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# MINIMAL SKEW-PRODUCT HOMEOMORPHISMS AND COALESCENCE

by

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## 0. Introduction

J. Auslander has raised the following question: Are minimal distal homeomorphisms of compact metric spaces coalescent? (For definitions see § 1). We construct an example which shows that the answer is, in general, negative. For certain classes of minimal distal homeomorphisms an affirmative answer is known. e.g. for minimal homeomorphisms with discrete spectrum and for totally minimal homeomorphisms with quasi-discrete spectrum. This is implicit in [5] and [3] respectively. An interesting study of a class of distal homeomorphisms, called skew-products (and first introduced in [1]), appears in [2]. Our work shows that minimal skew-products on finite-dimensional tori are coalescent (§ 2) but we construct an example of a non-coalescent, minimal skew-product on an infinite-dimensional torus (§ 3). In § 1 we show that minimal homeomorphisms of finite dimensional tori with generalised discrete spectrum are skew products.

## 1. Definitions

Let $(X, d)$ be a compact metric space and $T: X \to X$ a homeomorphism. $T$ is minimal if $TE = E$, $E$ closed, implies $E = \emptyset$ or $X$. $T$ is distal if whenever $x \neq y$ there exists $\delta > 0$ with $d(T^n x, T^n y) > \delta$ for all $n \in Z$. This condition is independent of the metric $d$. $T$ is coalescent if $S : X \to X$ is continuous and $ST = TS$ imply $S$ is a homeomorphism of $X$. If $T$ is minimal and $ST = TS$ for $S: X \to X$ continuous then $S$ maps $X$ onto $X$, since $S(X)$ is a non-empty closed set with $TS(X) = S(X)$. Therefore to show that a minimal homeomorphism is coalescent we need only show that each continuous map commuting with it is $1-1$.

$K^n$ will denote the $n$-dimensional torus, written multiplicatively. A skew-product homeomorphism of $K^n$ is a map $T : K^n \to K^n$ of the form $T(z_1, \cdots, z_n) = (\alpha z_1, \varphi_1(z_1)z_2, \varphi_2(z_1, z_2)z_3, \cdots, \varphi_{n-1}(z_1, \cdots, z_{n-1})z_n)$ where $\alpha \in K$ and $\varphi_i : K^i \to K = K^1$ are continuous $1 \leqq i \leqq n-1$. Define a metric on $K$ by $d_1(e^{2\pi ix}, e^{2\pi iy}) = \inf_{m\in Z}|x-y+m|$ $x, y \in R$, and a metric on $K^n$ by $d_n(z, w) = \sum_{i=1}^{n} d_1(z_i, w_i)$ if $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n) \in K^n$. $T$ is distal, for if $z \neq w$ let $i$ be the least integer with $z_i \neq w_i$. Then $d_1((T^p z)_i, (T^p w)_i) = d_1(z_i, w_i)$ for all $p \in Z$ and therefore $d_n(T^p z, T^p w) \geqq d_1(z_i, w_i)$ for all $p \in Z$.

A skew-product on an infinite-dimensional torus $K^\infty$ is a map of the form $T(z_1, z_2, z_3, \ldots) = (\alpha z_1, \varphi_1(z_1)z_2, \varphi_2(z_1, z_2)z_3, \cdots)$ where $\alpha \in K$ and $\varphi_i : K^i \to K$ are continuous $i \geqq 1$. Such a map is distal as is easily seen using the metric $d_\infty(z, w) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_1(z_i, w_i)$. If $T$ has the special form $(z_1, z_2, \cdots) \to (\alpha z_1, \varphi_1(z_1)z_2, \varphi_2(z_1)z_3, \cdots)$ then $T$ is minimal if and only if $\alpha$ is not a root of unity and for every $n \geqq 1$ and $(0, \cdots, 0) \neq (m_1, \cdots m_n) \in Z^n$ there is no continuous map $F : K \to K$ satisfying $F(\alpha z_1)/F(z_1) = \varphi_1(z_1)^{m_1} \varphi_2(z_1)^{m_2} \cdots \varphi_n(z_1)^{m_n}$ ([2], see also [4]).

A minimal skew product homeomorphism $T$ on a finite dimensional torus $X$ has generalised discrete spectrum of finite order in the following sense: If $H_0 = K$ and $H_{i+1} = \{f \in C(X, K) | f(Tx)/f(x) \in L(H_i)\}$, where $L(H_i)$ denotes the closed linear span of $H_i$ in the uniform topology of $C(X)$, then for some positive integer $k$ $L(H_k) = C(X)$. ($C(X, K)$ denotes the group of continuous maps from $X$ to $K$ and $C(X)$ the space of all complex-valued continuous maps of $X$). The members of $H_i$ are called the $i$-th generalised eigenfunctions. Theorem 9 of [4] asserts there are closed subgroups of $X$, $X = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$ such that $G_{i-1} = \{g \in X | T(gx)\, g^{-1}\, (Tx)^{-1} \in G_i\}$ and $L(H_i) = \{f \in C(X) | f(gx) = f(x)$ for all $g \in G_i\}$. Our first result shows that these subgroups are all tori and that any minimal homeomorphism of a finite-dimensional torus with generalised discrete spectrum (necessarily of finite order) is a skew product for some basis of the torus.

THEOREM 1.

*If $T$ is a minimal homeomorphism of a finite-dimensional torus $X$ with generalised discrete spectrum of finite order then $T$ is a skew product. Moreover the groups $G_i$ determined by Theorem 9 of* [4] *(see above) are all tori, and in the representation $X = X/G_{k-1} \times G_{k-1}$, $T$ has the form $(y, g) \to (T_0 y, \varphi(y)g)$ where $T_0 : X/G_{k-1} \to X/G_{k-1}$ is a skew-product and $\varphi : X/G_{k-1} \to G_{k-1}$ is continuous.*

PROOF:

Let $H_0 = K$ and $H_{i+1} = \{f \in C(X, K) | f(Tx)/f(x) \in L(H_i)\}$ where $k$ is the least positive integer such that $L(H_k) = C(X)$. By Theorem 9 of [4] there are closed subgroups of $X$, $X = G_0 \supset G_1 \supset \cdots \supset G_k = \{e\}$, such that $G_{i-1} = \{g \in X | T(gx)g^{-1}(Tx)^{-1} \in G_i\}$ and $L(H_i) = \{f \in C(X) | f(gx) = f(x)$ for all $g \in G_i\}$. We shall show that $G_{k-1}$ cannot be finite.

For suppose $i$ is the least integer such that $G_i$ is finite where $i \leqq k-1$ and let $G$ be the connected component of $G_{i-1}$. Then $(x, g) \to T(gx)g^{-1}(Tx)^{-1}$ maps $X \times G$ to $G$ and since $X \times G$ is connected it maps $X \times G$ to $e$ (the identity element of $X$) i.e. $T(gx) = gT(x)$ for $g \in G$ and $G$ is contained in $G_{k-1}$ contradicting the finiteness of $G_{k-1}$. If $G'$ is the connected component of $G_{k-1}$ then by considering $X/G'$ we see, by the above, that $G' = G_{k-1}$. Similarly each $G_i$ is connected and hence a torus. The proof that $T$ is a skew product is now completed by induction on the dimension of $X$. Since $G_{k-1}$ is a non-trivial torus, $T$ induces a skew product on $X/G_{k-1}$ and $X$ is the direct product of the tori $X/G_{k-1}$ and $G_{k-1}$.

## 2. Coalescence of finite dimensional skew products

In this section we show that minimal skew products of finite-dimensional tori are coalescent. $C(Y, R^r)$ denotes the collection of all continuous maps from $Y$ to $R^r$, equipped with the supremum norm.

LEMMA 1.
*Let $Y$ be a compact space and $S : Y \to Y$ a homeomorphism. Suppose $C$ is a non-singular linear transformation of $R^r$ with no eigenvalues of absolute value one. Then the mapping $\psi(y) \to \psi(Sy) - C\psi(y)$ is an invertible linear transformation of $C(Y, R^r)$.*

PROOF:
Write $E$ instead of $C(Y, R^r)$. $E$ is a Banach space with a supremum norm $\|\cdot\|$. Define $L : E \to E$ to be the continuous linear map $(L\psi)(y) = C\psi(S^{-1}y)$. If $V_1 = \{x \in R^r | C^n x \to 0 \text{ as } n \to +\infty\}$ and $V_2 = \{x \in R^r | C^n x \to 0 \text{ as } n \to -\infty\}$ then $V_1$ and $V_2$ are closed subspaces of $R^r$ and $V_1 \oplus V_2 = R^r$. There exists $0 < \lambda < 1$ and a norm $|\cdot|$ on $R^r$ such that $|Cx| \leqq \lambda|x|$ if $x \in V_1$ and $|C^{-1}x| \leqq \lambda|x|$ if $x \in V_2$. We suppose that the norm $\|\cdot\|$ on $E$ is defined using the norm $|\cdot|$ on $R^r$.

Let $E_1$ denote the collection of all continuous maps $\psi : Y \to V_1$ and $E_2$ the collection of all continuous maps $\psi : Y \to V_2$. $E_1$ and $E_2$ are closed subspaces of $E$ and $E_1 \oplus E_2 = E$. We have $\|L\psi\| \leqq \lambda\|\psi\|$ if $\psi \in E_1$ and $\|L^{-1}\psi\| \leqq \lambda\|\psi\|$ if $\psi \in E_2$. Let $L_1 : E_1 \to E_1$ and $L_2 : E_2 \to E_2$ denote the restrictions of $L$. Then $(I-L_1)^{-1} = \sum_{k=0}^{\infty} L_1^k$ exists and $(I-L_2)^{-1} = -\sum_{k=1}^{\infty} L_2^{-k}$ exists. Hence $(I-L)^{-1}$ exists and the lemma follows.

THEOREM 2.
*Let $T$ be a minimal skew product homeomorphism of a finite-dimensional torus $X$. Then $T$ is coalescent.*

PROOF:

If $H_i$ are the groups of generalised eigenfunctions of $T$ and $L(H_k) = C(X)$ then for any continuous map $S: X \to X$ with $ST = TS$ we have $U_S H_i \subset H_i$ where $U_S : C(X) \to C(X)$ is defined by $(U_S f)(x) = f(Sx)$. In order to show that $S$ is a homeomorphism we have to show that $U_S$ maps $H_i$ onto $H_i$ for $o \leqq i \leqq k$. Assume this is not the case and let $l$ be the least positive integer such that $U_S$ maps $H_l$ properly into $H_l$. By considering $X/G_l$ instead of $X$, if necessary, there is no loss in generality in assuming $l = k$. In other words; we may assume that $U_S$ maps $H_{k-1}$ onto $H_{k-1}$ ($S$ induces a homeomorphism on $X/G_{k-1}$) but $U_S$ maps $H_k$ properly into $H_k$ ($S$ is not a homeomorphism of $X$). By Theorem 1 we can write $T$ as $T(y, g) = (T_0 y, \varphi(y)g)$ where $X = Y \times G_{k-1}$, $Y = X/G_{k-1}$, and $\varphi : Y \to G_{k-1}$ is a continuous map. It is not difficult to deduce from this that $S(y, g) = (S_0 y, \psi(y)\,\sigma(G))$ where $S_0$ is a homeomorphism of $Y$, $\psi : Y \to G_{k-1}$ is continuous and $\sigma$ is a proper endomorphism of $G_{k-1}$ onto itself.

Let $S$ be written uniquely as $S(x) = a(x)\,A(x)$ where $A$ is an endomorphism of $X$ and $a = \pi \circ \alpha$ where $\pi : R^n \to X \cong R^n/Z^n$ is the natural projection and $\alpha : X \to R^n$ is continuous. By the above $S(gx) = a(gx)$ $A(gx) = \sigma(g)\ S(x)$ if $g \in G_{k-1}$ so that $a(gx) = a(x)$ and $A(g) = \sigma(g)$. Hence $A$ is a proper endomorphism of $X$. In the same way $T(x) = b(x)$ $B(x)$ where $B$ is an automorphism of $X$, $b = \pi \circ \beta$, $\beta : X \to R^n$ and $b(gx) = b(x)$, $B(gx) = gB(x)$ for $g \in G_{k-1}$.

Since $A$ is a proper endomorphism the characteristic polynomial (with integer coefficients) has a factor $p(\lambda)$ irreducible over $Z(\lambda)$ and with unit leading coefficient and constant term of absolute value greater than one. As was pointed out to us by R. W. Carter, such a polynomial $p(\lambda)$ has no roots of absolute value one. On the other hand, it is not difficult to see from the skew product form of $T$ that $B$ has characteristic polynomial $(\lambda - 1)^n$. Hence the homomorphism $B_1(x) = x^{-1}Bx$ is nilpotent. Let $H = \{x \in X | B_1^l(x) \in p(A)X\}$ where $l+1$ is the least positive integer such that $B_1^{l+1}(X) \subset p(A)X$. ($p(A)x = A^m x \cdot (A^{m-1}x)^{a_{m-1}} \cdots (Ax)^{a_1} x^{a_0}$ where $p(\lambda) = \lambda^m + a_m \lambda^{m-1} + \cdots + a_0$, $|a_0| > 1$). $H$ is a closed subgroup of $X$.

Since $ST = TS$ we have $AB = BA$ and $a(Tx)\,A(b(x)) = b(Sx)\,B(a(x))$. Let $q$ be the natural map of $X$ to $X/H$. Then $q(Bx) = q(x)$ and $a'(Tx)$ $A'b'(x) = b'(Sx)\,a'(x)$ where $a' = qa$, $b' = qb$ and $A'$ is the endomorphism of $X/H$ induced by $A$. Clearly the characteristic polynomial of $A'$ over $Z$ is a power of $p(\lambda)$. Since $a(x)$ and $b(x)$ are $G_{k-1}$ invariant we have

$$a'(T_0 y)\,A'b'(y) = b'(S_0 y)\,a'(y) \text{ for } y \in Y = X/G_{k-1}$$

where $a'$, $b'$ are regarded as maps from $Y$ to $X/H$. Since we may write

$a' = \pi' \circ \alpha'$, $b' = \pi' \circ \beta'$ where $\pi'$ is the covering projection of some $R^r$ to $X/H$ and $\alpha', \beta' : Y \to R^r$, we have

$$\beta'(S_0 y) - C\beta'(y) = \alpha'(T_0 y) - \alpha'(y) + v$$

where $v \in Z^r$ and $C$ covers $A'$. Let $v = (C-I)\ (\mu)$ where $\mu = \rho/m$, $\rho \in Z^r$ and $m$ is a positive integer. Then $m\beta'(S_0 y) - Cm\ \beta'(y) - (I-C)\rho = m\alpha'(T_0 y) - m\alpha'(y)$ and by lemma 1 $m\beta'(y) - \rho = f(T_0 y) - f(y)$ for some continuous $f : Y \to R^r$. Hence $\pi f(y) \cdot b'(y)^m = \pi f(T_0 y)$. Writing $g(x) = \pi f(xG_{k-1})$ we have $g(Tx) = b'(x)^m g(x)$. Since $q(Tx) = b'(x)q(x)$ the minimality of $T$ implies that $g(x) = c\, q^m(x)$ where $c$ is a constant. But $g(x)$ was defined by a covering map and hence is null homotopic whereas $q^m(x)$ clearly is not. This contradiction shows that $A$ is an automorphism and therefore $\sigma$ is an automorphism. Hence $S$ is a homeomorphism and $T$ is coalescent.

## 3. A counter-example on the infinite-dimensional torus

We now construct an example of a non-coalescent minimal distal homeomorphism of an infinite dimensional torus. The example is a skew-product and has generalised discrete spectrum of order two. If $z, w \in K$ we shall write $|z-w|$ to mean the distance $d_1(z, w)$ defined in § 1.

Let $T : K^\infty \to K^\infty$ be defined by

$$T(z, w_1, w_2, w_3, \cdots) = (\alpha z, \varphi(z)w_1, \varphi(\beta z)w_2, \varphi(\beta^2 z)w_3, \varphi(\beta^3 z)w_4 \cdots)$$

$$S(z, w_1, w_2, w_3, \cdots) = (\beta z, w_2, w_3, w_4, \cdots)$$

where $\alpha, \beta \in K$ and the continuous map $\varphi : K \to K$ will be chosen to make $T$ minimal. $T$ is a homeomorphism, $S$ is not $1-1$ and $ST = TS$. Let $w \in K$ be non-algebraic. Then $p(w) \neq 0$ for every polynomial $p(\lambda)$ over $Z$. Let $\beta$ be not a root of unity. Choose a sequence of integers $N = \{1 < n(1) < n(2) < n(3) < \cdots\}$ so that $|\beta^{n(i)} - w| < 1/i$ for $i \geqq 1$. Then choose $\alpha$, not a root of unity, so that $|\alpha^n - 1| \leqq 1/n^2$ for $n$ running through some subsequences of $N$. (We shall show below how such an $\alpha$ can be constructed.) Without loss of generality we can suppose this subsequence is $N$. Put $a_n = \alpha^n - 1$ if $n \in N$, $a_n = \alpha^n - 1$ if $n \in -N = \{-n(i)\}_{i \geqq 1}$, and $a_n = 0$ if $n \notin N \cup -N$. Since $\Sigma |a_n| \leqq 2\Sigma\, 1/n^2 < \infty$, $\Sigma\, a_n z^n$ converges uniformly to a continuous function $H(z)$, which is real-valued since $a_n = \bar{a}_n$. Define $\varphi : K \to K$ by $\varphi(z) = \exp\,[2\pi i\, H(z)]$.

It remains to show that $T$ is minimal. $T$ will be minimal if for every integer $k \geqq 0$ and every $k+1$-tuple of integers $(m_0, m_1, \cdots, m_k) \neq (0, 0, \cdots, 0)$ the equation

$$\frac{F(\alpha z)}{F(z)} = \varphi(z)^{m_0}\, \varphi(\beta z)^{m_1} \cdots \varphi(\beta^k z)^{m_k} \tag{3.1}$$

has no continuous solution $F: K \to K$. Fix $k \geqq 0$ and $(m_0, m_1, \cdots, m_k) \in Z^{k+1}$. Since each continuous $F: K \to K$ is of the form $F(z) = z^l \exp[2\pi i f(z)]$ where $l \in Z$ and $f: K \to R$ is continuous, equation (3.1) has a solution if and only if there exists $v \in Z$ such that $l\alpha + f(\alpha z) - f(z) = m_0 H(z) + m_1 H(\beta z) + \cdots + m_k H(\beta^k z) + v$ has a solution for $l$ and $f$. Let $f(z) \sim \Sigma b_n z^n$ ($L^2$-convergence). If $f$ satisfies the above equation and $n \neq 0$ then $(\alpha^n - 1)b_n = a_n(m_0 + m_1 \beta^n + \cdots + m_k \beta^{nk}) = a_n p(\beta^n)$. If $n \in N$ then $b_n = p(\beta^n) \to p(w) \neq 0$ and therefore $\Sigma |b_n|^2 = \infty$. The above equation has no solution for $f$ and hence (3.1) has no solution for $F$.

We now give an indication of how $\alpha$ can be constructed. Let $N = \{1 < n(1) < n(2) < \cdots\}$ be given. We wish to find $\alpha$, not a root of unity, such that $|\alpha^{m(i)} - 1| \leqq 1/m(i)^2$ for some subsequence $\{m(i)\}_{i \geqq 1}$ of $N$. Equivalently we wish to find an irrational $a \in [0, 1]$ such that for all $i \geqq 1$ there exists $p(i) \in Z$ with $|a - p(i)/m(i)| < 1/m(i)^3$. Put $m(1) = n(1)$, $I_1(j) = [j/m(1) - 1/m(1)^3,\ j/m(1) + 1/m(1)^3]$ $j = 1, \cdots, m(1)-1$ and $F_1 = \bigcup_{j=1}^{m(1)-1} I_1(j)$. Choose $n(i_2)$ so that $2/n(i_2) < 1/m(1)^3$. Put $m(2) = n(i_2)$. $I_2(k) = [k/m(2) - 1/m(2)^3,\ k/m(2) + 1/m(2)^3]$ $1 \leqq k \leqq m(2)-1$ and $F_2 = \bigcup_{k=1}^{m(2)-1} I_2(k)$. Each $I_1(j)$ contains at least two intervals of the form $I_2(k)$. Choose $n(i_3)$ so that $2/n(i_3) < 1/m(2)^3$. Put $m(3) = n(i_3)$, $I_3(l) = [l/m(3) - 1/m(3)^3,\ l/m(3) + 1/m(3)^3]$ $1 \leqq \rho \leqq m(3)-1$ and $F_3 = \bigcup_{l=1}^{m(3)-1} I_3(l)$. Each $I_2(k)$ contains at least two intervals of the form $I_3(l)$. Proceeding in this way we construct $m(i)$ so that $2/m(i) < 1/m(i-1)^3$, $I_i(q) = [q/m(i) - 1/m(i)^3,\ q/m(i) + 1/m(i)^3]$, $i \leqq q \leqq m(i)-1$ and $F_i = \bigcup_{q=1}^{m(i)-1} I_i(q)$. Then $\bigcap_{i=1}^{\infty} F_i$ is a subset of [0, 1] with the same cardinality as the real line and therefore contains an irrational number.

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