

# COMPOSITIO MATHEMATICA

D. W. CURTIS

**Property  $Z$  for function-graphs and finite-dimensional sets in  $I^\infty$  and  $s$**

*Compositio Mathematica*, tome 22, n° 1 (1970), p. 19-22

[http://www.numdam.org/item?id=CM\\_1970\\_\\_22\\_1\\_19\\_0](http://www.numdam.org/item?id=CM_1970__22_1_19_0)

© Foundation Compositio Mathematica, 1970, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## Property Z for function-graphs and finite-dimensional sets in $I^\infty$ and $s$

by

D. W. Curtis

For each  $i > 0$ , let  $I_i = [-1/i, 1/i]$ ,  ${}^0I_i = (-1/i, 1/i)$ . The Hilbert cube  $I^\infty$  is the product  $\prod_{i>0} I_i$ , with pseudo-interior  $s = \prod_{i>0} {}^0I_i$  (also denoted by  ${}^0I^\infty$ ). Let  $I^\alpha = \{x \in I^\infty : x_i = 0 \text{ if } i \notin \alpha\}$ , similarly for  ${}^0I^\alpha$ , and let  $\pi_i, \tau^\alpha$  denote the projections onto  $I_i, I^\alpha$ , respectively. For  $\alpha = \{1, \dots, n\}$  we write  $I^\alpha = I^n, \tau^\alpha = \tau^n$ . We use the metric  $d(x, y) = \max_{i>0} |x_i - y_i|$  in  $I^\infty$ , and the corresponding supremum metric  $d^*$  for maps into  $I^\infty$ .

**DEFINITION** (Anderson [1]). A closed subset  $K$  of  $X$  has *Property Z* in  $X$  if for every nonempty, homotopically trivial open set  $U$  in  $X$ ,  $U \setminus K$  is nonempty and homotopically trivial.

We show that the graph  $G(f)$  of a continuous function  $f: I^\infty \rightarrow I^\infty$  has Property Z in  $I^\infty \times I^\infty$ , and that a closed subset  $K \subset s$  with dimension  $k$  and finite deficiency  $2k+1$  has Property Z in  $s$ .

**LEMMA 1.** *Let  $K$  be a closed subset of  $s$ . Suppose that for every  $n \geq 0$ , every map  $f: I^n \rightarrow s$ , and every  $\varepsilon > 0$ , there exists a map  $g: I^n \rightarrow s$  such that  $d^*(f, g) < \varepsilon$  and  $g(I^n) \cap K = \emptyset$ . Then  $K$  has Property Z.*

**PROOF.** We show that for every map  $f: I^n \rightarrow s$  with  $f(Bd I^n) \cap K = \emptyset$ , and every  $\varepsilon > 0$ , there exists a map  $g: I^n \rightarrow s$  such that  $f/Bd I^n = g/Bd I^n$ ,  $d^*(f, g) < \varepsilon$ , and  $g(I^n) \cap K = \emptyset$ . Clearly, this implies Property Z. Choose an  $n$ -cube  $J^n$  in  ${}^0I^n$  such that  $d(f(I^n \setminus J^n), K) > 0$ . Let  $0 < \eta < \min \{d(f(I^n \setminus J^n), K), \varepsilon\}$ , and let  $g_0: J^n \rightarrow s$  be a map such that  $d^*(f/J^n, g_0) < \eta$  and  $g_0(J^n) \cap K = \emptyset$ . Extend the map  $f/Bd I^n \cup g_0$  to a map  $g: I^n \rightarrow s$  such that  $d^*(f, g) < \eta$ . Then  $g(I^n) \cap K = \emptyset$ .

The condition of the lemma is actually equivalent to Property Z. Moreover, since a closed subset  $K$  of  $I^\infty$  has Property Z in  $I^\infty$  if (and only if)  $K \cap s$  has Property Z in  $s$  (Anderson [1]), and since there exist arbitrarily small maps of  $I^\infty$  into  $s$ , the lemma remains true when  $s$  is everywhere replaced by  $I^\infty$ .

**LEMMA 2.** *Let maps  $f, g : I^n \rightarrow I^{3^n}$ , and  $\varepsilon > 0$  be given. Then there exists a map  $h : I^n \rightarrow I^{3^n}$  such that  $d^*(g, h) < \varepsilon$  and  $f(x) \neq h(x)$  for every  $x$ .*

**PROOF.** Assume  $\varepsilon < 1$ . Choose  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $d(f(x), f(y)), d(g(x), g(y)) < \varepsilon/3^{n+1}$ . On each interval  $I_i$ ,  $i = 1, \dots, n$ , take a regular subdivision with mesh less than  $\delta/n$ , and consider the resulting product subdivision  $\{J_1^n, \dots, J_r^n\}$  of  $I^n$  into  $n$ -cubes. Each cube has diameter less than  $\delta$ , and no cube meets more than  $3^n - 1$  other cubes. Thus to each cube  $J_j^n$  we may assign an integer  $i_j$ ,  $1 \leq i_j \leq 3^n$ , such that  $i_k = i_m$  only if  $J_k^n \cap J_m^n = \emptyset$ . For each cube  $J_j^n$ , select a point  $p_j$  in the corresponding interval  $I_{i_j}$ , such that  $p_j \notin \pi_{i_j} f(J_j^n)$  and

$$d(p_j, \pi_{i_j} g(J_j^n)) < \varepsilon/3^{n+1}.$$

Let  $V$  be the collection of vertices of the cubes  $\{J_j^n\}$ , and define a function  $h_0 : V \rightarrow I^{3^n}$  as follows:  $\pi_{i_j} h_0(v) = p_j$  if  $v \in J_j^n$ , otherwise  $\pi_{i_j} h_0(v) = \pi_{i_j} g(v)$ . Extend  $h_0$  piecewise-linearly to  $h : I^n \rightarrow I^{3^n}$ .

The lemma holds true with  $I^{3^n}$  replaced by  $I^{n+1}$ , but for our purposes the sharper result is unnecessary.

**THEOREM 1.** *Let  $f : I^\infty \rightarrow I^\infty$  be a map. Then  $G(f)$  has Property  $Z$  in  $I^\infty \times I^\infty$ .*

**PROOF.** Let  $\tau_1^\infty, \tau_2^\infty, \tau_2^n$  be the projections of  $I^\infty \times I^\infty$  onto  $I^\infty \times 0, 0 \times I^\infty, 0 \times I^n$ , respectively. Suppose a map  $g : I^n \rightarrow I^\infty \times I^\infty$  and  $\varepsilon > 0$  are given. Consider the maps  $\tau_2^{3^n} \circ f \circ \tau_1^\infty \circ g, \tau_2^{3^n} \circ g : I^n \rightarrow I^{3^n}$ . By Lemma 2 there exists a map  $h : I^n \rightarrow I^{3^n}$  such that  $d^*(\tau_2^{3^n} \circ g, h) < \varepsilon$  and  $h(x) \neq (\tau_2^{3^n} \circ f \circ \tau_1^\infty \circ g)(x)$  for every  $x$ . Consider the map  $\bar{g} : I^n \rightarrow I^\infty \times I^\infty$ , defined by  $\tau_1^\infty \circ \bar{g} = \tau_1^\infty \circ g, \tau_2^{3^n} \circ \bar{g} = h$ , and  $\pi_i \circ \tau_2^\infty \circ \bar{g} = \pi_i \circ \tau_2^\infty \circ g$  for  $i > 3^n$ . We have  $d^*(g, \bar{g}) < \varepsilon$  and  $\bar{g}(I^n) \cap G(f) = \emptyset$ . By the remark following Lemma 1,  $G(f)$  has Property  $Z$ .

In  $I^\infty$  (and in  $s$ ), homeomorphisms between closed subsets with Property  $Z$  can be extended to space homeomorphisms (Anderson [1]). Since  $I^\infty \times 0$  has Property  $Z$  in  $I^\infty \times I^\infty$ , the above result guarantees the existence of a homeomorphism  $H$  of  $I^\infty \times I^\infty$  onto itself, such that  $H/G(f) = \tau_1^\infty/G(f)$ . In general, we cannot require that  $\tau_1^\infty \circ H = \tau_1^\infty$ . (With such an  $H$  for  $f = id$ , the map  $h : I^\infty \rightarrow I^\infty$  defined by  $h(x) = \tau_2^\infty \circ H^{-1}(x, p)$ , where  $0 \neq p \in I^\infty$ , would have no fixed point).

A closed subset  $K$  of  $I^\infty$  or  $s$  has *finite deficiency*  $k$  if  $\tau^k(K) = 0$ .

**COROLLARY 1.** *If  $K \subset I^\infty$  has finite deficiency  $k$  and can be imbedded in  $I^k$ , then it has Property  $Z$ .*

**PROOF.** Let  $h : K \rightarrow {}^0I^k$  be an imbedding, and let  $H$  be a homeomorphism of  $I^\infty$  onto itself, such that  $\pi_i \circ H = \pi_i$  for  $i > k$  and  $\tau^k \circ H/K = h$ . Partition the set of positive integers into two infinite classes  $\alpha, \beta$ , with  $\{1, \dots, k\} \subset \alpha$ , and consider  $I^\infty = I^\alpha \times I^\beta$ . Clearly,  $H(K) \subset G(f)$  for an appropriate map  $f : I^\alpha \rightarrow I^\beta$ , and since Property  $Z$  is hereditary (Lemma 1),  $H(K)$ , and therefore  $K$ , has Property  $Z$ .

**COROLLARY 2.** *A closed subset  $K$  in  $I^\infty$  with dimension  $k$  and finite deficiency  $2k+1$  has Property  $Z$ .*

This result for the case  $k = 0$  relates to the question, raised by Anderson, of whether a closed set with Property  $Z$  in  $I^\infty$  (or  $s$ ) intersects a hyperplane of deficiency one in a set with Property  $Z$  in the hyperplane. Pełczynski suggested as a possible counterexample a wild Cantor set in the hyperplane  $\{0\} \times \prod_{i>1} I_i$ , and Anderson verified that such a set does have Property  $Z$  in  $I^\infty$ .

Since, with  $K$  as above,  $K \cap s$  has dimension less than or equal to  $k$  and finite deficiency  $2k+1$ , the following theorem may be regarded as a generalization of Corollary 2.

**THEOREM 2.** *A closed subset  $K$  in  $s$  with dimension  $k$  and finite deficiency  $2k+1$  has Property  $Z$ .*

**PROOF.** Let a map  $f : I^n \rightarrow s$  and  $\varepsilon > 0$  be given. Let  $\gamma = \{2k+2, \dots, 2k+n+2\}$ ,  $\gamma' = \{1, \dots, 2k+1\} \cup \{2k+n+3, \dots\}$ . Choose  $0 < \delta < \min \{1/2(2k+n+2), \varepsilon/12\}$ . Since  $(s, d)$  is totally bounded there exists a finite open cover of  $K$  with mesh less than  $\delta$  and  $\text{ord} \leq k$ . Let  $L$  be a realization in

$$U = \{x \in {}^0I^{2k+1} : d(x, 0) < \varepsilon/4\}$$

of the abstract nerve of the cover. Let  $\bar{m} : \prod_{i>2k+1} {}^0I_i \rightarrow U$  be an extension of the barycentric map  $m : K \rightarrow L$ . Then  $\bar{m}$  defines a homeomorphism  $h$  of  $s$  onto itself, such that  $d^*(h, id) \leq \varepsilon/4$ ,  $\pi_i \circ h = \pi_i$  for  $i > 2k+1$ , and  $\tau^{2k+1} \circ h/K = m$ . Let  $g : I^n \rightarrow s$  be a piecewise-linear map with  $d^*(g, h \circ f) < \varepsilon/4$ . We construct a map  $r : g(I^n) \rightarrow s$  such that  $d^*(r, id) < \varepsilon/4$  and  $r(g(I^n)) \cap h(K) = \emptyset$ ;  $r$  will change coordinates only in the directions given by  $\gamma$ , and will be independent of the coordinates in the directions given by  $\{2k+n+3, \dots\}$ . Let  $\{\sigma_i\}_1^q$  be an ordering of the simplices of  $L$ , with  $\dim \sigma_i \leq \dim \sigma_j$  if  $i \leq j$ . Choose a closed cover  $\{B_i\}_1^q$  of

$L$ , such that  $B_i \subset {}^0st \sigma_i$ , and  $B_i \cap B_j \neq \emptyset$  only if  $\sigma_i \subset \sigma_j$  or  $\sigma_j \subset \sigma_i$ . Let  $\eta > 0$  be such that  $d(B_i, B_j) > 2\eta$  if  $B_i \cap B_j = \emptyset$ . For any subset  $D$  of  ${}^0I^\gamma$  with  $\text{diam } D < \delta$ , let  $[D] = \prod_{i \in \gamma} J_i$ , where  $J_i = [\inf \pi_i(D) - \delta, \sup \pi_i(D) + \delta] \cap I_i$ . Let

$$\alpha(D) = \{i \in \gamma : \inf J_i = -1/i\},$$

and

$$\beta(D) = \{i \in \gamma : \sup J_i = 1/i\};$$

then  $\alpha \cap \beta = \emptyset$ . If  $\alpha \cup \beta = \emptyset$ , and  $y \in {}^0[D] = \prod_{i \in \gamma} {}^0J_i$ , let  $p(y) : [D] \setminus y \rightarrow Bd[D]$  be the projection from  $y$ . Otherwise, for each  $0 < \xi < \delta$ , define

$$[D]_\xi = \prod_{i \in \alpha} [-1/i + \xi, \sup J_i] \times \prod_{i \in \beta} [\inf J_i, 1/i - \xi] \times \prod_{i \notin \alpha \cup \beta} J_i,$$

and let  $p(\xi) : [D]_\xi \rightarrow Bd[D]_\xi \cap Bd[D]$  be the projection from a point in  ${}^0[D] \setminus [D]_\xi$ . In either case, there exists a homotopy  $\{p_t\}_{t \geq 0}$  of  $[D] \setminus y$  into  $[D] \setminus y$ , of  $[D]_\xi$  into  $[D]_\xi$ , with  $p_0 = p(y)$ ,  $p(\xi)$ , respectively, and  $p_t = id$  for  $t \geq 1$ . Extend each  $p_t$  by the identity on  ${}^0I^\gamma \setminus [D]$ . For each  $\sigma_i \subset L$ , let  $D_i = \tau^\gamma(m^{-1}({}^0st \sigma_i))$ . We successively define maps

$$r_1 : g(I^n) \rightarrow s, \dots, r_a : (r_{a-1} \circ \dots \circ r_1 \circ g)(I^n) \rightarrow s$$

as follows. If  $\alpha(D_i) \cup \beta(D_i) = \emptyset$ , choose

$$y \in {}^0[D_i] \setminus \tau^\gamma(r_{i-1} \circ \dots \circ r_1 \circ g)(I^n).$$

Otherwise, choose  $0 < \xi < \delta$  such that

$$\tau^\gamma(r_{i-1} \circ \dots \circ r_1 \circ g)(I^n) \cap [D_i] \subset [D_i]_\xi.$$

In either case, let  $\{p_t\}_{t \geq 0}$  be the homotopy noted above, and define  $r_i : (r_{i-1} \circ \dots \circ r_1 \circ g)(I^n) \rightarrow s$  by  $\tau^\gamma(r_i(x)) = \tau^\gamma(x)$ , and  $\tau^\gamma(r_i(x)) = p_t(\tau^\gamma(x))$ , where

$$t = d(\tau^{2k+1}(x), B_i)/\eta.$$

Let  $r = r_a \circ \dots \circ r_1$ . Then  $(h^{-1} \circ r \circ g)(I^n) \cap K = \emptyset$  and

$$d^*(h^{-1} \circ r \circ g, f) < \varepsilon.$$

#### REFERENCE

R. D. ANDERSON

[1] On topological infinite deficiency, Michigan Math. J. 14 (1967), 365—383.

(Oblatum 5-IV-69)

Department of Mathematics  
Louisiana State University  
Baton Rouge, Louisiana USA