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ON THE UNIVALENCE OF POLYNOMIALS

by

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Let $f(z)$ be a polynomial of degree n ($n \geqq 2$). If the moduli of all the zeros of $f'(z)$ are greater than or equal to cosec (π/n) then by a theorem of S. Kakeya [1] $f(z)$ is univalent in $|z| < 1$. M. Robertson [2] gave a necessary and sufficient condition for $f(z)$ to have the radius of univalence exactly equal to 1. I [3] formed a counter example to show that this result is not sufficient. In connection with the same problem I will now prove another necessary and sufficient condition, Theorem 1, and by this proof will also deduce Kakeya's result. Then I will consider a result given by L. N. Čakalov, which will follow from Theorem 1, and will give some improved results.

THEOREM 1. *Let $f(z)$ be a polynomial of degree n ($n \geqq 2$). If the moduli of all the zeros of $f'(z)$ are greater than or equal to* cosec (π/n) *then the necessary and sufficient condition for $f(z)$ to have the radius of univalence exactly equal to* 1 *is that all the zeros of $f'(z)$ should be concentrated at the same point on* $|z| =$ cosec (π/n).

The condition is sufficient for all $n \geqq 2$. If $n = 2$ this can be seen by the polynomial z^2+2z, since cosec $(\pi/n) = 1$ and the derivative vanishes at $z = -1$. If $n > 2$ let us consider $f'(z) = (z - \operatorname{cosec}(\pi/n))^{n-1}$. Then $f(z)$ takes the same value at the points $\alpha = e^{\pi(2-n)i/2n}$, $\beta = e^{\pi(n-2)i/2n}$, because if we put $w = e^{(2\pi i)/n}$, then α, β satisfy the equation

$$\operatorname{cosec}\frac{\pi}{n} = \frac{\alpha-\beta w}{1-w},$$

which implies that

$$\left(\frac{\alpha\text{-}\operatorname{cosec}\dfrac{\pi}{n}}{\beta\text{-}\operatorname{cosec}\dfrac{\pi}{n}}\right)^n = 1.$$

In order to prove that the condition is also necessary I will use the principle of apolarity of polynomials. Therefore first I will define this principle and state a theorem of Grace about apolarity.

Definition. If the coefficients of two polynomials

$$f(z) = a_0+C_n^1 a_1 z+C_n^2 a_2 z^2+ \cdots +a_n z^n$$ [1]
$$g(z) = b_0+C_n^1 b_1 z+C_n^2 b_2 z^2+ \cdots +b_n z^n$$

of degree n satisfy the condition

$$a_0 b_n-C_n^1 a_1 b_{n-1}+C_n^2 a_2 b_{n-2}- \cdots +(-1)^n a_n b_0 = 0 \qquad (1)$$

then $f(z)$ and $g(z)$ are called apolar polynomials.

Let

$$f(z) = a_0+C_n^1 a_1 z+C_n^2 a_2 z^2+ \cdots +a_n z^n,$$

where the coefficients satisfy a linear relation

$$a_0 l_n+C_n^1 a_1 l_{n-1}+C_n^2 a_2 l_{n-2}+ \cdots +a_n l_0 = 0$$

then

$$g(z) = l_0-C_n^1 l_1 z+C_n^2 l_2 z^2- \cdots +(-1)^n l_n z^n$$

is apolar to $f(z)$. If we write the same relation for the particular polynomial

$$F(x) = (x-z)^n = x^n-C_n^1 x^{n-1} z+ \cdots,$$

regarding x as a parameter, we find that

$$g(z) = 0.$$

Therefore if the coefficients of a polynomial $f(z)$ satisfy a linear relation then we can obtain a polynomial $g(z)$ apolar to $f(z)$ directly from this relation [2]. I will use this fact in order to prove Theorem 1.

For the relative location of the zeros of apolar polynomials we have the following theorem of Grace [3].

Theorem 2. *If two polynomials are apolar then any circular domain* [4] *containing all the zeros of one of these polynomials contains at least one zero of the other.*

By using arguments similar to those used in the proof of Theorem 2 I will prove the following result.

Theorem 3. Let $f(z)$ and $g(z)$ *be apolar polynomials of degree* $n \geqq 2$. *Let* C *be the circle* $|z| = r$ *such that one zero of* $f(z)$ *is on* C *and this is*

[1] C_n^p denotes the $(p+1)$th coefficient of the n'th power of the binomial, i.e.

$$C_n^p = \frac{n!}{p!(n-p)!}.$$

[2] See, e.g. [4] p. 19–20.

[3] [5], see e.g., [4] p. 16–19.

[4] By a circular domain we mean the interior or exterior of a circle or half plane.

not a zero of $f'(z)$; and $n-1$ zeros of $f(z)$ lie in the interior of C. If all the zeros of $g(z)$ are not concentrated at the same point on C then there exists at least one zero of $g(z)$ in the interior of C.

PROOF. Let two polynomials

$$f(z) = a_0+C_n^1a_1 z+C_n^2a_2 z^2+ \cdots +a_n z^n \tag{2}$$
$$g(z) = b_0+C_n^1b_1 z+C_n^2b_2 z^2+ \cdots +b_n z^n$$

of degree n be apolar. Then their coefficients satisfy the condition of apolarity (1). We denote the zeros of $f(z)$ by $\alpha_1, \alpha_2, \cdots, \alpha_n$ and the zeros of $g(z)$ by $z_1, z_2, \ldots, z_n$.

Putting

$$S_k = (-1)^k C_n^k \frac{b_{n-k}}{b_n},$$

the relation (1) can be written as

$$a_0 S_0+a_1 S_1+ \cdots +a_n S_n = 0, \tag{3}$$

where

$$S_0 = 1$$
$$S_1 = z_1+z_2+ \cdots +z_n$$
$$\cdots\cdots$$
$$S_n = z_1 z_2 \cdots z_n$$

In this way we associate the relation (3) with the equation (2). We will show that, if $z_1, z_2, \ldots, z_n$ is a system of solutions of equation (3) and if all z_k's are not concentrated at the same point on C then at least one z_k lies in the interior of C. We may assume that at least one of the points z_k is exterior to or on C, otherwise there is nothing to prove. Supposing that this point is $z_n = \zeta$ we will show that one of the points $z_1, z_2, \ldots, z_{n-1}$ lies in the interior of C. Let us put

$$s_0 = 1$$
$$s_1 = z_1+z_2+ \cdots +z_{n-1}$$
$$\cdots\cdots$$
$$s_{n-1} = z_1 z_2 \cdots z_{n-1}.$$

Then

$$S_0 = s_0$$
$$S_1 = s_1+\zeta s_0,$$
$$\cdots\cdots$$
$$S_n = \zeta s_{n-1}.$$

By substituting these values in (3) then we obtain the relation

$$(a_0+a_1\zeta)s_0+(a_1+a_2\zeta)s_1+\ldots+(a_{n-1}+\zeta a_n)s_{n-1}=0;$$

but this relation is associated with the equation

$$G(z)=(a_0+a_1\zeta)+C^1_{n-1}(a_1+a_2\zeta)z+\cdots+(a_{n-1}+\zeta a_n)z^{n-1}=0$$ [5]

Now it is sufficient to show that all the roots of this last equation are in the interior of C. Since $z_1, z_2, \cdots, z_{n-1}$ are the zeros of a polynomial apolar to $G(z)$, then by Grace's theorem at least one z_k lies in the interior of C. Writing $G(z)$ as

$$G(z)=a_0+C^1_{n-1}a_1z+\cdots+a_{n-1}z^{n-1}$$
$$+\zeta(a_1+C^1_{n-1}a_2z+\cdots+a_nz^{n-1}),$$

since

$$f'(z)=n[a_1+C^1_{n-1}a_2z+\cdots+a_nz^{n-1}],$$

we have

$$nG(z)=\zeta f'(z)+n[a_0+C^1_{n-1}a_1z+\cdots+a_{n-1}z^{n-1}],$$

and subtracting

$$nf(z)=na_0+nC^1_na_1z+\cdots+na_nz^n$$

we obtain

$$nG(z)=nf(z)+(\zeta-z)f'(z). \tag{4}$$

Division by $f(z)$ gives

$$h(z)=n+(\zeta-z)\frac{f'(z)}{f(z)}.$$

First we will show that if $G(z)$ had a zero, z_0, outside or on C then we would have $f(z_0)\neq 0$, and so $G(z_0)=0$ would imply that $h(z_0)=0$. Then we will show that $h(z)$ cannot have any zero outside or on C which will complete the proof. Now let us suppose that z_0 is outside or on C and $G(z_0)=0$, $f(z_0)=0$. Then by equation (4) we have either $f'(z_0)=0$ or $\zeta=z_0$. If z_0 is on C, since there exists just one zero of $f(z)$ on C which is not a zero of $f'(z)$, then $f'(z_0)\neq 0$. If z_0 is exterior to C, then z_0 cannot be a zero of $f'(z)$ because the circle C encloses all the zeros of $f(z)$ and therefore encloses all the zeros of $f'(z)$ [6]. The second possibility, $\zeta=z_0$, does not hold either since at the beginning we can choose ζ such that $f(\zeta)\neq 0$. This can be done because the zeros of $g(z)$ are not

[5] $-a_{n-1}/a_n$ lies in the interior of C because

$$\frac{-C^{n-1}_n a_{n-1}}{a_n}=\alpha_1+\alpha_2+\cdots+\alpha_n, \text{ and so } -\frac{a_{n-1}}{a_n}=\frac{\alpha_1+\alpha_2+\cdots+\alpha_n}{n},$$

and since all the zeros of $f(z)$ do not have the same modulus then $|a_{n-1}/a_n|$, is less than the maximum modulus. Therefore $\zeta\neq -a_{n-1}/a_n$ and so $G(z)$ is a polynomial of degree n—1.

[6] [6], p. 15, Thm. (6.2).

concentrated at the same point, therefore there exists at least one zero of $g(z)$ which lies either in the interior of C, when there is nothing to prove, or lies on or outside C and is not a zero of $f(z)$. So we can choose this zero as ζ.

Now $h(z)$ cannot have a zero exterior to or on C, for suppose that z_0 is a zero exterior to or on C, then

$$\frac{f'(z_0)}{f(z_0)} = \sum_{i=1}^{n} \frac{1}{z_0 - \alpha_i},$$

and

$$h(z_0) = \sum_{i=1}^{n} \left(1 + \frac{\zeta - z_0}{z_0 - \alpha_i}\right) = \sum_{i=1}^{n} \frac{\zeta - \alpha_i}{z_0 - \alpha_i} = 0 \tag{5}$$

The image of the interior of C under the transformation

$$Z = \frac{\zeta - z}{z_0 - z}$$

is a convex domain. Let us denote this domain by Γ. Since ζ is not in the interior of C, $Z = 0$ is exterior to Γ. By (5) the sum of the transforms of α_i's is 0; so $Z = 0$ is their centre of gravity. But the transform of at least one α_i is in Γ; thus $Z = 0$ is also in Γ [7]. This contradiction completes the proof.

Now I will prove the necessary condition of Theorem 1, and by the same proof I will also deduce Kakeya's Theorem.

Proof. Let $f(z)$ be a polynomial of degree $n \geqq 2$ which attains the same value at two distinct points z_1, z_2 in the closed unit circle. The relation

$$f(z_1) = f(z_2)$$

is a linear relation between the coefficients of $f'(z)$. As it is explained in the argument following the definition of apolarity, by writing the same linear relation between the coefficients of

$$F(x) = (x - z)^{n-1}$$

we find that

$$y(z) = \int_{z_1}^{z_2} (x - z)^{n-1} dx = 0, \tag{6}$$

where $y(z)$ and $f'(z)$ are apolar polynomials. Now let Z be the zero of $y(z)$ of maximum modulus. By writing equation (6) as

[7] If n points are located in or on the boundary of a convex domain and if at least one of them lies in the domain then the centre of gravity of these points also lies in the domain.

$$\left(\frac{z_1-z}{z_2-z}\right)^n = 1,$$

we have

$$z = \frac{z_1-wz_2}{1-w}$$

where w is an n'th root of unity different from 1. If we allow z_1, z_2 to vary in the closed unit circle, then we have

$$\max\left|\frac{z_1-wz_2}{1-w}\right| = \frac{2}{\min|1-w|} = \frac{2}{\left|1-\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n}\right|}$$

$$= \frac{2}{\sqrt{\left(1-\cos\frac{2\pi}{n}\right)^2+\sin^2\frac{2\pi}{n}}}$$

$$= \frac{2}{\sqrt{2\left(1-\cos\frac{2\pi}{n}\right)}} = \frac{2}{\sqrt{4\sin^2\frac{\pi}{n}}} = \operatorname{cosec}\frac{\pi}{n}.$$

Thus $|Z| \leqq \operatorname{cosec}\pi/n$. Suppose that $n > 2$ and $[Z| = \operatorname{cosec}\pi/n$, then $y(z)$ satisfies the conditions of $f(z)$ in Theorem 3 [8]. Therefore if all the zeros of $f'(z)$ are not concentrated at the same point on $|z| = \operatorname{cosec}\pi/n$ then there exists at least one zero of $f'(z)$ in the interior of $|z| = \operatorname{cosec}\pi/n$. Thus if the zeros of $f'(z)$ are not concentrated at the same point on $|z| = \operatorname{cosec}(\pi/n)$, and if $f'(z)$ does not vanish in $|z| < \operatorname{cosec}(\pi/n)$ then $f(z)$ cannot attain the same value at two distinct points in $|z| \leqq 1$. Hence by an argument of J. Dieudonné [9] the radius of univalence of $f(z)$ is greater than 1 [10]. Thus the necessary condition of Theorem 1 follows.

In the above proof if we allow z_1, z_2 to vary only in the interior of the unit circle then by applying Grace's Theorem we deduce Kakeya's Theorem.

L. N. Čakalov [8, Theorem 2] formed a special type of distribution of the zeros of $f'(z)$ outside the unit disc, for which he showed that $f(z)$

[8] Max. modulus of the zeros of $y'(z)$ cannot be greater than cosec $(\pi/n—1)$, and since $n > 2$ then cosec $(\pi/n—1) <$ cosec (π/n).

[9] [7], p. 309–310. If $|z| = R$ is the largest circle about the origin in which a polynomial $f(z)$ is univalent then either $f(z)$ takes the same value at two distinct points on $|z| = R$ or $f'(z)$ vanishes on $|z| = R$. Otherwise $f(z)$ is univalent in a larger circle.

[10] If $n = 2$ then cosec $(\pi/n) = 1$, and if the derivative does not vanish inside or on the unit circle then by Kakeya's Theorem the radius of univalence of $f(z)$ is greater than 1.

is univalent in a larger circle than that given by Kakeya's Theorem. His result is as follows:

THEOREM 4. *Suppose that m is a non-negative integer less than* $(n+1)/2$, *and let*

$$R = \sin\frac{\pi}{n+1} : \sin\frac{(n+1-2m)\pi}{(n-m)(2n+2)}.$$

Let m of the zeros of the polynomial $Q(z) = \prod_{k=1}^{n}(1-(z/z_k))$ *lie in the annulus* $1 \leqq |z| \leqq R$ *and the remaining* $n-m$ *be situated in the region* $|z| > R$. *Then the polynomial* $P(z) = \int Q(z)dz$ *is univalent in* $|z| < r_0$ *where* r_0 *is larger than the radius* $\sin[\pi/(n+1)]$ *given by the theorem of Kakeya.*

Now this result follows from Theorem 1 since the zeros of $Q(z)$ are not concentrated at the same point on the unit cicle. By using arguments similar to Čakalov's, we can obtain the following improved results, Theorems 5 and 6.

THEOREM 5. *Suppose that* $n > 1$, x *is a real number such that* $1/n^2 < x < 1/n$ *and*

$$R = \sin\frac{x\pi n}{xn+1} : \sin\frac{(1-xn)\pi}{2(n-1)(xn+1)}.$$

Let one of the zeros of the polynomial $Q(z) = \prod_{k=1}^{n}(1-(z/z_k))$ *lie in the annulus* $1 \leqq |z| \leqq R$ *and the remaining* $n-1$ *be situated in the region* $|z| > R$. *Then the polynomial* $P(z) = \int Q(z)dz$ *is univalent in* $|z| < r_0$ *where*

$$r_0 > \sin\frac{x\pi n}{xn+1} > \sin\frac{\pi}{n+1}.$$ [11]

THEOREM 6. *Suppose that* $n > 1$, x *is a real number such that* $x > 1$, k *is an integer such that* $0 < k < n$ *and*

$$R = \sin\frac{x\pi n}{2k(xn+1)} : \sin\frac{\pi}{2(n-k)(xn+1)}.$$

Let k of the zeros of the polynomial $Q(z) = \prod_{k=1}^{n}(1-(z/z_k))$ *lie in the annulus* $1 \leqq |z| \leqq R$ *and the remaining* $n-k$ *be situated in the region* $|z| > R$. *Then the polynomial* $P(z) = \int Q(z)dz$ *is univalent in* $|z| < r_0$ *where*

$$r_0 > \sin\frac{x\pi n}{2k(xn+1)}.$$ [12]

[11] If x is near to $1/n$ then R becomes large and $\sin x\pi n/(xn+1)$ is near to 1.

[12] For $k = 1$ and for large x, R becomes large, and $\sin(x\pi n/2k(xn+1))$ is near to 1.

If $k < (xn^2+xn)/2(xn+1)$ *we have further*

$$r_0 > \sin\frac{x\pi n}{2k(xn+1)} > \sin\frac{\pi}{n+1}\,.$$ [13]

I wish to thank Professor F. R. Keogh for suggesting that I should prove whether or not the condition of Theorem 1 is necessary.

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[13] In [9, Theorem 2], by considering the distribution of the zeros of a polynomial $g(z)$ relative to an annulus, I obtained the best possible results for the minimum radius of univalence and the minimum radius of starlikeness of $f(z) = zg(z)$. These results are valid for every annulus about the origin and for every type of distribution of the zeros relative to an annulus.