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REGULAR MINIMAL SETS
II: THE PROXIMALLY EQUICONTINUOUS CASE

by

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Introduction

The regular minimal sets were introduced in [3]. These are defined to be 'universal' minimal sets for 'admissible' properties; that is, every minimal set satisfying the admissible property is a homomorphic image of the universal minimal set. Another characterization, which we will use in this paper, expresses a kind of homogeneity of regular minimal sets: a minimal set (X, T) is regular if and only if, whenever $x, y \in X$, there is an endomorphism φ of (X, T) such that $\varphi(x)$ is proximal to y. The class $\mathscr{R}(T)$ of regular minimal sets with phase group T is a complete lattice, where the partial ordering is defined by the existence of a homomorphism ([3], theorem 5).

In this paper, we will intensively study the proximally equicontinuous regular minimal sets. Proximally equicontinuous means that the proximal relation P is a closed equivalence relation, and that the quotient minimal set $(X/P, T)$ is equicontinuous, ([3], [4]). It is shown in § 1 that proximally equicontinuous is an admissible, divisible property. The main results are contained in § 2, where the structure of proximally equicontinuous regular minimal sets with phase group discrete abelian is completely determined. From a proximally equicontinuous regular minimal set (X, T) one obtains, in a natural way, a compact Hausdorff space C, a compact abelian group G, and a class of subsets of G satisfying certain conditions (theorem 2). Conversely, (theorem 3) if such a C, G and subsets of G satisfying these conditions are given, a proximally equicontinuous regular minimal set may be constructed. Unfortunately, these conditions are extremely complicated. They are applied in § 3 to the case $G = S^1$ and C a finite set. The concluding sections examine the relation between proximal equicontinuity and local almost periodicity, and also consider homomorphisms of proximally equicontinuous regular minimal sets.

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1. Admissibility and divisibility

In this section we show that the class of proximally equicontinuous regular minimal sets is a reasonable subclass of all regular minimal sets.

THEOREM 1. (i) *'Proximally equicontinuous' is an admissible, divisible property.*

(ii) *The proximally equicontinuous regular minimal sets form a sublattice of $\mathscr{R}(T)$.*

PROOF. Note that (ii) is an immediate consequence of (i) and the definition of the lattice operations in $\mathscr{R}(T)$, ([3], theorems 6 and 7).

To prove (i) we require two lemmas:

LEMMA 1. *Let (X_i, T) $(i \in \mathscr{I})$ be a family of transformation groups such that $P(X_i)$ is an equivalence relation, for each $i \in \mathscr{I}$. Let $X = \times_{i \in \mathscr{I}} X_i$, and let $x = (x_i)$, $y = (y_i)$ be in X. Then $(x, y) \in P(X)$ is and only if $(x_i, y_i) \in P(X_i)$ $(i \in \mathscr{I})$.*

PROOF. The necessity is obvious. We prove sufficiency. Since each $P(X_i)$ is an equivalence relation, so is $P(X)$, ([1], theorem 2). Let $\pi_i : X \to X_i$ be the natural projection, and let $\theta_i : E(X) \to E(X_i)$ be the induced homomorphism of the enveloping semigroups. Let I, I_i be the unique minimal right ideals in $E(X)$, $E(X_i)$ respectively, ([6], theorem 2). Now if $(x_i, x'_i) \in P(X_i)(i \in \mathscr{I})$, $x = (x_i)$, $x' = (x'_i)$, in $\times_i X_i$, and $p \in I$, then $\pi_i(xp) = x_i\theta_i(p) = x'_i\theta_i(p) = \pi_i(x'p)$. Since this equality holds for each $i \in \mathscr{I}$, $xp = x'p$ and $(x, x') \in P(X)$.

LEMMA 2. *Let (X, T) and (Y, T) be minimal sets, let $\pi : (X, T) \to (Y, T)$ be a homomorphism, and let $\hat{\pi} : (X \times X, T) \to (Y \times Y, T)$ be the induced homomorphism. Then $\hat{\pi}(P(X)) = P(Y)$.*

PROOF. It is clear that $\hat{\pi}(P(X)) \subset P(Y)$. Let $(y_1, y_2) \in P(Y)$. Then there is an idempotent u' in a minimal right ideal I' of $E(Y)$ such that $y_2 = y_1 u'$. Let $\theta : E(X) \to E(Y)$ be the semigroup homomorphism induced by π, let I be a minimal right ideal in $E(X)$ such that $\theta(I) = I'$, and let u be an idempotent in I with $\theta(u) = u'$. Choose $x_1 \in X$ such that $\pi(x_1) = y_1$ and let $x_2 = x_1 u$. Then $(x_1, x_2) \in P(X)$, and $\pi(x_2) = \pi(x_1 u) = \pi(x_1)\theta(u) = y_1 u' = y_2$. Thus $\hat{\pi}(x_1, x_2) = (y_1, y_2)$.

Now we return to the proof of (i) in theorem 1. We show that 'proximally equicontinuous' is productive. Since it is clearly hereditary, this will show admissibility. Now, each $P(X_i)$ is closed, hence by lemma 1 $P(X)$ is closed and is therefore an equivalence relation, [1]. Now $(\times_i X_i/P(X_i), T)$ is a product of equicontinuous transformation groups, and is therefore equicontinuous. Moreover, if $x = (x_i) \in X$, the homomorphism $x \to ([x_i])$ of X onto $\times_i X_i/P(X_i)$ (where $[x_i]$ is the image of x in $X_i/P(X_i)$) induces a homomorphism $[x] \to ([x_i])$ of X/P onto $\times_i X_i/P(X_i)$ which, with the aid of lemma 1, is easily seen to be one to one. Hence $(X/P, T)$ is isomorphic with $(\times_i X_i/P(X_i), T)$, and is therefore equicontinuous.

To prove divisibility of 'proximally equicontinuous', let (X, T) and (Y, T) be minimal with (X, T) proximally equicontinuous, and let $\pi : X \to Y$ be a homomorphism. $P(X)$ is a closed equivalence relation. Then $P(Y) = \hat{\pi}(P(X))$ (lemma 2) is closed and hence an equivalence relation. Then π induces a homomorphism $\hat{\pi} : X/P(X) \to Y/P(Y)$. Since $X/P(X)$ is equicontinuous, so is $Y/P(Y)$, and the proof is completed.

A theorem similar to theorem 1 is proved in [4]. Lemma 2 is also proved in [8].

2. The structure of proximally equicontinous regular minimal sets

In this section it is assumed that T is discrete abelian. We first discuss briefly the equicontinuous minimal sets. For convenience we suppose that T acts effectively. If (X, T) is an equicontinuous minimal set, then the enveloping semigroup E is an abelian group of self homeomorphisms of X, and is identical with the automorphism group $A(X)$ of (X, T), [2]. Since $E(X) = A(X)$ acts transitively on X, (X, T) is certainly regular. Indeed, in this case, (X, T) may be given the structure oı a compact abelian group in which T is embedded in a one-one continuous manner as a dense subgroup. We may choose any point x_0 as the identity, define multiplication on the orbit of x_0 by $(x_0 t)(x_0 t') = x_0 tt'$, and then use the assumed equicontinuity of (X, T) to extend the multiplication to all of X.

Thus the study of equicontinuous minimal sets is reduced to the study of compact abelian groups of the type described. Our basic strategy in the study of proximally equicontinuous minimal sets is to use the natural projection $\eta : X \to X/P$, to 'pull back', as much as possible, the desirable properties of $(X/P, T)$ to (X, T).

Now, let (X, T) be a proximally equicontinuous regular minimal set. Then, if $x \in X$, $\eta^{-1}(\eta(x)) = P(x)$, the set of points proximal to x. If φ is an automorphism of (X, T), then it is easy to see that $\varphi(P(x)) = P(\varphi(x)$ for $x \in X$. (This shows that all the 'fibers' $P(x)$ are homeomorphic.) Moreover, since $\varphi_1, \varphi_2 \in A(X)$ with $\varphi_1 \neq \varphi_2$ implies $\varphi_1(x)$ and $\varphi_2(x)$ are distal ([2], theorem 2), it follows that, if $x, y \in X$, there is exactly one $\varphi \in A(X)$ such that $(\varphi(x), y) \in P$.

Now, choose an identity element e in X/P, and let G denote the topological group obtained by this choice. We may identify G with $A(X)$ as follows. If $g \in G$, $x \in X$, we define $g(x) = \varphi(x)$, where φ is the unique element of $A(X)$ for which $\eta(\varphi(x)) = g\eta(x)$ (the group product in G). If $t \in T$, $g\eta(xt) = g\eta(x)t = \eta(\varphi(x))t = \eta\varphi(xt)$, so if $y \in X$, $g\eta(y) = \eta\varphi(y)$, and φ depends only on g. It is easily verified that this correspondence defines an (algebraic) isomorphism of G with $A(X)$.

Let $C = \eta^{-1}(e) \subset X$. Consider the map $\psi : X \to G \times C$ defined by $\psi(x) = (\eta(x), \eta(x)^{-1}(x))$. It is easily verified that ψ is one to one and onto, so that, set theoretically, X may be identified with $G \times C$. Now, if $t \in T$, $\psi(xt) = (\eta(x)t, \eta(x)^{-1}(x))$. Since xT is dense in X, for each $x \in X$, ψ is not continuous (unless C consists of one point).

Looking at this in another way, if we identify X with $G \times C$ as in the preceding paragraph, and define the action of T on $G \times C$ by $(g, c)t = (gt, c)$, there is a topology for $G \times C$ (obviously *not* the product topology) such that, for every $(g, c) \in G \times C$, the orbit of (g, c), $(g, c)T = [(gt, c)|t \in T]$ is dense in $G \times C$. Using this representation, the action of G as the automorphism group of $(G \times C, T)$ is given by $g'(g, c) = (g'g, c)$.

Thus, we have determined the set theoretic, or 'algebraic' structure of proximally equicontinuous regular minimal sets. We now turn to the discussion of their topological properties.

LEMMA 3. *Let $\{g_n\}$, $\{x_n\}$ $(n \in D)$ be nets in G and X respectively, such that $g_n \to g \in G$, and $x_n \to x \in X$. Then $g_n(x_n) \to P(g(x))$.*

PROOF. It is sufficient to show $\eta(g_n(x_n)) \to \eta(g(x))$. Now $\eta(g_n(x_n)) = g_n\eta(x_n)$, and $\eta(g(x)) = g\eta(x)$. Since η is continuous, and G is a topological group, the conclusion follows immediately.

Let $\mathscr{U}$ denote the uniformity of X, and let $\mathscr{V}$ be the uniformity which C acquires as a subspace of X. If $\alpha \in \mathscr{U}$, $c, c' \in C$, consider the subset $O_\alpha(c, c')$ of G which is defined by $O_\alpha(c, c') = [g \in G|(g(c'), c) \in \alpha]$.

THEOREM 2. *The sets $O_\alpha(c, c')$ have the following properties:*

1. *$e \in O_\alpha(c, c)$ for all $c \in C$ and all $\alpha \in \mathscr{U}$.*
2. *If $\alpha \in \mathscr{U}$, $c' \in C$, let $O_\alpha(c') = \bigcup_{c \in C} O_\alpha(c, c')$, $O_\alpha^* = \bigcap_{c' \in C} O_\alpha(c')$, and $O_\alpha = \bigcup_{c' \in C} O_\alpha(c') = \bigcup_{c, c' \in C} O_\alpha(c, c')$. Then, both $\{O_\alpha^*\}_{\alpha \in \mathscr{U}}$ and $\{O_\alpha\}_{\alpha \in \mathscr{U}}$ constitute fundamental systems of neighborhoods of e.*
3. *If $\beta, \gamma \in \mathscr{U}$, $c \in C$, then there is an $\alpha \in \mathscr{U}$ such that $O_\alpha(c, c') \subset O_\beta(c, c') \cap O_\gamma(c, c')$ for all $c' \in C$.*
4. *If $c, c_1, c_2 \in C$, then $O_\alpha(c, c_1) \cap O_\alpha(c, c_2)$ meets every orbit in (G, T).*
5. *If $c_1, c_2 \in C$ with $c_1 \neq c_2$, then there is an $\alpha \in \mathscr{U}$ such that $O_\alpha(c_1, c) \cap O_\alpha(c_2, c) = \emptyset$ for all $c \in C$.*
6. *If $\alpha \in \mathscr{U}$, $c, c' \in C$, and $g \in O_\alpha(c, c')$, then there is a $\beta \in \mathscr{U}$ such that $gO_\beta(c', c'') \subset O_\alpha(c, c'')$, for all $c'' \in C$.*

7. *If $\alpha \in \mathscr{U}$, then there is a $\beta \in \mathscr{U}$, and a $\delta \in \mathscr{V}$ such that if $c, c', c'' \in C$ with $(c', c) \in \delta$, then $O_\beta(c', c'') \subset O_\alpha(c, c'')$.*

8. *Let $c_1, \cdots, c_k \in C, \alpha_1, \cdots, \alpha_k \in \mathscr{U}$ such that for all $c' \in C$, $e \in \bigcup_{j=1,\cdots,k} O_{\alpha_j}(c_j, c')$. Then there is an $\alpha \in \mathscr{U}$ such that if $c \in C$, there is a j, $1 \leqq j \leqq k$, such that $O_\alpha(c, c') \subset O_{\alpha_j}(c_j, c')$, for all c'.*

PROOF. The proofs of 1, 3, 5, 7 and 8 follow easily from the definition of the sets $O_\alpha(c, c')$, the facts that X and C are compact Hausdorff spaces, and elementary properties of uniformities.

If $g \in G$, $c_1, c_2 \in C$, then $g(c_1)$ and $g(c_2)$ are proximal. This implies 4.

We prove 2. Note that O^*_α consists of those $g \in G$ for which all $g(c')$ are in an 'α neighborhood' of the set C, and O_α consists of those g such that some $g(c')$ has this property. Since $O^*_\alpha \subset O_\alpha$, it is sufficient to show that O^*_α is a neighborhood of e, and that if $\mathscr{V}$ is any neighborhood of e, there is an $\alpha \in \mathscr{U}$ such that $O_\alpha \in \mathscr{V}$.

If O^*_α were not a neighborhood of e, there would exist a net $\{g_n\}$ in G, with $g_n \to e$ such that $g_n \notin O^*_\alpha$. Then there would be $c'_n \in C$ such that $(g_n(c'_n), c) \notin \alpha$, for all $c \in C$. But this contradicts lemma 3.

Now, let U be a neighborhood of e in G, and let $\alpha_n \in \mathscr{U}$ such that $\cap\, \alpha_n = \Delta$ the diagonal of $X \times X$. If no O_{α_n} is contained in U, then there are c_n, $c'_n \in C$ and $g_n \in O_{\alpha_n}(c_n, c'_n)$ with $g_n \notin U$. We may suppose $g_n \to g \in G - U$. Then $(g_n(c'_n), c_n) \in \alpha_n$. Suppose $c_n \to c \in C$. Then $g_n(c'_n) \to c$. But then, by lemma 3, $g_n \to e$, and $g = e$. This is a contradiction.

To prove 6, choose $\alpha_0 \in \mathscr{U}$ such that if $(z, g(c')) \in \alpha_0$, then $(z, c) \in \alpha$. Let $\beta \in \mathscr{U}$ such that $(x, y) \in \beta$ implies $(g(x), g(y)) \in \alpha_0$. Now let $g' \in O_\beta(c', c'')$. Then $(g'(c''), c') \in \beta$, and $(gg'(c''), g(c')) \in \alpha_0$. Since $g \in O_\alpha(c, c')$, we have $(g(c'), c) \in \alpha$, and therefore $(gg'(c''), c) \in \alpha$. That is $gO_\beta(c', c'') \subset O_\alpha(c, c'')$. The proof is completed.

Note that properties $1-8$ are expressed entirely in terms of the topological group G, and the compact Hausdorff space C. Now, suppose we are given such a G and C, and a collection of subsets $O_\alpha(c, c')$ of G which satisfy $1-8$. The next theorem shows how to construct a proximally equicontinuous regular minimal set (X, T) out of G and C. That is, the sets $O_\alpha(c, c')$ completely determine the structure of the minimal set.

THEOREM 3. *Let T be a discrete abelian group, and let (G, T) be an equicontinuous minimal set. Let G be given the structure of a compact abelian group. Let C be a compact Hausdorff space, with uniformity $\mathscr{V}$. Suppose there is an index set $\mathscr{U}$, such that, for all $\alpha \in \mathscr{U}$, $c, c' \in C$, there are subsets $O_\alpha(c, c')$ of G with properties $1-8$ of theorem 2. If $(h, c) \in G \times C$, let $N_\alpha(g, c) = [(g', c') | g^{-1}g' \in O_\alpha(c, c')]$.*

Then, the sets $N_\alpha(g, c)$ constitute a base for a compact Hausdorff topology of $X = G \times C$. If T acts on X by $(g, c)t = (gt, c)$, then (X, T) is a *proximally equicontinuous regular minimal set, with each $P(x)$ homeomorphic to C, and $(X/P, T)$ isomorphic with (G, T).*

PROOF. To show that the sets $N_\alpha(g, c)$ are a base for a topology on X, we first observe that $(h, c) \in N_\alpha(g, c)$ if and only if $g^{-1}h \in O_\alpha(c, c')$. For, $(h, c') \in N_\alpha(g, c)$ means that $h = gg_0$, where $(g_0, c') \in N_\alpha(e, c)$ and this is equivalent to $g_0 \in O_\alpha(c, c')$ or $g^{-1}h \in O_\alpha(c, c')$.

Now, suppose $(h, c') \in N_\alpha(g, c)$ or $g^{-1}h \in O_\alpha(c, c')$. By 6, we may choose $\beta \in \mathscr{U}$ so that $g^{-1}hO_\beta(c', c'') \subset O_\alpha(c, c'')$, for all $c'' \in C$. Now, let $(k, c'') \in N_\beta(h, c')$. Then $h^{-1}k \in O_\beta(c', c'')$, and $g^{-1}k = g^{-1}hh^{-1}k \in g^{-1}hO_\beta(c', c'') \subset O_\alpha(c, c'')$, or $(k, c'') \in N_\alpha(g, c)$. That is, $N_\beta(h, c') \subset N_\alpha(g, c)$.

If $(h, c'') \in N_\alpha(g, c) \cap N_\gamma(g', c')$, choose $\beta, \mathscr{G} \in \mathscr{U}$ such that $N_\beta(h, c'') \subset N_\alpha(g, c)$ and $N_\mathscr{G}(h, c'') \subset N_\gamma(g', c')$. Now choose $\lambda \in \mathscr{U}$, so that $O_\lambda(c'', c_0) \subset O_\beta(c'', c_0) \cap O_\mathscr{G}(c'', c_0)$, for all $c_0 \in C$ (property 3). It follows that $N_\lambda(h, c'') \subset N_\beta(h, c'') \cap N_\mathscr{G}(h, c'') \subset N_\alpha(g, c) \cap N_\gamma(g', c')$.

This shows that the $N_\alpha(g, c)$ define a topology on $X = G \times C$. We show that this topology is compact Hausdorff. Suppose $(g, c) \neq (g', c')$. If $g \neq g'$, choose $\alpha \in \mathscr{U}$ such that $gO_\alpha \cap g'O_\alpha = \emptyset$. Now, if $(h, c'') \in N_\alpha(g, c) \cap N_\alpha(g', c')$, then $h \in gO_\alpha(c, c'') \cap g'O_\alpha(c', c'') \subset gO_\alpha \cap g'O_\alpha$; hence $N_\alpha(g, c)$ and $N_\alpha(g', c')$ must be disjoint. If $g = g'$ but $c \neq c'$, choose $\alpha \in \mathscr{U}$ such that $O_\alpha(c, c'') \cap O_\alpha(c', c'') = \emptyset$, for all $c'' \in C$. Then it follows that $N_\alpha(g, c) \cap N_\alpha(g, c') = \emptyset$. Thus the topology is Hausdorff.

To prove compactness, suppose first that (g_n, c_n') is a net in $G \times C$ such that $g_n \to e$. Let $\alpha \in \mathscr{U}$. We show that there is a $c \in C$ such that $(g_n, c_n') \in N_\alpha(e, c)$, for $n \geqq n_0$. Let $\beta \in \mathscr{U}$, $\delta \in \mathscr{U}$, as in 7. Since O_β is a neighborhood of e, $g_n \in O_\beta$ for $n \geqq n_0$. Then $g_n \in O_\beta(c_n')$ and $g_n \in O_\beta(c_n, c_n')$, for some $c_n \in C$. Suppose $c_n \to c \in C$, so $(c_n, c) \in \delta$, for $n \geqq n_1 \geqq n_0$. By 7, $O_\beta(c_n, c_n') \subset O_\alpha(c, c_n')$, and $(g_n, c_n') \in N_\alpha(e, c)$.

Now, suppose no subnet of (g_n, c_n') converges. Then if $c \in C$, there is an $\alpha(c) \in \mathscr{U}$ and an $n_0 \in D$ such that $(g_n, c_n') \notin N_{\alpha(c)}(e, c)$ or $g_n \notin O_{\alpha(c)}(c, c_n')$, for $n \geqq n_0$.

Now $\{e\} \times C \subset \bigcup_{c \in C} N_{\alpha(c)}(e, c)$. From property 7, it follows easily that $\{e\} \times C$ is homeomorphic with C, and is therefore compact. Let $c_1, \cdots, c_k$ in C such that $\{e\} \times C \subset \bigcup_{j=1,\cdots,k} N_{\alpha_j}(e, c_j) = N^*$, where we write α_j for $\alpha(c_j)$. Then there is an $n_1 \in D$ such that $(g_n, c_n') \notin N^*$ for $n \geqq n_1$. Let $\alpha \in \mathscr{U}$ be chosen to satisfy 8. Then, if $(g, c') \in N_\alpha(e, c)$, $g \in O_\alpha(c, c') \subset O_{\alpha_j}(c_j, c')$, for some j $(1 \leqq j \leqq k)$, by 8, and $(g, c') \in N_{\alpha_j}(e, c_j)$. That is, $\bigcup_{c \in C} N_\alpha(e, c) \subset N^*$. But the preceding paragraph tells us that $(g_n, c_n') \in N_\alpha(e, c)$, for some c, and therefore $(g_n, c_n') \in N^*$, for all $n \geqq n_0$. This is a contradiction, and therefore a subnet of $(g_n, c_n') \to (e, c)$, for some $c \in C$.

Finally, let (g_n, c_n') be any net in $G \times C$. We may suppose $g_n \to g$ in G.

Then $h_n = g^{-1}g_n \to e$, and, by the discussion just concluded, a subnet of $(h_n, c_n') \to (e, c)$. It follows immediately that the corresponding subnet of $(g_n, c_n') \to (g, c)$.

Let $\pi : X \to G$ be the first coordinate projection. Then π is continuous. For, let $\{(g_n, c_n)\}$ be a net in X with $(g_n, c_n) \to (g, c)$. If $\alpha \in \mathscr{U}$, the net $\{(g_n, c_n)\}$ is eventually in $N_\alpha(g, c)$. Therefore, $g^{-1}g_n \in O_\alpha(c, c_n) \subset O_\alpha$, for $n \geqq n_0$. Since α is arbitrary, and the $\{O_\alpha\}$ are a fundamental system of neighborhoods of e, by 2, we have $g^{-1}g_n \to e$, or $g_n \to g$.

It is immediate that the maps $(g, c) \to (g, c)t = (gt, c)$ are continuous. Now let $g, h \in G$, $c_0, c, c' \in C$, and $\alpha \in \mathscr{U}$. By property 4, there is a $t \in T$ such that $h^{-1}gt \in O_\alpha(c_0, c) \cap O_\alpha(c_0, c')$. This says that $(g, c)t \in N_\alpha(h, c_0)$ and $(g, c')t \in N_\alpha(h, c_0)$. This shows that (g, c) and (g, c') are proximal, and also that (X, T) is minimal.

Conversely, if (g, c) and (g', c') are proximal, then if $(h, c_0) \in X$, there is a net $\{t_n\}$ in T such that $(g, c)t_n \to (h, c_0)$ and $(g', c')t_n \to (h, c_0)$. Then $gt_n \to h$ and $g't_n \to h$. This can only happen if $g = g'$. This shows that $P(g, c) = \{g\} \times C$.

From this it follows easily that proximal is a closed equivalence relation and that $(X/P, T)$ is isomorphic with (G, T). Finally, if $h \in G$, it defines an automorphism of (X, T) by $h(g, c) = (hg, c)$. Thus, if (g, c), $(g', c') \in X$, then $g'g^{-1}(g, c) = (g', c)$ is proximal with (g, c). This shows (X, T) is regular, and the proof is completed.

3. Examples

In order to use theorem 3 to construct proximally equicontinuous regular minimal sets, we must construct the sets $O_\alpha(c, c')$ for a given G and C. We do this when $G = S^1$, and C is a finite set with the discrete topology.

An example of a minimal set on S^1 is obtained when $T = Z$, the additive group of integers, and a generating homeomorphism is a rotation through an irrational multiple of π.

Since it is only necessary to define the sets $O_\alpha(c, c')$ in a neighborhood of the identity of $G = S^1$, we may work with a neighborhood of 0 on the real line.

Let $C = \{c_1, \cdots, c_n\}$, where $n \geqq 2$, and let k be an integer with $1 \leqq k < n$. Let $C_1 = \{c_1, \cdots, c_k\}$ and $C_2 = \{c_{k+1}, \cdots, c_n\}$. Consider two sequences of real numbers, $\{a_{ir}\}$ and $\{b_{ir}\}$ $(i = 1, \cdots, n, r = 1, 2, \cdots)$ which approach 0 as $r \to \infty$, and which satisfy

$$\begin{aligned} b_{11} > a_{11} = b_{21} > a_{21} = b_{31} > a_{31} = b_{41} > \cdots > a_{k-1,1} \\ = b_{k,1} > a_{k,1} = b_{12} > a_{12} = b_{22} > a_{22} = b_{32} > \cdots > 0; \end{aligned}$$

and

$$a_{k+1,1} < b_{k+1,1} = a_{k+2,1} < b_{k+2,1} = a_{k+3,1} < \cdots < b_{n-1,1}$$
$$= a_{n,1} < b_{n,1} = a_{k+1,2} < b_{k+1,2} = a_{k+2,2} < \cdots < 0.$$

Let I_{ir} be the open interval (a_{ir}, b_{ir}).

We are now almost ready to define the sets $O_\alpha(c, c')$. The index set will in this case be the set N of natural numbers. If $m \in N$, and $1 \leqq i \leqq n$, let $O^*_m(c_i) = \bigcup_{r \geqq m} I_{ir}$. This is illustrated for $n = 5$ and $k = 3$ in figure 1.

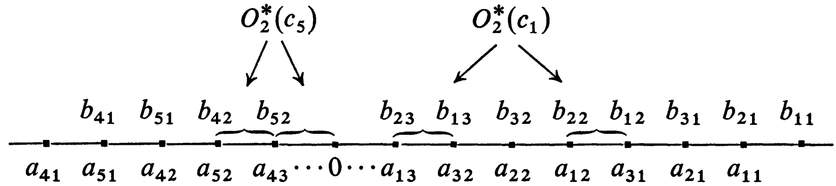

Figure 1

The sets $O_m(c_i, c_j)$ are to be $O^*_m(c_i)$ together with certain endpoints of the intervals I_{ir}. To be precise, let

$$O_m(c_i, c_j) = \begin{cases} O^*_m(c_i) \cup \{a_{ir}\}(r \geqq m), \text{ if } 1 \leqq j \leqq k \text{ and } i \neq j \\ O^*_m(c_i) \cup \{a_{ir}\} \cup \{0\}, \ (r \geqq m\} \text{ if } 1 \leqq j = i \leqq k. \\ O^*_m(c_i) \cup \{b_{ir}\}(r \geqq m) \text{ if } k+1 \leqq j \leqq n, \text{ and } i \neq j \\ O^*_m(c_i) \cup \{b_{ir}\} \cup \{0\}, \ (r \geqq m), \text{ if } k+1 \leqq j = i \leqq n. \end{cases}$$

That is, to obtain $O_m(c_i, c_j)$ from $O^*_m(c_i)$, we add left endpoints of I_{ij} if $c_j \in C_1$ and right endpoints if $c_j \in C_2$. Also, 0 is included in every $O_m(c_i, c_i)$.

We verify that properties $1-8$ of theorem 2 are satisfied. Property 1 (that $0 \in O_m(c_i, c_i)$) is true by construction. To see that property 2 is satisfied, note that $O_m(c_j) = \bigcup_{i=1,\cdots,n} O_m(c_i, c_j) = [a_{k+1,m}, b_{1,m})$ or $(a_{k+1,m}, b_{1,m}]$ according as c_j is in C_1 or C_2. Thus $O^*_m = \cap_j O_m(c_j) = (a_{k+1,m}, b_{1\ m})$ and $O_m = \bigcup_{i,j} O_m(c_i, c_j) = [a_{k+1,m}, b_{1,m}]$. For 3, let $m, m' \in N$, and choose $m'' > m$ and m'. Property 4 holds, since $O_m(c, c_1) \cap O_m(c, c_2) \supset O^*_m(c)$, which has non empty interior.

Let $1 \leqq i < j \leqq n$. If $i \leqq k < j$, then clearly $O_m(c_i, c) \cap O_m(c_j, c) = \emptyset$. Suppose $1 \leqq i < j \leqq k$ or $k+1 \leqq i < j \leqq n$. Then, by definition $O^*_m(c_i) \cap O^*_m(c_j) = \emptyset$, so it is only necessary to check the added points. Note that 0 is in at most one of the sets $O_m(c_i, c)$ and $O_m(c_j, c)$. Moreover, the endpoints which are added are all left endpoints or all right endpoints (depending on whether c is in C_1 or C_2), so in these cases also $O_m(c_i, c) \cap O_m(c_j, c) = \emptyset$. Therefore 5 holds.

Before proving 6, we rephrase it in additive terminology. We must show: if $m \in N$, $c, c' \in C$ and $g \in O_m(c, c')$, then there is a $k \in N$ such that

$g+O_k(c', c'') \subset O_m(c, c'')$, for all $c'' \in C$. If $g = 0$ or if g is an interior point of $O_m(c, c')$, this is easy. If g is an endpoint of one of the intervals I_{ir}, there are two special cases to consider. If $c' \in C_1$, then g is a left endpoint and $O_k(c', c'')$ is a subset of R^+, the positive reals (or, if $c'' = c'$, of $R^+ \cup \{0\}$). Then, if k is chosen sufficiently large, $g+O_k(c', c'') \subset O_m(c, c'')$. If $c' \in C_2$ then g is a right endpoint, and $O_k(c', c'') \subset R^-$ or $R^- \cup \{0\}$. Again, for sufficiently large k, $g+O_k(c', c'') \subset O_m(c, c'')$. Thus 6 is proved.

Since C is discrete, 7 is immediate. Finally, we verify 8. If $O \in \bigcup_{j=1,\cdots,s} O_{m_j}(c_{i_j}, c')$ for all $c' \in C$, we must have $s = n$, and there fore the set $\{c_{i_j}\}$ $(j = 1, \cdots, n)$ is equal to all of C. Let $m = \max m_j$. Then $O_m(c, c') \subset O_{m_j}(c, c')$, and the proof is completed.

4. Proximal equicontinuity and local almost periodicity

A transformation group (X, T) is said to be *locally almost periodic* if, for every $x \in X$ and neighborhood U of x, there is a syndetic subset A of T and a neighborhood V of x such that $VA \subset U$, ([7], 3.38). Locally almost periodic transformation groups are proximally equicontinuous ([6], theorem 3). The converse is not true, even for minimal sets, as we will show. However, the examples constructed in § 3 are locally almost periodic. This is a consequence of a general theorem (theorem 5), and partially answers a question of Ellis ([5], remark 9). In this section we do not require T discrete abelian, or (X, T) regular.

LEMMA 4. *Let (X, T) be a transformation group for which P is a closed equivalence relation. Let $x \in X$ and let U be a neighborhood of $P(x)$. Then there is a neighborhood U_1 of $P(x)$ such that $y \in U_1$ implies $P(y) \subset U$.*

PROOF. If the conclusion is false, then there are nets $x_n \to P(x)$ and $y_n \in P(x_n)$ such that $y_n \notin U$. We may suppose $x_n \to x' \in P(x)$, and $y_n \to y' \in X$. Since P is closed, $(x', y') \in P$, so $y' \in P(x') = P(x) \subset U$. This is a contradiction.

LEMMA 5. *Let (X, T) be minimal, let proximal be an equivalence relation in X, and let $x_1, \cdots, x_n$ be a finite set of mutually proximal points. Suppose U is a non empty open set in X. Then there is a $t \in T$ such that $x_j t \in U$ $(j = 1, \cdots, n)$.*

PROOF. Let I be the unique minimal right ideal in $E(X)$. Since $xI = X$, for all $x \in X$, there is a $p \in I$ such that $x_n p \in U$. Then $x_1 p = x_2 p = \cdots = x_n p \in U$. Since $I \subset E$, which is the closure of T in X^X, there is a $t \in T$ for which $x_j t \in U$ $(j = 1, \cdots, n)$.

THEOREM 4. *Let (X, T) be minimal and proximally equicontinuous.*

Suppose there is an $x \in X$ for which $P(x)$ is finite. Then (X, T) is locally almost periodic.

PROOF. It is sufficient to show that T is locally almost periodic at x ([7], 4.11). Let U be a neighborhood of x, and let $t \in T$ such that $P(xt) = P(x)t \subset U$. Let U_1 be a neighborhood of $P(xt)$ such that $P(U_1) = \bigcup_{y \in U_1} P(y) \subset U$.

Now, let $\eta : (X, T) \to (X/P, T)$ be the natural projection. Let $y = \eta(x)$. Since $P(xt) \subset U_1$, it follows easily that $U^* = \eta(U_1)$ is a neighborhood of yt. Since $(X/P, T)$ is almost periodic ([7], 4.38), it is locally almost periodic. Let V^* be a neighborhood of yt and A a syndetic subset of T such that $V^*A \subset U^*$. Let $V = \eta^{-1}(V^*) \subset X$. Then $VA \subset \eta^{-1}(U^*) = \eta(U_1) \subset U$. Now, V is a neighborhood of xt, so Vt^{-1} is a neighborhood of x. Then $Vt^{-1}tA = VA \subset U$. Since tA is syndetic, this shows that T is locally almost periodic at x.

Here is an example of a proximally equicontinuous regular minimal set which is not locally almost periodic. Let $X = S^1$, and let T be the total homeomorphism group of X. (Let T be given the discrete topology.) Note that $P(X) = X \times X$ so that (X, T) is proximally equicontinuous and regular. Let $x \in X$ and let U be a connected neighborhood of x with interior $(X-U) \neq \emptyset$. Let V be a connected open neighborhood of x and let $A = A(V) = [t \in T | Vt \subset U]$. We show that A is not syndetic. For this it is sufficient to show if K is a finite subset of T (say $K = \{k_1, \cdots, k_n\}$), then $T \neq AK$. That is, we are to find a $t \in T$ such that $tk_j^{-1} \notin A$, or, what is the same thing, $Vt \not\subset Uk_j$ $(j = 1, \cdots, n)$. For $j = 1, \cdots, n$, let W_j be open connected and non empty, such that $W_j \subset X - Uk_j$ and such that arc length $W_j < 1/2^j$. Let $t \in T$ such that $Vt \supset W_j$ $(j = 1, \cdots, n)$. Then certainly $Vt \not\subset Uk_j (j = 1, \cdots, n)$. The proof is completed.

5. Homomorphisms

In this section we assume again that T is discrete abelian. The next theorem describes the structure of homomorphisms of proximally equicontinuous regular minimal sets. The proof is completely straightforward, and is therefore omitted.

THEOREM 5. *Let (G, T) be an equicontinuous minimal set. Suppose that C and D are compact Hausdorff spaces. Let $\{O_\alpha(c, c')\}$ and $\{O_\lambda(d, d')\}$ $(\alpha \in \mathscr{U}_C, \lambda \in \mathscr{U}_D, c, c' \in C, d, d' \in D)$ be subsets of G which satisfy properties $1-8$ of theorem 2. Let $G \times C$ and $G \times D$ be topologized as in theorem 4. (So $(G \times C, T)$ and $(G \times D, T)$ are proximally equicontinuous regular minimal sets.) Then*

(i) *Every homomorphism of $(G \times C, T)$ to $(G \times D, T)$ is of the form*

$\pi(g, c) = (\psi(g), \sigma(c))$, where ψ is an automorphism of (G, T) and $\sigma : C \to D$ is continuous and onto.

ii) *If ψ is an automorphism of (G, T)* and *$\sigma : C \to D$ is continuous and onto, then $\pi : G \times C \to G \times D$ as defined in* (i) *is continuous if and only if, for every $\lambda \in \mathscr{U}_D$, there is an $\alpha \in \mathscr{U}_C$ such that $\psi(O_\alpha(c, c')) \subset O_\lambda(\sigma(c), \sigma(c'))$.*

Now, let $G = S^1$ and let C be a finite set. Define sets $\{O_m(c, c')\}$ and $\{\tilde{O}_m(c, c')\}$ satisfying $1-8$ so that, for *no* $c_1, c_2, c_3, c_4 \in C$ and for *no* positive integers m and r is it the case that either $O_m(c_1, c_2) \subset \tilde{O}_r(c_3, c_4)$ or $\tilde{O}_r(c_3, c_4) \subset O_r(c_1, c_2)$. (This can be achieved, for example, by defining sequences of real numbers $\{a_{ir}\}$, $\{b_{ir}\}$, $\{\tilde{a}_{ir}\}$, and $\{\tilde{b}_{ir}\}$ such that $a_{ir} < \tilde{a}_{ir} < b_{ir} < \tilde{b}_{ir}$ and then defining $O_m(c_i, c_j)$ and $\tilde{O}_m(c_i, c_j)$ as in § 3). Then, if we denote by $\mathscr{T}_1$ and $\mathscr{T}_2$ the topologies defined on $G \times C$ (as in theorem 3) from $O_m(c, c')$ and $\tilde{O}_m(c, c')$, theorem 5 tells us that the minimal sets $(G \times C, \mathscr{T}_1, T)$ and $(G \times C, \mathscr{T}_2, T)$ are not isomorphic. Thus, G and C do not uniquely determine a proximally equicontinuous regular minimal set. Indeed, it is easy to see that infinitely many non-isomorphic minimal actions can occur.

We may also use theorem 5 to obtain a positive result on homomorphisms, and thereby to construct a new regular minimal set.

Let C and D be finite sets, and let $C = C_1 \cup C_2$, $D = D_1 \cup D_2$, where $C_1 \cap C_2 = D_1 \cap D_2 = \emptyset$. Let $\sigma : C \to D$ such that $\sigma(C_i) = D_i$ $(i = 1, 2)$. Using the decomposition $C = C_1 \cup C_2$, let $O_m(c, c')$ be defined as in § 3, and topologize $S^1 \times C$ so that $(S^1 \times C, T)$ is a proximally equicontinuous regular minimal set. For $d, d' \in D$, let $O_m(d, d') = \cup [O_m(c, c') | \sigma(c) = d, \sigma(c') = d']$. It is readily verified that the sets $O_m(d, d')$ satisfy properties $1-8$. (The condition $\sigma(C_i) = D_i$ is necessary for property 7). Then, by theorem 5 (if $S' \times D$ is topologized appropriately) the map $(g, c) \to (g, \sigma(c))$ is a homomorphism from $(S^1 \times C, T)$ to $(S^1 \times D, T)$.

Now, for $n \geqq 2$, let C_n be a set with cardinality n, let $C_n = C_{1n} \cup C_{2n}$, where $C_{1n} \cap C_{2n} = \emptyset$ and let $\sigma_n : C_{n+1} \to C_n$ such that $\sigma(C_{i,n+1}) = C_{i,n}$ $(i = 1, 2)$. Then if $X_n = S^1 \times C_n$, there are, by the discussion in the preceding paragraph, homomorphisms $\pi_n : (X_{n+1}, T) \to (X_n, T)$. We may regard the collection of minimal sets and homomorphisms $\{(X_n, T), \pi_n\}$ as an inverse system. It is easy to see that the inverse limit of this system, namely, the subset of $\times_{n=2,3,\cdots} X_n$ of points $(x_2, x_3, \cdots)$ satisfying $\pi_n(x_{n+1}) = x_n$ is a minimal set, and therefore, by theorem 6 of [3], is the regular minimal set $(\vee X_n, T)$.

Let $X^* = \vee X_n$, and let $\psi_n : X^* \to X_n$ be the n^{th} projection. We may represent points of X^* by $x^* = (g, c_2, c_3, \cdots)$ $(g \in S^1, c_j \in C_j$ with $\sigma_1(c_{j+1}) = c_j)$. It follows from lemma 1 that $P(x^*)$ consists of those points $y^* = (g, c'_1, c'_2, \cdots)$ in X^* with $c'_j \in C_j$. Clearly $P(x^*)$ is compact, self-dense, and metrizable. Moreover, $P(x^*)$ is totally disconnected. For if

$y^* = (g, c_2', c_3', \cdots)$ and $z^* = (g, c_2'', c_3'', \cdots)$ are in a connected component of $P(x^*)$, then $\psi_n(y^*) = \psi_n(z^*)$, for $n = 2, 3, \cdots$ and $y^* = z^*$. Therefore $P(x^*)$ is homeomorphic to the Cantor discontinuum.

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