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REPRESENTATIONS OF SEMI DIRECT PRODUCTS OF GROUPS

by

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Introduction

Let G_1 be a locally compact Abelian group, G_2 a locally compact group of continuous automorphisms of G_1. In this paper we characterise all pairs of unitary representations ρ and σ of G_1 and G_2 respectively in a Hilbert space $\mathfrak{H}$, where ρ is cyclic and

$$\sigma(\alpha)\rho(x)\sigma(\alpha^{-1}) = \rho(\alpha[x]),\ \alpha \in G_2, x \in G_1. \tag{*}$$

A set of necessary and sufficient conditions for a pair (ρ_1, σ_1) to be unitarily equivalent to a pair (ρ_2, σ_2) is given.

It can be shown that the commutation relations (*) define a system of imprimitivity for the representation σ. In [[3] § 14.] Mackey investigates these representations, from a different point of view from ours, primarily as an application of his theory of induced representations.

I would like to thank the referee for his helpful comments.

1. Preliminaries

DEFINITION 1.1. Let G be a locally compact group. A unitary representation of G is a homomorphism $\Pi : g \to \Pi(g)$ of G into the group of unitary transformations of a Hilbert space $\mathfrak{H}(\Pi)$, such that Π is continuous in the weak topology for operators. A closed linear manifold $\mathfrak{M}$ is called an invariant subspace for Π if $\Pi(g)\,\xi \in \mathfrak{M}$ for all $g \in G$ and all $\xi \in \mathfrak{M}$. An invariant subspace $\mathfrak{M}$ is said to be a cyclic subspace for Π if there is an element ξ_0 in $\mathfrak{H}(\Pi)$, such that the smallest invariant subspace for Π containing ξ_0 is $\mathfrak{M}$. ξ_0 is called a relative cyclic vector for Π. If $\mathfrak{H}(\Pi)$ is a cyclic subspace, then Π is said to be cyclic. The intertwining algebra of a representation Π is the set

$$R(\Pi, \Pi) = \{T : T\Pi(g) = \Pi(g)T,\ g \in G\},$$

T being bounded, everywhere defined, linear transformations on $\mathfrak{H}(\Pi)$.

DEFINITION 1.2. Let $\mathfrak{H}$ be a Hilbert space. A^{-*} algebra $\mathfrak{A}$ of (bounded, everywhere defined, linear) transformations of $\mathfrak{H}$ is called a von Neumann algebra, if $\mathfrak{A}$ is closed in the weak topology for operators. A closed linear manifold $\mathfrak{M}$ is called an invariant subspace for $\mathfrak{A}$, if $A\xi \in \mathfrak{M}$ for all $A \in \mathfrak{A}$ and all $\xi \in \mathfrak{M}$. An invariant subspace $\mathfrak{M}$ is said to be a cyclic subspace for $\mathfrak{A}$ if there is an element ξ_0 in $\mathfrak{H}$, such that the smallest invariant subspace for $\mathfrak{A}$ containing ξ_0 is $\mathfrak{M}$. If $\mathfrak{H}$ is an invariant subspace for $\mathfrak{A}$, then $\mathfrak{A}$ is said to be cyclic. The commutant of $\mathfrak{A}$ is the set

$$\mathfrak{A}' = \{T : TA = AT, A \in \mathfrak{A}\},$$

T being bounded, everywhere defined, linear transformations on $\mathfrak{H}$.

It is easy to prove that a closed linear manifold $\mathfrak{M} \subseteq \mathfrak{H}(\Pi)$ (resp. $\mathfrak{M} \subseteq \mathfrak{H}$) is a cyclic subspace for Π (resp. $\mathfrak{A}$) if and only if there is an element $\xi_0 \in \mathfrak{H}(\Pi)$ (resp. $\xi_0 \in \mathfrak{H}$) such that the closed linear manifold generated by $(\Pi(g)\xi_0 : g \in G)$ (resp. $(A\xi_0 : A \in \mathfrak{A})$) is $\mathfrak{M}$.

If S is a set of elements in a Hilbert space the closed linear manifold generated by S is denoted by $[s : s \in S]$.

Let $\Pi : g \to \Pi(g)$ be a representation of a locally compact group G. We shall often use the following well-known results

LEMMA 1.1.

(i) *$R(\Pi, \Pi)$ is a von Neumann algebra;*
(ii) *$R(\Pi, \Pi)'$ is the smallest von Neumann algebra containing the operators $(\Pi(g) : g \in G)$;*
(iii) *$\mathfrak{M}$ is an invariant subspace for Π (resp. $R(\Pi, \Pi)'$) if and only if P, the projection whose range is $\mathfrak{M}$, belongs to $R(\Pi, \Pi)$.*
(iv) *A closed linear manifold $\mathfrak{M}$ is a cyclic subspace for Π if and only if $\mathfrak{M}$ is a cyclic subspace for $R(\Pi, \Pi)'$.*

DEFINITION 1.3. Let X be a locally compact space, μ a finite regular measure defined on the σ-ring of Borel subsets of X. We denote by $L(X)$ the set of all continuous functions with compact support; $C(X)$ the set of all continuous functions on X. If $f \in L(X)$ we denote by M_f the operator on $L^2(X, \mu)$ defined by $(M_f h)(x) = f(x)h(x)$, where $h \in L^2(x, \mu)$.

LEMMA 1.2. *Let G_1 be a locally compact Abelian group, $\hat{G}_1$ the character group of G_1 and μ a finite regular measure defined on the σ-ring of Borel subsets of $\hat{G}_1$. The mapping $M : x \to M_x$, where $(M_x f)(\tau) = x(\tau)f(\tau), f \in L^2(\hat{G}_1, \mu)$, $x \in G_1$ is a cyclic representation of G_1.*

PROOF. It is easy to verify that $M : x \to M_x$ is a weakly continuous unitary representation of G_1. We shall show that M is cyclic.

Let e be the function on $\hat{G}_1$, $e(\tau) = 1$. Since μ is a finite measure on $\hat{G}_1$, e belongs to $L^2(\hat{G}_1, \mu)$ and therefore $x = M_x e \in L^2(\hat{G}_1, \mu)$ for all $x \in G_1$. Denote by F the set of all finite linear combination of elements of G_1. We recall [[4] § 31, cor. 4] that every continuous function on $\hat{G}_1$ can be approximated uniformly on compact sets by members of F. If f, h_1, h_2 are continuous functions with compact supports and $\varepsilon > 0$, we can find $s \in F$ such that

$$|f(\tau)-s(\tau)| < \frac{\varepsilon}{||h_1|| \, ||h_2||} \text{ for all } \tau \in k_1 \cap k_2$$

where K_i is the support of h_i. Hence

$$|((M_f - M_s)h_1, h_2)| = \left| \int_{\hat{G}_1} (f(\tau)-s(\tau))h_1(\tau)\overline{h_2(\tau)} d\mu(\tau) \right|$$
$$< \frac{\varepsilon}{||h_1|| \, ||h_2||} ||h_1|| \, ||h_2|| = \varepsilon.$$

This is true for all $h_1 \in L(\hat{G}_1)$ and $h_2 \in L(\hat{G}_1)$. Since $L(\hat{G}_1)$ is dense in $L^2(\hat{G}_1, \mu)$, we have proved that $(M_f : f \in L(\hat{G}_1))$ belongs to the weakly closed algebra generated by $(M_x : x \in G_1)$. From Lemma 1.1 (ii) we deduce that $(M_f : f \in L(\hat{G}_1)) \subseteq R(\Pi, \Pi)'$ and therefore from the (iv) of Lemma 1.1. we deduce that $f = M_f e \in [M_x e : x \in G_1]$. That is $L(\hat{G}_1) \subseteq [M_x e : x \in G_1]$. We complete the proof by observing that $L(\hat{G}_1)$ is dense in $L^2(G_1, \mu)$.

LEMMA 1.3. *Let $\Pi : x \to \Pi(x)$ be a cyclic representation of a locally compact Abelian group G_1. There is a regular finite measure μ on $\hat{G}_1$, and a linear isometry $S : \mathfrak{H}(\Pi) \to L^2(\hat{G}_1, \mu)$ such that $S\Pi(x)S^{-1} = M_x$, where $M : x \to M_x$ is the representation of G_1 defined in Lemma* 1.2.

PROOF. Let ξ_0 be a cyclic element for the cyclic representation Π, and let $\Phi(x) = (\Pi(x)\xi_0, \xi_0)$. There is a positive functional P on $R(G_1)$, the group algebra of G_1, which corresponds to the continuous positive definite function Φ. Since $R(G_1)$ is a commutative Banach algebra, the positive functional P can be represented in the form

$$P(f) = \int_{\Delta} f(\tau) d\mu(\tau).$$

The spectrum Δ of $R(G_1)$ is homeomorphic to $\hat{G}_1 \cup \{L^1(G_1)\}$ and $\mu(\{L^1(G_1)\}) = 0$. Therefore, the measure μ may be considered as a measure defined on $\hat{G}_1$ [[4] § 31, sec. 3].

The Gelfand isomorphism theorem allows us to regard P as a positive functional on $C(\Delta)$, where $C(\Delta)$ is the set of all continuous functions on Δ. The positive functional P defines a representation of $C(\Delta)$ which is equivalent to the representation $M : f \to M_f$ on $L^2(\Delta, \mu)$, where

$$(M_f g)(\delta) = f(\delta)g(\delta), \; g \in L^2(\Delta, \mu).$$

[[4]. ch. 4. § 17]. Since $G_1 \subseteq C(\Delta)$, we obtain a representation $M : x \to M_x$ of G_1 in

$$L^2(\Delta, \mu) = L^2(\hat{G}_1, \mu),$$

where

$$(M_x g)(\tau) = x(\tau)g(\tau).$$

Since the representations M and Π of G_1 define the same representation of $R(G_1)$, namely the representation defined by the positive functional P, the representations M and Π are equivalent. [[4] § 29, sec. 3].

2. Semi-direct products

Let G be a locally compact group, G_2 a locally compact group of automorphisms of G such that the mapping $(g, \alpha) \to \alpha[g]$ of $G \times G_2$ into G is continuous in both variables. The semi-direct product $G \circledS G_2$ is the set of all pairs (g, α), $g \in G$, $\alpha \in G_2$, whose group operation is defined by

$$(g, \alpha)(h, \beta) = (g\alpha[h], \alpha\beta).$$

$G \circledS G_2$ is a locally compact group in the product topology. The mapping $g \to (g, \varepsilon)$ where ε is the identity of G_2 is an isomorphism between G and a closed normal subgroup of $G \circledS G_2$. The mapping $\alpha \to (e, \alpha)$ where e is the identity element of G is an isomorphism between G_2 and a closed subgroup of $G \circledS G_2$. Finally, $(g, \alpha) = (g, \varepsilon)(e, \alpha)$. [[2] pp. 6–7, 58-59, [3] § 14]. The proof of the following lemma is routine.

LEMMA 2.1. *Let $\rho : g \to \rho(g)$ and $\sigma : \alpha \to \sigma(\alpha)$ be representations of G and G_2 respectively in a Hilbert space $\mathfrak{H}$. The mapping $\Pi : (g, \alpha) \to \Pi(g, \alpha)$, where $\Pi(g, \alpha) = \rho(g)\sigma(\alpha)$ is a representation of $G \circledS G_2$ if and only if*

$$\sigma(\alpha)\rho(g)\sigma(\alpha^{-1}) = \rho(\alpha[g]).$$

In the following pages let G_1 be a locally compact Abelian group, $\hat{G}_1$ the character group of G_1, G_2 a locally compact group of continuous automorphisms of G_1 such that the mapping $(x, \alpha) \to \alpha[x]$ of $G_1 \times G_2$ to G_1 is continuous in both variables. The group G_2 acts as a group of automorphisms of $\hat{G}_1$, if we define $[\tau]\alpha$ by the equation $([\tau]\alpha)(x) = \tau(\alpha[x])$, $x \in G_1$. [[2] 26.9].

DEFINITION 2.1. Let μ be a finite Borel measure defined on $\hat{G}_1$, and for each $\alpha \in G_2$ let μ_α be the measure on $\hat{G}_1$ defined by $\mu_\alpha(B) = \mu([B]\alpha)$. The measure μ is said to be G_2-quasi invariant if μ_α is absolutely continuous with respect to μ for all $\alpha \in G_2$.

LEMMA 2.2. *Let μ be a G_2-quasi invariant measure on $\hat{G}_1$. The mapping*

$$\Pi : (x, \alpha) \to \Pi(x, \alpha) = \Pi(x, \varepsilon)\Pi(e, \alpha),$$

where

$$(\Pi(x, \varepsilon)f)(\tau) = x(\tau)f(\tau)$$

and

$$(\Pi(e, \alpha)f)(\tau) = \sqrt{\frac{d\mu_\alpha}{d\mu}}(\tau)f([\tau]\alpha), \; f \in L^2(G_1, \mu)$$

is a representation of $G_1 \circledS G_2$ *in* $L^2(\hat{G}_1, \mu)$.

As the proof consists of a routine verification of the condition given in lemma 2.1, we omit the proof.

THEOREM 2.1. *Let* $\Pi : (x, \alpha) \to \Pi(x, \alpha)$ *be a representation of* $G_1 \circledS G_2$ *in a Hilbert space* $\mathfrak{H}(\Pi)$ *such that the representation* $\Pi(x, \varepsilon)$ *of* G_1 *in* $\mathfrak{H}(\Pi)$ *is cyclic. There is a* G_2*-quasi invariant measure* μ *on* $\hat{G}_1$ *and a linear isometry* S *from* $\mathfrak{H}(\Pi)$ *on to* $L^2(\hat{G}_1, \mu)$ *such that*

$$S\Pi(x, \varepsilon)S^{-1}f(\tau) = x(\tau)f(\tau)$$

and

$$S\Pi(e, \alpha)S^{-1}f(\tau) = a(\tau, \alpha)\sqrt{\frac{d\mu_\alpha}{d\mu}}(\tau)f([\tau]\alpha)$$

where $a(\tau, \alpha)$ *is a Borel function on* $\hat{G}_1 \times G_2$ *with the following properties*:

i $|a(\tau, \alpha)| = 1$ *almost everywhere, and*
ii $a(\tau, \alpha_1\alpha_2) = a(\tau, \alpha_1)\, a([\tau]\alpha_1, \alpha_2)$, *a.e.*

PROOF. Let $\rho(x) = \Pi(x, \varepsilon)$ and $\sigma(\alpha) = \Pi(e, \alpha)$. Since ρ is a cyclic representation of G_1 in $\mathfrak{H}(\Pi)$, it follows from Lemma 1.2 that there is a finite Borel measure μ on $\hat{G}_1$ and a linear isometry S from $\mathfrak{H}(\Pi)$ onto $L^2(\hat{G}_1, \mu)$ such that $S\rho(x)S^{-1}f(\tau) = x(\tau)f(\tau)$. The well-known Stone-Naimark-Ambrose-Godement theorem asserts that there is a projection valued measure $P : B \to P_B$ on the Borel subsets of $\hat{G}_1$ to the projections in the intertwining algebra $R(\rho, \rho)'$ such that [[4] § 31. Th. 6]

$$(\rho(x)\xi, \eta) = \int_{\hat{G}_1} x(\tau)d(P_\tau\xi, \eta) \tag{1}$$

for every pair of elements ξ and η in $\mathfrak{H}(\Pi)$. Moreover, if ξ_0 is a cyclic element for the representation ρ then the measure μ is equivalent to the measure ν where $\nu(B) = ||P_B\xi_0||^2$. Now

$$(\sigma(\alpha)\rho(x)\sigma(\alpha^{-1})\xi, \eta) = \rho(\alpha[x]\xi, \eta). \tag{2}$$

From (1) we have

$$\begin{aligned}(\sigma(\alpha)\rho(x)\sigma(\alpha^{-1})\xi, \eta) &= (\rho(x)\sigma(\alpha^{-1})\xi, \sigma(\alpha^{-1})\eta) \\ &= \int_{\hat{G}_1} x(\tau)d(P_\tau\sigma(\alpha^{-1})\xi, \sigma(\alpha^{-1})\eta) \\ &= \int_{\hat{G}_1} x(\tau)d(\sigma(\alpha)P_\tau\sigma(\alpha^{-1})\xi, \eta) \qquad (3)\end{aligned}$$

Also,

$$\begin{aligned}(\rho(\alpha[x])\xi, \eta) &= \int_{\hat{G}_1} \alpha[x](\tau)d(P_\tau\xi, \eta) \\ &= \int_{\hat{G}_1} x([\tau]\alpha)d(P_\tau\xi, \eta) = \int_{\hat{G}_1} x(\tau)d(P_{[\tau]\alpha-1}\xi, \eta) \qquad (4)\end{aligned}$$

It follows from (2), (3) and (4) that

$$\sigma(\alpha)P_B\sigma(\alpha^{-1}) = P_{[B]\alpha^{-1}}. \qquad (5)$$

Now $\mu(B) = 0$ implies $\nu(B) = 0$ and consequently $P_B\xi_0 = 0$. Since $P_B \in R(\rho, \rho)$, the equation $0 = TP_B\xi_0 = P_BT\xi_0$, $T \in R(\rho, \rho)'$ implies $P_BE = 0$ where E is the projection on the closed linear manifold generated by $(T\xi_0 : T \in R(\rho, \rho)')$. However, $E = I$ because ξ_0 is a cyclic element for ρ. Therefore $P_B = 0$. Thus $\mu(B) = 0$ implies $P_B = 0$, and from (5) it follows that $P_{[B]\alpha^{-1}} = 0$. That is, $\mu(B) = 0$ implies $\nu([B]\alpha^{-1}) = 0$. Since μ and ν are equivalent, $\nu([B]\alpha^{-1}) = 0$, implies $\mu([B]\alpha^{-1}) = 0$.

Hence $\mu_{\alpha^{-1}}$ is absolutely continuous with respect to μ. Since $\alpha \in G_2$ is arbitrary, we have shown that μ is G_2 quasi invariant.

Let

$$\sigma_0(\alpha)f(\tau) = \sqrt{\frac{d\mu_\alpha}{d\mu}}(\tau)f([\tau]\alpha),\ f \in L^2(\hat{G}_1, \mu)$$

and

$$\sigma_1(\alpha) = S\sigma(\alpha)S^{-1}\sigma_0(\alpha^{-1})$$

where S is the linear isometry $\mathfrak{H}(\Pi) \to L^2(\hat{G}_1, \mu)$ introduced in the first paragraph of this proof. It is clear that $\sigma_1(\alpha)$ is a unitary transformation. Now, from the relation $\sigma_0(\alpha^{-1})M_x = M_{\alpha^{-1}[x]}\sigma_0(\alpha^{-1})$, we have

$$\begin{aligned}\sigma_1(\alpha)S\rho(x)S^{-1}(\tau) &= S\sigma(\alpha)S^{-1}\sigma_0(\alpha^{-1})M_xf(\tau) \\ &= S\sigma(\alpha)S^{-1}M_{\alpha^{-1}[x]}\sigma_0(\alpha^{-1})f(\tau) \\ &= S\sigma(\alpha)S^{-1}S\rho(\alpha^{-1}[x])S^{-1}\sigma_0(\alpha^{-1})f(\tau) \\ &= S\sigma(\alpha)\rho(\alpha^{-1}[x])S^{-1}\sigma_0(\alpha^{-1})f(\tau) \\ &= S\rho(\alpha\alpha^{-1}[x])\sigma(\alpha)S^{-1}\sigma_0(\alpha^{-1})f(\tau) \\ &= S\rho(x)\sigma(\alpha)S^{-1}\sigma_0(\alpha^{-1})f(\tau) \\ &= S\rho(x)S^{-1}S\sigma(\alpha)S^{-1}\sigma_0(\alpha^{-1})f(\tau) \\ &= S\rho(x)S^{-1}\sigma_1(\alpha)f(\tau).\end{aligned}$$

This shows that $\sigma_1(\alpha)$ commutes with $S\rho(x)S^{-1} = M_x$ and consequently $\sigma_1(\alpha)$ commutes with the von Neumann algebra generated by M_x. It is known [[5] cor. 1.1] that a commutative von Neumann algebra with a cyclic vector is maximal Abelian. Therefore $\sigma_1(\alpha)$ belongs to the von Neumann algebra generated by $(M_x : x \in G_1)$ which is the algebra of multiplication by essentially bounded measurable functions on $(\hat{G}_1, \mu)$. Hence $\sigma_1(\alpha)f(\tau) = a(\tau, \alpha)f(\tau)$ where $a(\tau, \alpha)$ is, for each α a measurable essentially bounded function of modulus 1. We introduce the operator M_a in $L^2(\hat{G}_1, \mu)$ where $(M_a f)(\tau) = a(\tau)f(\tau)$.

From the equation $S\sigma(\alpha)S^{-1}\sigma_0(\alpha^{-1}) = M_\alpha$ we obtain $S\sigma(\alpha)S^{-1} = M_a\sigma_0(\alpha)$: that is

$$S\sigma(\alpha)S^{-1}f(\tau) = a(\tau, \alpha)\sqrt{\frac{d\mu_\alpha}{d\mu}}(\tau)f([\tau]\alpha).$$

Finally,

$$S\sigma(\alpha_1\alpha_2)S^{-1}f(\tau) = a(\tau, \alpha_1\alpha_2)\sqrt{\frac{d\mu_{\alpha_1\alpha_2}}{d\mu}}(\tau)f([\tau]\alpha_1\alpha_2)$$

$$\begin{aligned}
S\sigma(\alpha_1)\sigma(\alpha_2)S^{-1}f(\tau) &= S\sigma(\alpha_1)S^{-1}S\sigma(\alpha_2)S^{-1}f(\tau) \\
&= S\sigma(\alpha_1)S^{-1}a(\tau, \alpha_2)\sqrt{\frac{d\mu_{\alpha_2}}{d\mu}}(\tau)f([\tau]\alpha_2) \\
&= a(\tau, \alpha_1)\sqrt{\frac{d\mu_{\alpha_1}}{d\mu}}(\tau)\alpha([\tau]\alpha_1, \alpha_2). \\
&\quad \sqrt{\frac{d\mu_{\alpha_2}}{d\mu}}([\tau]\alpha_1)f([\tau]\alpha_1\alpha_2) \\
&= a(\tau, \alpha_1)a([\tau]\alpha_1, \alpha_2)\sqrt{\frac{d\mu_{\alpha_1\alpha_2}}{d\mu_{\alpha_1}}}(\tau) \\
&\quad \sqrt{\frac{d\mu_{\alpha_1}}{d\mu}}(\tau)f([\tau]\alpha_1\alpha_2) \\
&= a(\tau, \alpha_1)a([\tau]\alpha_1, \alpha_2)\sqrt{\frac{d\mu_{\alpha_1\alpha_2}}{d\mu}}(\tau)f([\tau]\alpha_1\alpha_2).
\end{aligned}$$

Since $S\sigma(\alpha_1\alpha_2)S^{-1} = S\sigma(\alpha_1)\sigma(\alpha_2)S^{-1}$ we have

$$\alpha(\tau, \alpha_1\alpha_2) = \alpha([\tau]\alpha_1, \alpha_2)\alpha(\tau, \alpha_1), \text{ a.e.}$$

This completes the proof of the theorem.

DEFINITION 2.2. A Borel measure μ on $\hat{G}_1$ is said to be G_2-ergodic if

1. μ is G_2-quasi invariant, and

2. the G_2-quasi-invariant non zero measures on $\hat{G}_2$ which are absolutely continuous with respect to μ are equivalent to μ.

THEOREM 2. *Let G_1 and G_2 be as in the paragraph preceding Definition 2.1. Let $\Pi : (x, \alpha) \to \Pi(x, \alpha) = \rho(x)\sigma(\alpha)$ be a representation of $G_1 \circledS G_2$. If the measure μ defined by the cyclic representation ρ is G_2 ergodic, then Π is irreducible.*

PROOF. Suppose a closed linear manifold $\mathfrak{M}$ of $\mathfrak{H}(\Pi)$ is invariant for Π. Then clearly $\mathfrak{M}$ is invariant for ρ and σ. Let E be the projection whose range is $\mathfrak{M}$. E belongs to $R(\rho, \rho)$. The representation ρ being cyclic, the von Neumann algebra $R(\rho, \rho)'$, generated by the operators $\rho(x) : x \in G_1$, is a commutative von Neumann algebra with a cyclic element. Consequently [[5]. cor. 1.1]. $R(\rho, \rho)'$ is maximal Abelian. Therefore $R(\rho, \rho') = R(\rho, \rho)$. Since every projection of $R(\rho, \rho)'$ is of the form P_B, where $P : B \to P_B$ is the projection valued measure defined by ρ, there is a Borel set B_0 of $\hat{G}_1$ such that $E = P_{B_0}$.

Let $\mu_0(B) = \mu(B_0 \cap B)$. Clearly μ_0 is absolutely continuous with respect to μ. We shall show that μ_0 is G_2-quasi invariant. The measure μ is equivalent to the measure ν where $\nu(B) = ||P_B \xi_0||^2$. We may for the purpose of this proof assume, without loss of generality, that $\mu(B) = ||P_B \xi_0||^2$. From equation (5) in the proof of Theorem 2.1 we have

$$\sigma(\alpha)P_{B_0 \cap B}\sigma(\alpha^{-1}) = P_{[B_0 \cap B]\alpha^{-1}}.$$

However,

$$\begin{aligned}\sigma(\alpha)P_{B_0 \cap B}\sigma(\alpha^{-1}) &= \sigma(\alpha)P_{B_0}P_B\sigma(\alpha^{-1})\\ &= \sigma(\alpha)P_{B_0}\sigma(\alpha)^{-1}\sigma(\alpha)P_B\sigma(\alpha^{-1}).\end{aligned}$$

Now suppose $\mu_0(B) = 0$. Then $\mu(B_0 \cap B) = 0$, and by the G_2-quasi invariance of μ, it follows that $\mu([B_0 \cap B]\alpha^{-1}) = 0$. Consequently,

$$\begin{aligned}0 &= ||P_{[B_0 \cap B]\alpha^{-1}}\xi_0||^2\\ &= ||P_{B_0 \cap [B]\alpha^{-1}}\xi_0||^2\\ &= \mu(B_0 \cap [B]\alpha^{-1}) = \mu_0([B]\alpha^{-1}).\end{aligned}$$

Since α in G_2 is arbitrary, we have shown that μ_0 is G_2-quasi invariant. The measure μ is G_2-ergodic. Therefore either μ_0 is equivalent to μ or μ_0 is the zero measure. That is either $B_0 = \hat{G}_1$ or $B_0 = \phi$. Consequently, $\mathfrak{M} = \mathfrak{H}$ or $\mathfrak{M} = \{0\}$.

This completes the proof.

DEFINITION 2.3. Let $\Pi_i : (x, \alpha) \to \Pi_i(x, \alpha) = \rho_i(x)\sigma_i(\alpha)$ be representations of G_1 s G_2 in $\mathfrak{H}(\Pi_i)$, $i = 1, 2$. Π_i is said to be equivalent to Π_2 if there is a linear isometry $S : \mathfrak{H}(\Pi_1) \to \mathfrak{H}(\Pi_2)$ such that

$$S\rho_1(x)S^{-1} = \rho_2(x) \quad S\sigma_1(\alpha)S^{-1} = \sigma_2(\alpha).$$

THEOREM 2.3. *Let $\Pi_i : (x, \alpha) \to \Pi_i(x, \alpha)$ be representations of $G_1 \circledS G_2$ on $\mathfrak{H}$ (Π_i) where*

$$\Pi_i(\circ, \varepsilon) : x \to \Pi_i(x, \varepsilon) \quad i = 1, 2$$

are cyclic. A set of necessary and sufficient condition that Π_1 is equivalent to to Π_2 is

1. *μ^1 is equivalent to μ^2 where μ^i is the measure on $\hat{G}_i$ defined by Π_i $(\circ, \varepsilon)$ $i = 1, 2$; and*
2. *there exists a Borel function b on $\hat{G}_1$ with the properties*
   2.1. *$|b(\tau)| = 1$ almost everywhere, and*
   2.2. *$a_2(\tau, \alpha) = b(\tau)a_1(\tau, \alpha)b^{-1}([\tau]\alpha)$ where $a_i(\tau, \alpha)$ is the function associated with $\Pi_i(e, \circ) : \alpha \to \Pi_i(e, \alpha)$ in theorem 2.1.*

PROOF. It is evident from Theorem 2.1 that, Π_1 is equivalent to Π_2 if and only if the following is true: (*) there is a linear isometry $S : L^2(\hat{G}_1, \mu^1) \to L^2(\hat{G}_1, \mu^2)$ such that $S\rho_1(x) = \rho_2(x)S$ where

$$\rho_i(x)f(\tau) = x(\tau)f(\tau),\ f \in L^2(\hat{G}_1, \mu^i)$$

and $S\sigma_1(\alpha) = \sigma_2(\alpha)S$, where

$$\sigma_i(\alpha)(\tau) = a_i(\tau, \alpha)\sqrt{\frac{d\mu^i_\alpha}{d\mu^i}}(\tau)f([\tau]\alpha),\ i = 1, 2.$$

Assume that the conditions (*) are satisfied. We recall that $L(\hat{G}_1)$, the set of all continuous functions with compact support, is dense in $L^p(G_1, \mu^i)$ where $p = 1, 2$ and $i = 1, 2$. In the course of the proof of lemma 1.2 we saw that the operators $\rho_i(g)$ where $(\rho_i(g)f)(\tau) = g(\tau)f(\tau)$, $g \in L(\hat{G}_1)$ and $f \in L^2(\hat{G}_1, \mu^i)$ belong to $R(\rho_i, \rho_i)'$. It is easily verified that $S\rho_1(x)S^{-1}$ $\rho_2(x)$ implies $S\rho_1(g)S^{-1} = \rho_2(g)$ for all $g \in L(\hat{G}_1)$.

Since $S\rho_1(x)S^{-1} = \rho_2(x)$ for all x in G_1, the commutative von Neumann algebra $R(\rho_1, \rho_1)'$ generated by $(\rho_1(x) : x \in G)$ is unitarily equivalent to the von Neumann algebra $R(\rho_2, \rho_2)'$ generated by $(\rho_2(x) : x \in G_1)$. Since ρ_i are cyclic representations, the commutative von Neumann algebras $R(\rho_i, \rho_i)'$ are cyclic. A commutative von Neumann algebra with a cyclic vector is maximal Abelian ([5] corollary 1.1) and is unitarily equivalent to a multiplication algebra ([5] Lemma 1.2). Consequently, the multiplication algebra on $L^2(\hat{G}_1, \mu^1)$ is unitarily equivalent to the multiplication algebra on $L^2(\hat{G}_1, \mu^2)$ and therefore ([6] Theorem 4.1) μ^1 is equivalent to μ^2.

The function e, where $e(\tau) = 1$ for all $\tau \in \hat{G}_1$, belongs to $L^2(\hat{G}_1, \mu^1)$. Let $Se = c \in L^2(\hat{G}_1, \mu^2)$. We shall show that c is an essentially bounded function. If $g \in L(\hat{G}_1)$, we have

$$Sg = Sge = S\rho_1(g)e = S\rho_1(g)S^{-1}Se$$
$$= \rho_2(g)Se = \rho_2(g)c. \qquad \text{(i)}$$

Let
$$C(g) = \int_{\hat{G}_1} |c(\tau)|^2 g(\tau) d\mu^2(\tau)$$
$$|C(g)| = (gc, c) = (\rho_2(g)c, c) = (\rho_2(g)Se, Se)$$
$$= (S^{-1}\rho_2(g)Se, e) = (\rho_2(g)e, e)$$
$$= \int_{\hat{G}_1} g(\tau) d\mu^1(\tau). \qquad \text{(ii)}$$

Hence $|C(g)| \leqq ||g||_1$ (the L^1-norm of $g \in L^1(\hat{G}_1, \mu^1)$.

That is, $C(g)$ is bounded on a dense linear subset $L(\hat{G}_1)$ of $L^1(\hat{G}_1, \mu^1)$, and can therefore be extended to $L^1(\hat{G}_1, \mu^1)$. Hence $C \in L^\infty(\hat{G}_1, \mu^1)$, and therefore c is essentially bounded with respect to μ^1. Since μ^1 and μ^2 are equivalent it follows that c is essentially bounded with respect to μ^2.

Since the function c is essentially bounded the equation (i) can be written in the form $Sg = M_c g$ where M_c is the operation of multiplying by c. Since M_c is a bounded operator and $L(\hat{G}_1)$ is dense in $L^2(\hat{G}_1, \mu)$, the equation $Sg = M_c g$ holds for all g in $L^2(\hat{G}_1, \mu^1)$. It follows from the equivalence of μ^1 and μ^2 and the equation (ii) that

$$\int_{\hat{G}_1} g(\tau)|c(\tau)|^2 d\mu^2(\tau) = \int_{\hat{G}_1} g(\tau) d\mu^1(\tau)$$
$$= \int_{\hat{G}_1} g(\tau) \frac{d\mu^1}{d\mu^2}(\tau) d\mu^2(\tau).$$

Hence
$$|c(\tau)|^2 = \frac{d\mu^1}{d\mu^2}(\tau)$$

almost everywhere, and

$$c(\tau) = b(\tau)\sqrt{\frac{d\mu^1}{d\mu^2}}(\tau)$$

where
$$|b(\tau)| = 1$$

almost everywhere.

Now,

$$S\sigma_1(\alpha)(\tau) = M_c a_1(\tau, \alpha)\sqrt{\frac{d(\mu^1)_\alpha}{d\mu^1}}(\tau) g([\tau]\alpha)$$
$$= b(\tau)\sqrt{\frac{d\mu^1}{d\mu^2}}(\tau) a_1(\tau, \alpha)\sqrt{\frac{d(\mu^1)_\alpha}{d\mu^1}}(\tau) g([\tau]\alpha)$$
$$= b(\tau)\alpha_1(\tau, \alpha)\sqrt{\frac{(d\mu^1)_\alpha}{d\mu^2}}(\tau) g([\tau]\alpha).$$

$$\begin{aligned}\sigma_2(\alpha)Sg &= \sigma_2(\alpha)b(\tau)\sqrt{\frac{d\mu^1}{d\mu^2}}(\tau)g(\tau)\\ &= a_2(\tau,\alpha)\sqrt{\frac{d(\mu^2)_\alpha}{d\mu^2}}(\tau)b([\tau]\alpha)\sqrt{\frac{d\mu^1}{d\mu^2}}([\tau]\alpha)g([\tau]\alpha)\\ &= a_2(\tau,\alpha)b(\tau\alpha)\sqrt{\frac{d(\mu^2)_\alpha}{d\mu^2}}(\tau)\sqrt{\frac{d(\mu^2)_\alpha}{d(\mu^2)_\alpha}}(\tau)g(\tau\alpha)\\ &= a_2(\tau,\alpha)b([\tau]\alpha)\sqrt{\frac{d(\mu^1)_\alpha}{d\mu^2}}(\tau)g([\tau]\alpha).\end{aligned}$$

Hence the equation $S\sigma_1(\alpha)g = \sigma_2(\alpha)Sg$ yields

$$b(\tau)\alpha_1(\tau,\alpha) = \alpha_2(\tau,\alpha)b([\tau]\alpha), \text{ a.e.}$$

i.e.

$$\alpha_2(\tau,\alpha) = b(\tau)\alpha_1(\tau,\alpha)b^{-1}([\tau]\alpha) \text{ a.e.}$$

The converse is easy to verify and we omit the details.
This completes the proof.

The condition 2 of the last theorem can be reformulated in terms of a one dimensional cohomology group. To this end we observe first that G_2 as a group of automorphisms of $L^\infty(\hat{G}_1, \mu) : \alpha[g](\tau) = g([\tau]\alpha)$. Furthermore, the function $\alpha(\tau, \alpha)$ of Theorem 2.1 defines a mapping $\tilde{\alpha} : G_2 \to L^\infty(\hat{G}_1, \mu)$ where $(\tilde{\alpha}(\alpha))(\cdot) = \alpha(\cdot, \alpha)$. From ii of theorem 2.1 we see that $\tilde{\alpha}$ is a crossed homomorphism. It is evident that $(\tilde{b}(\alpha))(\cdot) = b(\cdot)$ $b^{-1}([\cdot]\alpha)$, where $b \in L^\infty$, is a principal crossed homomorphism. In view of these observations the condition 2 of Theorem 2.3 states α_1 and α_2 define the same element of the one dimensional cohomology group $H^1(G_2, L^\infty)$.

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