, \mathscr{F} \otimes \Omega(F^k)) \oplus H^r(Y_2, \mathscr{F} \otimes \Omega(F^k)) \\ &\to H^r(X, \mathscr{F} \otimes \Omega(F^k)) \to H^{r+i}(Z, \mathscr{F} \otimes \Omega(F^k)) \to \cdots. \end{aligned}$$

If k is large by the theorem of n . 3

$$H^r(Z, \mathscr{F} \otimes \Omega(F^k)) = 0 = H^{r+1}(Z, \mathscr{F} \otimes \Omega(F^k))$$

if $r < \operatorname{prof}_Z(\mathscr{F})-1$.

Since Y_1 is q-complete

$$H^r(Y_1, \mathscr{F} \otimes \Omega(F^k)) = 0$$

if $r > q$.

Therefore

$$H^r(X, \mathscr{F} \otimes \Omega(F^k)) \cong H^r(Y_2, \mathscr{F} \otimes \Omega(F^k)).$$

Now the second of these groups vanishes for large k if $r < \operatorname{prof}_{Y_2}(\mathscr{F})-q-1$ according to theorem 2.

Collecting together all the above information we get the statement of theorem 4.

EXAMPLE. Let

$$X = \{z \in \boldsymbol{C}^n : a < \sum_1^n{}_i\, z_i \bar{z}_i < b,\ a < b\}.$$

It can be obtained as intersection of two open sets in $\boldsymbol{P}_n(\boldsymbol{C})$, a ball and the complement of a concentric ball. Let F be the Hopf bundle. Since $F_{|X}$ is trivial we obtain for any coherent sheaf $\mathscr{F}$ on $\boldsymbol{P}_n(\boldsymbol{C})$

$$H^r(X, \mathscr{F}) = 0$$

for $0 < r < \operatorname{prof}_{\boldsymbol{P}_n}(\mathscr{F})-1$.

c) Another vanishing theorem for a q-corona is as follows.

Let X be a q-pseudoconvex space for which we adopt the notations of $n \cdot 4$. Let $c < \inf_K \phi$ and let

$$X_c = \{x \in X : \phi(x) > c\}.$$

If $\inf_K \phi > \inf_X \phi$ this is a q-corona. For this type of coronas we have:

THEOREM 5. *Let F be a metrically pseudoconcave line bundle over X and $\mathscr{F}$ any coherent sheaf on X. There exists an integer $k_0 = k_0(\mathscr{F}, F)$ such that*

$$H^r(X_c, \mathscr{F} \otimes \Omega(F^k)) = 0$$

if $k \geqq k_0$ and $q < r < \operatorname{prof} \mathscr{F} - q - 1$.

PROOF. First one establishes that for $\varepsilon > 0$ and sufficiently small

$$H^r(X_{c-\varepsilon}, \mathscr{F} \otimes \Omega(F^k)) \cong H^r(X_c, \mathscr{F} \otimes \Omega(F^k)) \tag{1}$$

for $r < \operatorname{prof} \mathscr{F} - q - 1$ (cf. a)) (actually we need only the surjectivity of the restriction map).

Secondly by theorem 1 we can find $k_0 = k_0(\mathscr{F}, F)$ such that

$$H^r(X, \mathscr{F} \otimes \Omega(F^k)) = 0 \tag{2}$$

for $k \geqq k_0$ and $r > q$.

Let

$$B_{c-\varepsilon/2} = \left\{x \in X : \phi(x) < c - \frac{\varepsilon}{2}\right\}.$$

Let $\xi \in H^r(X_c, \mathscr{F} \otimes \Omega(F^k))$. By (1) we can find

$$\eta \in H^r(X - B_{c-\varepsilon/2}, \mathscr{F} \otimes \Omega(F^k))$$

such that by the natural restriction map

$$r_{X_c}^{X-B_{c-\varepsilon/2}}(\eta) = \xi.$$

Now by theorem 15 of [1] if $r < \operatorname{prof} \mathscr{F} - q - 1$ we can find

$$\hat{\eta} \in H^r(X, \mathscr{F} \otimes \Omega(F^k))$$

such that

$$r_{X_c}^{X}(\hat{\eta}) = \xi.$$

By (2) $\hat{\eta} = 0$ thus also $\xi = 0$.

For instance in the exemple given in b) if we apply this theorem we get the same conclusion but under less restrictive conditions i.e. $\mathscr{F}$ needs to be defined only in the ball

$$B = \{z \in \mathbf{C}^n : \sum_1^n {}_i z_i \bar{z}_i < b\}$$

and the vanishing of $H^r(X, \mathscr{F})$ is for $0 < r < \text{prof}_B \mathscr{F} - 1$.

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(Oblatum 12–5–69)