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## Meromorphic functions of regular growth

by

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### 0

Let  $f(z)$  be a meromorphic function of order  $\rho$  ( $0 \leq \rho < \infty$ ) and lower order  $\lambda$  ( $0 \leq \lambda < \infty$ ). Let  $T(r, f)$ ,  $n(r, a)$ ,  $N(r, a)$ , ( $0 \leq a \leq \infty$ ) have their usual meanings.

### 1

Let  $\lambda(r)$  be proximate order  $H$  of  $f(z)$  satisfying the following conditions:

- 1.1  $\lambda(r)$  is non-negative, continuous function of  $r$  for  $r \geq r_0$ .
- 1.2  $\lambda(r)$  is differentiable except at isolated points at which  $\lambda'(r-0)$  and  $\lambda'(r+0)$  exist.
- 1.3  $\lambda(r) \rightarrow \lambda$  as  $r \rightarrow \infty$
- 1.4  $r\lambda'(r) \log r = o(\lambda(r))$  as  $r \rightarrow \infty$
- 1.5  $T(r, f) \geq r^{\lambda(r)}$  for all  $r \geq r_0$
- 1.6  $T(r, f) = r^{\lambda(r)}$  for a sequence of values of  $r \rightarrow \infty$ .

From 1.1–1.4 we can easily deduce the following properties

- 1.7  $r^{\lambda(r)}$  is an increasing function of  $r \geq r_0$  [3]
- 1.8  $(ur)^{\lambda(ur)} \sim u^\lambda r^{\lambda(r)}$  for  $r \geq r_0$ .

From 1.1–1.6 we can easily prove that

$$1.9 \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} = 1.$$

<sup>1</sup> This paper was presented in the 75th annual meeting of The American Mathematical Society, at New Orleans, Louisiana, 1969.

## 2

Let  $\rho(r)$  be proximate order  $D$  of  $f(z)$  satisfying the following conditions:

- 2.1  $\rho(r)$  is non-negative, continuous function of  $r$  for  $r \geq r_0$
- 2.2  $\rho(r)$  is differentiable except at isolated points at which  $\rho'(r-0)$  and  $\rho'(r+0)$  exist.
- 2.3  $\rho(r) \rightarrow \rho$  as  $r \rightarrow \infty$
- 2.4  $r\rho'(r) \log r = o(\rho(r))$  as  $r \rightarrow \infty$
- 2.5  $T(r, f) \leq r^{\rho(r)}$  for all  $r \geq r_0$
- 2.6  $T(r, f) = r^{\rho(r)}$  for a sequence of values of  $r \rightarrow \infty$

From 2.1–2.4 we can easily deduce the following properties.

- 2.7  $r^{\rho(r)}$  is an increasing function of  $r \geq r_0$  [3]
- 2.8  $(ur)^{\rho(ur)} \sim u^\rho r^{\rho(r)}$  for  $r \geq r_0$

From 2.1–2.6 we can easily prove

$$2.9 \quad \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} = 1.$$

Since in this paper we study the asymptotic properties, the behaviour of proximate orders plays no role on a finite interval and we shall assume that  $r^{\lambda(r)}$  and  $r^{\rho(r)}$  are monotonic functions for  $r \geq 0$  and we define them to be zero at origin, without any loss of generality.

For the existence of these proximate orders  $H$  and  $D$  for  $f(z)$  see [1]. There these are defined for  $\lambda > 0, \rho > 0$ . Here our definition is true for  $\lambda = \rho = 0$  as well, since we have introduced the condition 1.4 and 2.4 in place of

- 1.4'  $r\lambda'(r) \log r \rightarrow 0$  as  $r \rightarrow \infty$
- 2.4'  $r\rho'(r) \log r \rightarrow 0$  as  $r \rightarrow \infty$ .

For  $\lambda > 0, \rho > 0$ , the conditions 1.4 and 1.4', 2.4 and 2.4' are equivalent.

## 3

For convenience we set

$$3.1 \quad n(r) = n(r, a) + n(r, b)$$

$$3.2 \quad N(r) = N(r, a) + N(r, b) \\ \text{where } a \neq b, (0 \leq a \leq \infty), (0 \leq b \leq \infty)$$

$$3.3 \quad H(r) = \exp\left(\int_{r_0}^r \frac{\lambda(r)}{r} dr\right)$$

$$3.4 \quad D(r) = \exp\left(\int_{r_0}^r \frac{\rho(r)}{r} dr\right).$$

## 4

In [2] we proved

**THEOREM A.** *If*

$$4.1 \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} = \beta > 0$$

and

$$4.2 \quad \frac{N(r)}{r^{\rho(r)}} \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

Then for  $x \neq a, b$

$$4.3 \quad 0 < \beta \leq \liminf_{r \rightarrow \infty} \frac{N(r, n)}{r^{\rho(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq 1$$

and

**THEOREM B.** *If*

$$4.4 \quad \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} = \alpha < \infty$$

and

$$4.5 \quad \frac{N(r)}{r^{\lambda(r)}} \rightarrow 0 \quad \text{as } r \rightarrow 0$$

Then for  $x \neq a, b$

$$4.6 \quad 1 = \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \alpha < \infty.$$

If we analyse the conditions 4.1 and 4.4, we notice that they imply some restrictions of regularity on the growth of the function  $f(z)$ . In fact in proving theorem 1, we have shown that functions satisfying either 4.1 or 4.4 are functions of regular growth.

Apparently it looks that there is no direct relation between

$$\frac{T(r, f)}{r^{\lambda(r)}} \quad \text{and} \quad \frac{T(r, f)}{r^{\rho(r)}}.$$

From 1.9 and 2.9 we can conclude that

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} = 1 = \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}}.$$

It is interesting to know a relation between

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \quad \text{and} \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}}$$

Using the properties of proximate orders given in 1 and 2 we prove

**THEOREM 1.**

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{H(r)} < \infty \quad \text{iff} \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{D(r)} > 0.$$

**THEOREM 2.**

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{H(r)} = \infty \quad \text{iff} \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{D(r)} = 0.$$

*As an illustration let us consider the following example when*

$$f(z) = \prod_1^{\infty} \left(1 + \frac{z}{\Delta_n}\right)^{b p_n}$$

where

$$b = [\rho] + 1$$

$$\Delta_n = n^{n^n}$$

$$p_n = (\Delta_n)^{\rho + \varepsilon}.$$

Then

$$\limsup_{r \rightarrow \infty} \frac{N(r, 0)}{r^{\lambda(r)}} = \infty \quad [2. \text{ pp. } 194].$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} = \infty$$

so that by lemma 1, 5.1

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{H(r)} = \infty.$$

Therefore by Theorem 2,

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{D(r)} = 0$$

which we can verify directly.

## 5

First we prove the following easy lemmas.

**LEMMA 1.**

$$5.1 \quad H(r) \sim \frac{r^{\lambda(r)}}{r_0^{\lambda(r_0)}}$$

$$5.2 \quad D(r) \sim \frac{r^{\rho(r)}}{r_0^{\rho(r_0)}} \quad \text{as } r \rightarrow \infty.$$

**PROOF OF LEMMA 1.**

$$\begin{aligned} H(r) &= \exp \left( \int_{r_0}^r \frac{\lambda(r)}{r} dr \right) \\ &= \exp \left[ \lambda(r) \log r \Big|_{r_0}^r - \int_{r_0}^r \lambda'(r) \log r dr \right] \\ &= \exp \left[ \lambda(r) \log r \Big|_{r_0}^r - o \left( \int_{r_0}^r \frac{\lambda(r)}{r} dr \right) \right] \quad \text{by 1.4} \\ &\sim \frac{r^{\lambda(r)}}{r_0^{\lambda(r_0)}} \quad \text{as } r \rightarrow \infty. \end{aligned}$$

The proof of 5.2 is similar, where we use property 2.4 in place of 1.4.

If  $r_0 \rightarrow 1$ ,  $r_0^{\lambda(r_0)} \rightarrow 1$  and we deduce that as  $r_0 \rightarrow 1$

$$5.3 \quad H(r) \rightarrow r^{\lambda(r)}$$

$$5.4 \quad D(r) \rightarrow r^{\rho(r)}.$$

**LEMMA 2.**

$$5.5 \quad \int_{r_0}^r \frac{H(r)}{r^{\alpha+1}} dr \sim \frac{H(r)}{(\lambda-\alpha)r^\alpha} \quad 0 \leq \alpha < \lambda$$

$$5.6 \quad \int_{r_0}^r \frac{D(r)}{r^{\alpha+1}} dr \sim \frac{D(r)}{(\rho-\alpha)r^\alpha} \quad 0 \leq \alpha < \rho$$

$$5.7 \quad \int_r^\infty \frac{H(r)}{r^{\alpha+1}} dr \sim \frac{H(r)}{(\alpha-\lambda)r^\alpha} \quad \alpha > \lambda$$

$$5.8 \quad \int_r^\infty \frac{D(r)}{r^{\alpha+1}} dr \sim \frac{D(r)}{(\alpha-\rho)r^\alpha} \quad \alpha > \rho$$

as  $r \rightarrow \infty$ .

PROOF OF LEMMA 2. We have

$$5.9 \quad \begin{aligned} \frac{d}{dr} (r^{\lambda(r)-\alpha}) &= [\lambda(r)-\alpha+\lambda'(r)r \log r] r^{\lambda(r)-\alpha-1} \\ &\sim (\lambda-\alpha) r^{\lambda(r)-\alpha-1} \end{aligned} \quad \text{by 1.3 and 1.4}$$

and so if  $\alpha < \lambda$  we have by Lemma 1

$$\begin{aligned} \int_{r_0}^r \frac{H(r)}{r^{\alpha+1}} dr &\sim \int_{r_0}^r \frac{r^{\lambda(r)-\alpha-1}}{r_0^{\lambda(r_0)}} dr \\ &= \frac{r^{\lambda(r)-\alpha}}{(\lambda-\alpha)r_0^{\lambda(r_0)}} - \frac{r_0^{-\alpha}}{\lambda-\alpha} \quad \text{by 5.9} \\ &\sim \frac{H(r)}{(\lambda-\alpha)r^\alpha} \end{aligned}$$

we omit the proofs of 5.6, 5.7 and 5.8, since they are similar to the proof of 5.5.

LEMMA 3.

$$5.9 \quad H(kr) \sim k^\lambda H(r)$$

$$5.10 \quad D(kr) \sim k^\rho D(r).$$

PROOF.

$$\begin{aligned} \log \frac{H(kr)}{H(r)} &= \int_r^{kr} \frac{\lambda(r)}{r} dr \quad \text{by 3.3} \\ &= \lambda(r) \log r \Big|_r^{kr} - \int_r^{kr} \lambda'(r) \log r dr \\ &= \log (kr)^{\lambda(kr)} - \log (r)^{\lambda(r)} - o \left( \int_r^{kr} \frac{\lambda(r)}{r} dr \right) \quad \text{by 1.4.} \end{aligned}$$

Hence

$$\begin{aligned} [1+o(1)] \log \frac{H(kr)}{H(r)} &= [1+o(1)] \int_r^{kr} \frac{\lambda(r)}{r} dr = \log \left[ \frac{(kr)^{\lambda(kr)}}{r^{\lambda(r)}} \right] \\ \therefore \frac{H(kr)}{H(r)} &\sim \frac{(kr)^{\lambda(kr)}}{r^{\lambda(r)}} \sim k^\lambda \quad \text{by 1.8} \end{aligned}$$

and the lemma is proved.

The proof of 5.10 is similar.

## 6

PROOF OF THEOREM 1. Let

$$\begin{aligned}
 6.1 \quad & \limsup_{r \rightarrow \infty} \frac{T(r, f)}{H(r)} = \alpha < \infty \\
 & \therefore T(r, f) < (\alpha + \varepsilon)H(r) \quad \text{for all } r \geq r_0 \\
 & \sim (\alpha + \varepsilon) \frac{r^{\lambda(r)}}{r_0^{\lambda(r_0)}} \quad \text{by Lemma 1}
 \end{aligned}$$

so that

$$\log T(r, f) < \log(\alpha + \varepsilon) + \lambda(r) \log r - \lambda(r_0) \log r_0$$

or

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \leq \limsup_{r \rightarrow \infty} \lambda(r) = \lambda$$

i.e.  $\rho \leq \lambda$ .

But  $\rho \geq \lambda$  always

6.2 Hence  $\rho = \lambda$  if 6.1 is true, and function is of regular growth.

Since  $r^{\lambda(r)} \leq T(r, f) \leq r^{\rho(r)}$  for all  $r \geq r_0$

$$\begin{aligned}
 \frac{T(r, f)}{D(r)} & \geq \frac{r^{\lambda(r)}}{D(r)} \\
 & \sim r^{\lambda(r) - \rho(r)} r_0^{\rho(r_0)}
 \end{aligned}$$

by 5.2 of lemma 1

$$\begin{aligned}
 \therefore \liminf_{r \rightarrow \infty} \frac{T(r, f)}{D(r)} & \geq r_0^{\rho(r_0)} \liminf_{r \rightarrow \infty} [r^{\lambda(r) - \rho(r)}] \\
 & = r_0^{\rho(r_0)} \quad (\text{since } [\lambda(r) - \rho(r)] \rightarrow \lambda - \rho = 0 \text{ by 6.2}) \\
 & > 0.
 \end{aligned}$$

Conversely let

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{D(r)} = \beta > 0$$

$$\begin{aligned}
 6.3 \quad & \therefore T(r, f) > (\beta - \varepsilon)D(r) \quad \text{for all } r \geq r_0 \\
 & \sim (\beta - \varepsilon) \frac{r^{\rho(r)}}{r_0^{\rho(r_0)}} \quad \text{by 5.2 of lemma 1}
 \end{aligned}$$

so that

$$\log T(r, f) > \log(\beta - \varepsilon) + \rho(r) \log r - \rho(r_0) \log r_0$$

or



$$\liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \geq \rho$$

i.e.  $\lambda \geq \rho$

But  $\lambda \leq \rho$  always

6.4 Hence  $\lambda = \rho$  if 6.3 is true, and the function is of regular growth. Now

$$\begin{aligned} \frac{T(r, f)}{H(r)} &\leq \frac{r^{\rho(r)}}{H(r)} && \text{by 2.5, for all } r \geq r_0 \\ &\sim r^{\rho(r)-\lambda(r)} r_0^{\lambda(r_0)} && \text{by 5.1 of lemma 1} \\ \therefore \limsup_{r \rightarrow \infty} \frac{T(r, f)}{H(r)} &\leq r_0^{\lambda(r_0)} && \text{by 6.4} \\ &< \infty \end{aligned}$$

and the proof is complete.

Proof of theorem 2 follows from theorem 1 as corollary.

Now Theorem A and Theorem B, take the following form.

**THEOREM 3.** *If  $f(z)$  is a meromorphic function of regular growth and*

$$N(r) = o(r^{\rho(r)}) \text{ as } r \rightarrow \infty$$

*Then for  $x \neq a, b$*

$$0 < \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} \leq \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq 1.$$

**THEOREM 4.** *If  $f(z)$  is a meromorphic function of regular growth and*

$$N(r) = o(r^{\lambda(r)}) \text{ as } r \rightarrow \infty.$$

*Then for  $x \neq a, b$*

$$1 = \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} < \infty.$$

**PROOF OF THEOREM 3 AND 4.**

These can be proved by application of same arguments as used in proving theorem 1 and then Theorem A and Theorem B.

## 7

Now we prove some more theorems on the distribution of  $a$ -points of  $f(z)$  of order  $\rho > 0$  and lower order  $\lambda > 0$ .

**THEOREM 5.**

$$0 < \limsup_{r \rightarrow \infty} \frac{n(r)}{H(r)} < \infty \text{ iff } 0 < \limsup_{r \rightarrow \infty} \frac{N(r)}{H(r)} < \infty.$$

**THEOREM 6.**

$$\frac{n(r)}{H(r)} \rightarrow 0 \quad \text{iff} \quad \frac{N(r)}{H(r)} \rightarrow 0.$$

**THEOREM 7.**

$$0 < \limsup_{r \rightarrow \infty} \frac{n(r)}{D(r)} < \infty \quad \text{iff} \quad 0 < \limsup_{r \rightarrow \infty} \frac{N(r)}{D(r)} < \infty.$$

**THEOREM 8.**

$$\frac{n(r)}{D(r)} \rightarrow 0 \quad \text{iff} \quad \frac{N(r)}{D(r)} \rightarrow 0.$$

**PROOF OF THEOREM 5.** Let

$$\limsup_{r \rightarrow \infty} \frac{n(r)}{H(r)} = \alpha \quad 0 < \alpha < \infty.$$

Therefore

$$n(r) < (\alpha + \varepsilon)H(r) \quad \text{for all } r \geq r_0.$$

Now

$$\begin{aligned} N(r) &= A + \int_{r_0}^r \frac{n(r)}{r} dr \\ &< A + (\alpha + \varepsilon) \int_{r_0}^r \frac{H(r)}{r} dr \\ &\sim A + (\alpha + \varepsilon) \frac{H(r)}{\lambda} \quad \text{by 5.5 of lemma 2.} \end{aligned}$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{N(r)}{H(r)} \leq \frac{\alpha}{\lambda}.$$

Conversely let

$$\limsup_{r \rightarrow \infty} \frac{N(r)}{H(r)} = \beta \quad 0 < \beta < \infty$$

$$\therefore N(r) < (\beta + \varepsilon)H(r) \quad \text{for all } r \geq r_0.$$

Now

$$\begin{aligned} n(r) \log 2 &< \int_r^{2r} \frac{n(r)}{r} dr \\ &< N(2r) \\ &< (\beta + \varepsilon)H(2r) \quad \text{by} \\ &\sim (\beta + \varepsilon)2^\lambda H(r) \quad \text{by 5.9 of lemma 3.} \end{aligned}$$

Therefore

$$\limsup_{r \rightarrow \infty} \frac{n(r)}{H(r)} \leq \frac{\beta 2^\lambda}{\log 2}$$

and the theorem is proved.

We omit the proof of theorems 6, 7, 8 since the proofs are similar to the proof of theorem 5.

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