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ON REPRESENTATIONS AS AN INFINITE SERIES OF ISOLS *

by

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1. Introduction

In this paper we wish to present some properties of isols and regressive isols that are closely related to infinite series of isols. In [14] A. Nerode associated with each set α of non-negative integers, a particular set α_Λ of isols. The results presented in the paper were obtained when the following problem was considered: Can we characterize the regressive isols that belong to α_Λ? In [3] J. C. E. Dekker introduced and studied a special kind of isol, denoted by $\Sigma_T a_n$ and called an *infinite series of isols*. Of particular interest are the infinite series that represent regressive isols. Properties of some special infinite series of this kind were studied in [8], [4] and [6]. Let α be any set of non-negative integers. The principal aim of this paper is to show that the regressive isols that belong to α_Λ can be characterized as being those isols that are representable as a particular type of infinite series of isols.

2. Preliminaries

We let E denote the collection $(0, 1, 2, \ldots)$ and the members of E are called *numbers*. By a *set* we will mean a subset of E. A set α is *immune* if α is infinite and α does not contain any infinite recursively enumerable (written r.e.) subset. If f is a function from a subset of E into E then δf will denote the domain of f and ρf the range of f. A one-to-one function a_n from E into E is *regressive*, if there is a partial recursive function $p(x)$ such that

$$(1) \qquad \rho a \subseteq \delta p,$$

$$(2) \qquad p(a_0) = a_0 \text{ and } (\forall n)[p(a_{n+1}) = a_n].$$

It is readily seen that for every regressive function a_n there also exists a partial recursive function $p(x)$ which satisfies, besides (1) and (2) the conditions

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(3) $$\rho p \subseteq \delta p,$$

(4) $$(\forall x)[x \in \delta p \Rightarrow (\exists n)[p^{n+1}(x) = p^n(x)]].$$

If a_n is a regressive function, then every partial recursive function $p(x)$ satisfying (1), (2), (3) and (4) is called a *regressing* function of a_n. Let $p(x)$ be a partial recursive function satisfying (3) and (4). The function

$$p^*(x) = (\mu y)[p^{y+1}(x) = p^y(x)], \text{ for } x \in \delta p,$$

is the *index* function of $p(x)$. Note that if $p(x)$ is a regressing function of the regressive function a_n then $p^*(x)$ is partial recursive with δp as its domain and for each number n, $p^*(a_n) = n$. $j(x, y)$ will denote the familiar primitive recursive function that maps E^2 onto E in a one-to-one manner, and defined by

$$j(x, y) = x + (x+y)(x+y+1)/2.$$

For any number n and set α,

$$\nu(n) = \{x | x < n\} \quad \text{and} \quad j(n, \alpha) = \{j(n, y) | y \in \alpha\}.$$

Let A be any regressive isol and u_n any function from E into E. If A is finite and $A = k$, then

$$\Sigma_A u_n = u_0 + \cdots + u_{k-1}$$

if $k \geqq 1$ and equals 0 if $k = 0$. If A is an infinite regressive isol, then

$$\Sigma_A u_n = \text{Req} \sum_0^\infty j(a_n, \nu(a_n))$$

where a_n is chosen to be any regressive function that ranges over a set in A. While the value of $\Sigma_A u_n$ will depend on the particular regressive isol A and function u_n, it will not depend on the particular regressive function a_n that is chosen to range over a set in A, when A is infinite. In the special case that u_n is a recursive function then $\Sigma_A u_n$ will be a regressive isol by [1, Theorem 1]. Let a_n and u_n each be functions from E into E. Then $a_n \leqq * u_n$ will mean that mapping $a_n \to u_n$ has a partial recursive extension. If A is a regressive isol, then $A \leqq * u_n$ will mean that either A is finite or else A is infinite and there is a regressive function a_n that ranges over a set in A statisfying the property $a_n \leqq * u_n$. Note that when a_n is any regressive function then $a_n \leqq * n$. It is readily seen from this fact that when A is any regressive isol and u_n is any recursive function then $A \leqq * u_n$. In the special case that A is a regressive isol and $A \leqq * u_n$ then $\Sigma_A u_n$ will also be a regressive isol by [4, Propositions 5 and 7]. We let Λ denote the collection of all isols and Λ_R the collection of all regressive isols. Let α be any set of numbers. Then α_A will denote the

Nerode extension of α to the isols and α_R will denote the collection of regressive isols belonging to α_Λ, i.e., $\alpha_R = \Lambda_R \cap \alpha_\Lambda$. In the next section we will describe the extension procedure that leads to the definition of α_Λ. Throughout the paper we will assume that the reader is familiar with some of the basic properties of regressive isols.

Regarding references in the paper to formulas that do not first appear in the proofs of theorems, we adopt the following convention. In each section, when referring to a formula of the same section we simply write the formula number, and when referring to a formula of a different section we indicate this with the section number and a period before the formula number. For example, a formula labeled (6) in section § 3, would be referred to in § 3 by (6) and in § 4 by (3.6).

3. The extension procedure

Q will denote the collection of all finite sets and the mapping $\varphi : Q \to E$ will denote a fully effective Gödel numbering of Q. The value of $\varphi(\alpha)$ will also be written as α^*. A collection F of finite sets is a *frame* if

(1) $$\alpha, \beta \in F \Rightarrow \alpha \cap \beta \in F.$$

A set β is *attainable* from a frame F, denoted by $\beta \in \mathscr{A}(F)$, if for each finite set $\eta \subseteq \beta$ there is a set $\delta \in F$ such that

(2) $$\eta \subseteq \delta \subseteq \beta.$$

Because F is a frame there will correspond to the set η a unique smallest set $\delta \in F$ for which (2) holds, and this particular member of F is denoted by $C_F(\eta)$. Note that a finite set is attainable from a frame F if and only if the set belongs to F. In addition, it is easy to see that if H is a non-empty directed subset of a frame F, in the sense that

$$\delta, \eta \in H \Rightarrow \delta \subseteq \eta \text{ or } \eta \subseteq \delta,$$

then the set

$$\beta = \sum_{\delta \in H} \delta$$

will also be attainable from the frame F. An isol B is *attainable* from a frame F, denoted by $B \in \mathscr{A}(F)$, if $B = \operatorname{Req} \beta$ for some $\beta \in \mathscr{A}(F)$. Let α be any set and F a frame, then F is an *α-frame* if

(3) $$\delta \in F \Rightarrow \operatorname{card} \delta \in \alpha.$$

For F any frame, let

$$F^* = \{\delta^* | \delta \in F\},$$
$$\delta F = \{\beta | \exists \delta \in F : \delta \supseteq \beta\},$$
$$\delta F^* = \{\beta^* | \beta \in \delta F\}.$$

δF will be a collection of finite sets. A frame F is *recursive* if

(4) $$\delta F^* \text{ is r.e.,}$$

and the mapping

(5) $$\beta^* \to C_F(\beta)^*, \text{ for } \beta \in \delta F$$

is partial recursive.

Note that if F is a frame, then F^* r.e. implies that δF^* is also r.e.. In addition, the following property follows from (3), (4) and (5); if F is a recursive α-frame and

$$\omega = \{\text{card } \delta | \delta \in F\},$$

then ω is an r.e. subset of α and F is a recursive ω-frame. The extension procedure introduced by Nerode in [14] can now be defined. Let α be any set of numbers, then α_Λ is defined to be the collection of all isols that are attainable from some recursive α-frame. This extension procedure has the following properties,

(6) $$\phi_\Lambda = \phi \text{ and } E_\Lambda = \Lambda,$$

(7) $$\alpha \subseteq \alpha_\Lambda \text{ and } E \cap \alpha_\Lambda = \alpha,$$

(8) $$\alpha \subseteq \beta \Rightarrow \alpha_\Lambda \subseteq \beta_\Lambda,$$

(9) $$(\alpha \cap \beta)_\Lambda = \alpha_\Lambda \cap \beta_\Lambda,$$

(10) $$\alpha \text{ finite or immune} \Rightarrow \alpha_\Lambda = \alpha.$$

The reader is referred to [12] and [14] for examples and a more complete development of frames and properties of the extension procedure. In view of the definition of α_R, we note that each of the five properties given above will be true when Λ is replaced in the extensions by R. For example,

(11) $$\alpha \subseteq \alpha_R \text{ and } E \cap \alpha_R = \alpha,$$

(12) $$\alpha \subseteq \beta \Rightarrow \alpha_R \subseteq \beta_R.$$

4. The principal theorems

The purpose of this section is to prove four theorems; each theorem is related to characterizing the regressive isols belonging to α_R as infinite series of isols. We need some new definitions for this purpose.

In [14] and [15], Nerode associated with each recursive function $f: E \to E$ a function $D_f : \Lambda \to \Lambda^*$; where Λ^* denotes the collection of isolic integers. Let f be a strictly increasing recursive function and let α be its range. Then it need not be true that D_f maps Λ into Λ, yet each of the properties

$$(1) \qquad D_f : \Lambda_R \to \Lambda_R,$$

$$(2) \qquad \alpha_R = D_f(\Lambda_R),$$

holds by [1, Corollary 4] and [5, Proposition 3] respectively. The function $e(x)$ related to $f(x)$ by

$$e(0) = f(0),$$
$$e(n+1) = f(n+1) - f(n)$$

is the *e-difference* function of f; we will also write e_n for $e(n)$. Note that for every number x,

$$f(x) = e_0 + \cdots + e_x.$$

Because $f(x)$ is a strictly increasing recursive function it follows that $e(x)$ will also be a recursive function. In addition, by [1, Proposition 2], the following property is true,

$$(3) \qquad D_f(A) = \Sigma_{A+1}\, e_n, \text{ for } A \in \Lambda_R.$$

Let δ be any set of numbers. We associate with δ a particular collection δ_Σ of isols in the following way. If δ is a finite set then $\delta_\Sigma = \delta$. If δ is an infinite set, let $g(x)$ denote the strictly increasing function that ranges over δ and let $d(x)$ denote the e-difference function of $g(x)$. Then

$$\delta_\Sigma = \{\Sigma_A d_n | A \in \Lambda_R, A \geqq 1 \text{ and } A \leqq * d_n\}.$$

The collections δ_Σ play a fundamental role in the paper. By an earlier remark we know that δ_Σ will be a collection of regressive isols. Observe also that

$$(4) \qquad \delta \subseteq \delta_\Sigma.$$

When δ is a finite set then (4) is clear. When δ is an infinite set, let the functions $g(x)$ and $d(x)$ be related to δ as above and, then (4) follows by noting,

$$\begin{aligned} x \in \delta &\Leftrightarrow (\exists k)[x = g(k)] \\ &\Leftrightarrow (\exists k)[x = d_0 + \cdots + d_k] \\ &\Leftrightarrow x = \Sigma_A d_n, \text{ for } A = k+1. \end{aligned}$$

It can also be readily shown that a finite isol belongs to δ_Σ if and only if it belongs to δ.

We now prove four theorems. The first theorem is easy to obtain from some known results and we mention it here mostly for comparison with the other three.

THEOREM 1. *If α is a recursive set then $\alpha_R = \alpha_\Sigma$.*

PROOF. Let α be a recursive set. If α is a finite set then the theorem is easy, because

$$\alpha_R = \alpha = \alpha_\Sigma.$$

Let us assume now that α is an infinite set. Let $f(x)$ denote the principal function of α, and $e(x)$ the e-difference function of $f(x)$. Let $D_f(X)$ denote the extension of $f(x)$ to Λ. Then $f(x)$ will be a strictly increasing recursive function, and $e(x)$ will be a recursive function. From (4.2) and (4.3) respectively, we have

$$(1) \qquad \alpha_R = D_f(\Lambda_R),$$

$$(2) \qquad D_f(A) = \Sigma_{A+1}\, e_n, \text{ for } A \in \Lambda_R.$$

Whenever A is a regressive isol, $A+1$ is also a regressive isol, and because $e(x)$ is a recursive function, one will have true the relation $A+1 \leqq * \, e_n$. Combining this property with (1), (2) and the definition of α_Σ it follows that $\alpha_R = \alpha_\Sigma$, and this completes the proof.

THEOREM 2. *If α is any set, then*

$$\alpha_R \subseteq \sum_{\substack{\delta \subseteq \eta \subseteq \alpha \\ \eta \text{ r.e.}}} \delta_\Sigma.$$

PROOF. Let α be any set and let $B \in \alpha_R$. Then B will be a regressive isol and attainable from some recursive α-frame. Assume first that B is finite, and let $B = b \in E$. Let η denote the one-element set whose only member is b. Then η will be an r.e. set and, by (3.11), also $\eta \subseteq \alpha$. Since by definition $\eta_\Sigma = \eta$, we see that $b \in \eta_\Sigma$ and therefore also that,

$$B = b \in \sum_{\substack{\delta \subseteq \eta \subseteq \alpha \\ \eta \text{ r.e.}}} \delta_\Sigma.$$

Assume now that B is an infinite regressive isol. Let $B = \text{Req } \beta$ and let F be a recursive α-frame such that $\beta \in \mathscr{A}(F)$. β will then be an infinite regressive and immune set, and we let b_x denote a regressive function that ranges over β. Let

$$(1) \qquad \eta = \{\text{card } \delta | \delta \in F\}.$$

Since F is a recursive α-frame, we know by an observation in § 3 that

$$(2) \qquad \eta \subseteq \alpha \text{ and } \eta \text{ is an r.e. set.}$$

We want to describe a certain procedure that is based on properties of the set β. Consider any particular number b_n of β. By using the regressive property of the function b_x we can effectively find from b_n the list of numbers $b_0, \cdots, b_n$ and their respective indices. In addition, because F

is a recursive frame (refer to (3.5)), the members of the finite set $C_F((b_0, \cdots, b_n)) \in F$ can also be found. Since β is attainable from F and $(b_0, \cdots, b_n) \subseteq \beta$, it follows that

$$(3) \qquad (b_0, \cdots, b_n) \subseteq C_F((b_0, \cdots, b_n)) \subseteq \beta.$$

We can now check to see if $C_F((b_0, \cdots, b_n))$ contains any numbers b_k with $k \geqq n+1$, and if so then proceed to find both the numbers in the list $b_0, \cdots, b_k$ and the finite set $C_F((b_0, \cdots, b_k)) \in F$. Then as in the step before, we can now check to see if this member of F contained any new b_x values and if it did then continue to find another member of F. Our procedure is effective and will therefore enable us to generate, beginning with the number b_n, an r.e. subset of β. Since β is an immune set, the procedure will have to terminate after a finite number of steps. This will happen when we arrive at a step with $k \geqq n$ and

$$(4) \qquad C_F((b_0, \cdots, b_k)) = (b_0, \cdots, b_k).$$

There will have to be infinitely many numbers k for which (4) holds, and we let π denote this particular set of numbers. Let k_x denote the strictly increasing function that ranges over π. It follows as an easy consequence of the procedure just described that given any number $b_m \in \beta$ we can effectively find out whether or not $m \in \pi$. If it turns out that $m \notin \pi$ then we will be able to find from b_m, by using our procedure, the smallest number $t > m$ such that $t \in \pi$. In addition, if it should turn our that $m \in \pi$ then, because we would know the values of $b_0, \cdots, b_m$, we could then also find the next smaller value than m in π if there is one. For $m \in E$, let

$$\tilde{m} = (\mu t)[t \geqq m \text{ and } t \in \pi].$$

We can conclude from our previous remarks that each of the mappings

$$(5) \qquad b_m \to b_{\tilde{m}},$$

$$(6) \qquad b_{k_{m+1}} \to b_{k_m},$$

will have a partial recursive extension. Let, for $n \in E$

$$a_n = b_{k_n},$$
$$\delta_n = C_F((b_0, \cdots, b_{k_n})) = (b_0, \cdots, b_{k_n}),$$
$$f(n) = \text{card } \delta_n = 1 + k_n.$$

Let

$$\delta = (f(0), f(1), \cdots).$$

Regarding these definitions, the following properties are true,

(7) a_n is a regressive function,

(8) $\delta \subseteq \eta \subseteq \alpha$

(9) $f(x)$ is the principal function of δ.

Property (7) follows because the mapping in (6) has a partıal recursive extension. Concerning property (8), first note the following implications,

$$\begin{aligned} y \in \delta &\Rightarrow y = f(m), \text{ for some } m \in E \\ &\Rightarrow y = \text{card } \delta_m \text{ and } \delta_m \in F \\ &\Rightarrow y \in \eta. \end{aligned}$$

The first two implications are clear form the definitions of δ and δ_m, and the last one follows from (1). Together they imply that $\delta \subseteq \eta$. Combining this inclusion with (2) gives the desired property in (8). Property (9) follows as an easy consequence of the definitions of δ, δ_n and $f(x)$, and the fact that k_x is a strictly increasing function.

Our approach in the remainder of the proof ıs to show that the isol B that we began with belonging to α_R, also belongs to δ_Σ. In view of (2) and (8) this will establish the desired result. Let $e(x)$ denote the e-difference function of $f(x)$. Note that

$$\begin{aligned} (10) \qquad e_0 &= 1+k_0 \\ &= \text{card } (b_0, \cdots, b_{k_0}), \\ e_{n+1} &= k_{n+1}-k_n \\ &= \text{card } (b_{k_n+1}, \cdots, b_{k_{n+1}}). \end{aligned}$$

We now verify that

$$(11) \qquad a_n \leqq * \, e_n,$$

i.e., that the mapping

$$a_n \to e_n$$

has a partial recursive extension. For this purpose we will assume that the value of the number k_0 is known to us; in view of (10), it follows that the value of e_0 will also be known. Let the number a_n be given; from it we would like to find the value of e_n. Since b_x is a regressive function the number k_n such that $b_{k_n} = a_n$ can be found. Here $k_n \geqq k_0$ since k_x is strictly increasing. Knowing the value of k_0 we then can check to see if $k_n = k_0$. If $k_n = k_0$, then $n = 0$ and $e_n = e_0$. In this event the value of $e_n = e_0$ can be computed because it is known to us. Let us assume now that $k_n > k_0$. Then $n \geqq 1$, and in view of the effectiveness of the mapping in (6), we can compute from the number $a_n = b_{k_n}$ the value of $b_{k_{n-1}}$. Since b_x is a regressive function, the value of k_{n-1} can be found. Knowing both of the numbers k_n and k_{n-1}, we can now find the number

$$e_n = k_n - k_{n-1}.$$

It readily follows, in light of these remarks, that the mapping $a_n \to e_n$ will have a partial recursive extension. Therefore $a_n \leqq * e_n$, and this verifies (11). Let

$$\sigma = \sum_0^\infty j(a_n, v(e_n)). \tag{12}$$

Since a_n is a regressive function and $a_n \leqq * e_n$ it follows from the definition of δ_Σ that

$$\text{Req } \sigma \in \delta_\Sigma. \tag{13}$$

To complete the proof of the theorem, we now verify that $B = \text{Reg } \sigma$, or equivalently that

$$\beta \simeq \sigma. \tag{14}$$

It is known by [9, P9.b] that the relation of (14) will hold if and only if both

$$\beta \leqq * \sigma, \tag{15}$$

and

$$\sigma \leqq * \beta \tag{16}$$

hold. Here the relation $\beta \leqq * \sigma$ means that there is a partial recursive function defined at least on β, that maps β onto σ and that is one-to-one on β; similarly $\sigma \leqq * \beta$ is defined. To prove (14) we will verify both (15) and (16). Let

$$\begin{aligned} \beta_0 &= (b_0, \cdots, b_{k_0}), \\ \beta_{n+1} &= (b_{k_n+1}, \cdots, b_{k_{n+1}}); \quad \text{and} \\ \sigma_n &= j(a_n, v(e_n)) \\ &= (j(a_n, 0), \cdots, j(a_n, e_n - 1)). \end{aligned} \tag{17}$$

Regarding these definitions note that each of $\{\beta_n\}$ and $\{\sigma_n\}$ is a sequence of mutually disjoint sets, and

$$\beta = \sum_0^\infty \beta_n, \tag{18}$$

$$\sigma = \sum_0^\infty \sigma_n. \tag{19}$$

Also, in view of (10) and (17), for each number n,

$$\text{card } \beta_n = \text{card } \sigma_n = e_n. \tag{20}$$

RE (15). We define a mapping $g : \beta \to \sigma$ such that for each number n,

$$g : \beta_n \to \sigma_n.$$

Let $b \in \beta_n$. To define $g(b)$ two separate cases are considered.

CASE 1. $n = 0$. Then $\beta_0 = (b_0, \cdots, b_{k_0})$. From (10), $e_0 = 1+k_0$, and therefore $b \in \beta_0$ implies

$$b = b_r \text{ for some } 0 \leqq r < e_0.$$

Define

$$g(b) = j(a_0, r).$$

Note that g will map β_0 onto σ_0. In addition, observe that $a_0 = b_{k_0} = b_{\tilde{r}}$ and

$$\begin{aligned} b &= b_r \quad \text{with} \quad 0 \leqq r \leqq \tilde{r}, \quad \text{and} \\ g(b) &= j(b_{\tilde{r}}, r). \end{aligned} \tag{21}$$

Combining (21) and the fact that the mapping $b_m \to b_{\tilde{m}}$ has a partial recursive extension it readily follows that, given a number $b \in \beta_0$ we can effectively find the value of $g(b)$.

CASE 2. $n \geqq 1$. Then

$$\beta_n = (b_{k_{n-1}+1}, \cdots, b_{k_n}).$$

From (10), we know that $e_n = k_n - k_{n-1}$, and therefore $b \in \beta_n$ implies that

$$b = b_{k_{n-1}+1+r} \quad \text{for some} \quad 0 \leqq r < e_n.$$

Define

$$g(b) = j(a_n, r).$$

Note that g will map β_n onto σ_n when $n \geqq 1$. Also, observe that

$$a_n = b_{k_n} = b_{\tilde{s}} \quad \text{for} \quad s = k_{n-1}+1+r,$$

and

$$\begin{aligned} b &= b_s \quad \text{for} \quad s = k_{n-1}+1+r, \quad \text{and} \\ g(b) &= j(b_{\tilde{s}}, r). \end{aligned} \tag{22}$$

Combining (22), the regressive property of b_x, and the fact that the mapping $b_m \to b_{\tilde{m}}$ has a partial recursive extension, it readily follows that, given a number $b \in \beta_n$ with $n \geqq 1$, we can effectively find both of the values n and $g(b)$.

This completes the definition of the mapping $g(x)$. In view of the fact that $g(x)$ maps β_n onto σ_n for each number n, we see from (18), (19) and (20) that $g(x)$ will map β onto σ and in a one-to-one manner. In addition, it follows as an easy consequence of the observations made in each of the cases of the definition of $g(x)$, that given any number $b \in \beta$ one can effectively find the value of $g(b)$. We can conclude therefore that $g(x)$ will have a partial recursive extension, and hence also that $\beta \leqq * \sigma$. This is the desired result of (15).

RE (16). To verify (16) we simply show that the inverse of the one-to-one mapping

$$g : \beta \to \sigma$$

defined in the course of proving (15), will also have a partial recursive extension. For this purpose let the number $d \in \sigma$ be given. In view of the definition of $g(x)$ we know that

$$\begin{aligned} d &= j(b_{\tilde{s}}, r) \\ &= j(b_{k_n}, r) \\ &= j(a_n, r), \end{aligned}$$

for some particular numbers s, n and r, with $0 \leqq r < e_n$ and

$$\begin{aligned} s &= r, && \text{if } n = 0 \\ s &= k_{n-1}+1+r, && \text{if } n \geqq 1. \end{aligned}$$

Moreover, then

$$g^{-1}(d) = b_s;$$

and we would like to find the value of b_s. Since j is a one-to-one recursive function and b_x and a_x are each regressive functions the values of the three numbers b_{k_n}, n and r can be effectively found from d. We now test on the value of n.

CASE 1. $n = 0$. Then $s = r$. In addition, in view of (21), $\tilde{r} = \tilde{s} = k_n$ with $r \leqq \tilde{r}$. Since we know the value of r with $r \leqq \tilde{r}$ and the value of $b_{\tilde{r}} = b_{k_n}$, we can by regressing from the number $b_{\tilde{r}}$ find the value of b_r. Therefore the number

$$g^{-1}(d) = b_s = b_r$$

can be found.

CASE 2. $n \geqq 1$. Then

$$g^{-1}(d) = b_s \text{ with } s = k_{n-1}+1+r. \tag{23}$$

Also $a_n = b_{k_n}$ and since a_x is a regressive function we can find from the number b_{k_n} the value of $b_{k_{n-1}} = a_{n-1}$. From the value of $b_{k_{n-1}}$ we can determine the number k_{n-1}. We know that

$$s = k_{n-1}+1+r \leqq k_n,$$

and therefore by regressing from the number b_{k_n} we can find the value of b_s. In view of (23), it follows that the value of $g^{-1}(d)$ can be found. In light of the previous remarks we can conclude that the mapping

$$g^{-1} : \sigma \to \beta$$

will have a partial recursive extension. This verifies the property $\sigma \leqq * \beta$ and completes the proof of (16). It had already been observed earlier that the proof of the theorem would be complete when both (15) and (16) had been verified; and therefore we are done.

The next result is a lemma. It is useful in proving the third theorem and because the lemma can be readily verified by using the definitions of the concepts involved, we will state it without a proof.

LEMMA 1. *Let B be an infinite regressive isol. Let α and $\tilde{\alpha}$ each be infinite sets with $\alpha \subseteq \tilde{\alpha}$ and $\tilde{\alpha}-\alpha$ finite. Then*

$$B \in \tilde{\alpha}_R \Rightarrow B \in \alpha_R.$$

THEOREM 3. *If α is a recursively enumerable set then,*

$$\alpha_R = \sum_{\delta \subseteq \alpha} \delta_\Sigma.$$

PROOF. Let α be an r.e. set. From Theorem 2 it follows that

$$\alpha_R \subseteq \sum_{\delta \subseteq \alpha} \delta_\Sigma.$$

To complete the proof we now verify

(1) $$\sum_{\delta \subseteq \alpha} \delta_\Sigma \subseteq \alpha_R.$$

For this purpose let $\delta \subseteq \alpha$ and let $B \in \delta_\Sigma$. We would like to show that $B \in \alpha_R$. If B is finite then B will belong to δ. In this case B will then also belong to α_R, since $\delta \subseteq \alpha \subseteq \alpha_R$.

Assume now that B is an infinite regressive isol. Then δ will be an infinite set. Let $g(x)$ denote the principal function of δ and let $d(x)$ be the e-difference function of $g(x)$. Then $B \in \delta_\Sigma$ implies that there is a regressive isol $A \geqq 1$ such that

(2) $$B = \Sigma_A d_n \text{ and } A \leqq * d_n.$$

Because B is infinite, it is easy to see from (2) that A will also be infinite. To establish the property that $B \in \alpha_R$, we will show that B is attainable from a recursive α-frame. It is convenient to assume here that $0 \in \alpha$; by Lemma 1 it follows that this will not effect the general result. We let a_n be a regressive function that ranges over a set in A, and let $p(x)$ be a regressing function for a_n. Then $p(x)$ is a partial recursive function such that $\rho a_n \subseteq \delta p$ and

(3) $$p(a_0) = a_0 \text{ and } p(a_{n+1}) = a_n,$$

(4) $$\rho p \subseteq \delta p,$$

(5) $$x \in \delta p \Rightarrow (\exists k)[p^{k+1}(x) = p^k(x)].$$

Let $p^*(x)$ denote the index function that is associated with $p(x)$. Then $p^*(x)$ is a partial recursive function with $\delta p^* = \delta p$, and

$$p^*(x) = (\mu y)[p^{y+1}(x) = p^y(x)], \quad \text{for } x \in \delta p.$$

Note that for each number n,

$$p^*(a_n) = n.$$

In view of (2), it follows that

(6) $$a_n \leqq * \, d_n,$$

and

(7) $$\sum_0^\infty j(a_n, v(d_n)) \in B.$$

Let $q(x)$ be a partial recursive function that establishes the relation of (6), i.e.,

$$q : a_n \to d_n.$$

Let β denote the set appearing on the left side in (7). We now proceed to define a frame F, with the aim of showing that F is a recursive α-frame and $\beta \in \mathscr{A}(F)$.

DEFINITION A. Let $b \in \delta p$ and with $p^*(b) = k$. Let

$$\begin{aligned} b_0 &= p^k(b), \\ b_1 &= p^{k-1}(b), \\ &\vdots \\ b_k &= b. \end{aligned}$$

Then b is called an *admissible* number if

(a) $b_0, b_1, \cdots, b_k \in \delta q$, and

(b) $r_0, r_0+r_1, \cdots, r_0+r_1+\cdots+r_k \in \alpha$, where $r_i = q(b_i)$, for $i = 0, 1, \cdots, k$.

Note that the collection of all admissible numbers will be an r.e. set, since each of the functions p and q is partial recursive, and each of the sets δp, δq and α is r.e.. In addition, we also note that each of the numbers a_n will be admissible, and if $b = a_n$ in Definition A, then $p^*(a_n) = n$ and

$$b_i = a_i \text{ and } r_i = d_i \text{ for } 0 \leqq i \leqq n.$$

DEFINITION B. Let $b \in \delta p$ be an admissible number and let the numbers $k, b_0, \cdots, b_k$ and $r_0, \cdots, r_k$ be defined as in Definition A. Define

$$(8) \qquad \delta_b = \sum_0^k j(b_n, v(r_n)).$$

In view of the definition of δ_b, it is readily seen that from a given admissible number b we can effectively find, by using the functions p and q, all of the members of the finite set δ_b. In particular, we note that for $b = a_n$,

$$(9) \qquad \delta_{a_n} = \sum_0^n j(a_i, v(d_i)).$$

Let

$$(10) \qquad F = \{\phi\} \cup \{\delta_b | b \in \delta p \text{ and admissible}\}.$$

Let

$$H = \{\delta_{a_n} | n = 0, 1, 2, \cdots\}.$$

In view of the property that each of the numbers a_n is an admissible number, it follows from (9) and (10) that H will be a directed subset of F.

We now wish to show that F is a recursive α-frame and $\beta \in \mathscr{A}(F)$. For this purpose let b and c each be admissible numbers of δp, and let $p^*(b) = k$ and $p^*(c) = h$. Consider the following two lists of numbers,

$$\text{L}_1: \qquad b_0 = p^k(b), b_1 = p^{k-1}(b), \cdots, b_k = b,$$

$$\text{L}_2: \qquad c_0 = p^h(c), c_1 = p^{h-1}(c), \cdots, c_h = c.$$

Because b and c are each admissible numbers, it is easy to see that each of the numbers appearing in L_1 and L_2 will also be admissible. In view of Definitions A and B the following properties are readily seen to be true. If there is no number that occurs in both L_1 and L_2, then $\delta_b \cap \delta_c = \phi$; if the number c occurs in L_1 then $\delta_c \subseteq \delta_b$; and if the number b occurs in L_2 then $\delta_b \subseteq \delta_c$. Otherwise there will be a number $m < \min (k, h)$ such that $b_m = c_m$ and $b_{m+1} \neq c_{m+1}$. In this special case it follows that the number $d = b_m = c_m$ is admissible and $\delta_b \cap \delta_c = \delta_k$. By combining (3.1), (10) and these properties, it follows that F will be a frame. Also F will be an α-frame, for by combining Definitions A and B, and (10) we have the following implications,

$$\delta \in F \Rightarrow \begin{cases} \delta = \phi & \text{or} \\ \delta = \delta_b & \text{for an admissible number } b \end{cases}$$
$$\Rightarrow \begin{cases} \text{card } \delta = 0 \in \alpha \text{ or} \\ \text{card } \delta = r_0 + \cdots + r_k \in \alpha \end{cases}$$
$$\Rightarrow \text{card } \delta \in \alpha.$$

We wish to verify now that F is a recursive frame. It has been already noted that the collection of all admissible numbers is an r.e. set. Combining this fact with the definition of F it follows that the collection $\{\delta^* | \delta \in F\}$ will also be r.e. We recall from § 3 the definition of the collection δF;

$$\delta F = \{\eta | \exists \delta \in F : \delta \supseteq \eta\}.$$

Because $\{\delta^* | \delta \in F\}$ is r.e., it is easy to see that also

(11) $$\delta F^* = \{\eta^* | \eta \in \delta F\} \text{ is r.e..}$$

Property (11) gives one of the two properties that we need to verify in order to show that F is a recursive frame, namely (3.4). The other property we need to verify is (3.5), it is that the mapping

(*) $$\eta^* \to C_F(\eta)^*, \text{ for } \eta \in \delta F,$$

is a partial recursive function. For this purpose, assume that we are given the number η^*, for some $\eta \in \delta F$; we wish to find the number $C_F(\eta)^*$. From the value of η^*, we can find all the members of the set η if there are any. If $\eta = \phi$, then we would recognize this fact. Also, in this event $C_F(\eta) = \phi$, and we could then find the value of $C_F(\eta)^*$. If η is non-empty, then η will consist of numbers of the form $j(b, y)$ where b is an admissible number. Among the admissible numbers b such that $j(b, y)$ belongs to η we could find the unique one that has the maximum index; i.e., the maximum value of $p^*(b)$. Let $\tilde{b}$ denote this particular admissible number. Then it is easy to see that

(12) $$C_F(\eta) = \delta_{\tilde{b}}.$$

In addition, because we would know the value of $\tilde{b}$ we could effectively find the set $\delta_{\tilde{b}}$ and its Gödel number $\delta_{\tilde{b}}^*$. In view of (12), this means that we would be able to find the value of $C_F(\eta)^*$. We can conclude from these remarks that the mapping (*) is a partial recursive function; and therefore F will be a recursive frame. Finally we verify that $\beta \in \mathscr{A}(F)$. First recall that the collection

$$H = \{\delta_{a_n} | n = 0, 1, 2, \cdots\}$$

is a directed subset of the frame F. Combining this fact with (7) and (9), it follows that

$$\beta = \sum_0^\infty j(a_n, v(d_n))$$
$$= \sum_0^\infty \delta_{a_n} \in \mathscr{A}(F).$$

We have therefore shown that the regressive isol B that we began with belonging to δ_Σ, can be attained from a recursive α-frame, namely F, and therefore $B \in \alpha_R$. This verifies the inclusion of (1), and completes the proof.

THEOREM 4. *If α is any set, then*

$$\text{(1)} \qquad \alpha_R = \sum_{\substack{\delta \subseteq \alpha \\ \delta \text{ r.e.}}} \delta_R .$$

PROOF. Let α be any set. The direction of inclusion $\subseteq$ in (1), follows from Theorems 2 and 3. The direction of inclusion $\supseteq$ in (1), follows readily by noting that, by (3.12),

$$\delta \subseteq \alpha \Rightarrow \delta_R \subseteq \alpha_R .$$

5. Some remarks about infinite series

(A) One of the advantages of considering infinite series of isols of the form $\Sigma_A e_n$ when $A \leqq * e_n$ is that it is easy to obtain and view regressive enumerations of representatives belonging to $\Sigma_A e_n$. When A is finite, this is easy since $\Sigma_A e_n$ will also be finite. In the special case that A is an infinite regressive isol and $A \leqq * e_n$, then it can be readily shown that for a_n any regressive function that ranges over a set in A, one will have

$$j(a_0, 0), \cdots, j(a_0, e_0 \dot{-} 1), j(a_1, 0), \cdots, j(a_1, e_1 \dot{-} 1), \cdots,$$

(where no terms of the form $j(a_m, y)$ would appear if $e_m = 0$) represent a regressive enumeration of a set belonging to $\Sigma_A e_n$. With this feature in mind it is easy to establish some properties of the minimum and maximum of infinite series of this kind; for definitions of the minimum and maximum of two regressive isols see [9] and [7]. For example,

I. Let A and B be regressive isols such that $A \leqq * e_n$ and $B \leqq * e_n$ Then min $(A, B) \leqq * e_n$, and

$$\min (\Sigma_A e_n, \Sigma_B e_n) = \Sigma_{\min (A, B)} e_n .$$

II. Let A and B be two regressive isols such that $A + B \in \Lambda_R$, $A \leqq * e_n$. and $B \leqq * e_n$. Then max $(A, B) \leqq * e_n$ and

$$\max (\Sigma_A e_n, \Sigma_B e_n) = \Sigma_{\max (A, B)} e_n .$$

Concerning the extensions α_Σ and α_R considered in § 4, the following theorem is readily obtained from properties I and II, Theorem 3 and the definition of α_Σ.

THEOREM A. Let α be any set. Then

(1) $A, B \in \alpha_\Sigma \Rightarrow \min(A, B) \in \alpha_\Sigma$,

(2) $A, B \in \alpha_\Sigma$ and $A+B \in \Lambda_R \Rightarrow \max(A, B) \in \alpha_\Sigma$,

(3) $A, B \in \alpha_R \Rightarrow \min(A, B) \in \alpha_R$, for α recursive,

(4) $A, B \in \alpha_R$ and $A+B \in \Lambda_R \Rightarrow \max(A, B) \in \alpha_R$, for α recursive.

(B) The only types of infinite series of isols that we have considered in the paper were those of the form $\Sigma_A e_n$ with $A \leqq * e_n$. These include as a special case the event when e_n is a recursive function. We wish to mention that, while $\Sigma_A e_n$ when $A \leqq * e_n$ will always be a regressive isol, it is possible to have the value of $\Sigma_B d_n$ be a regressive isol even when the relation $B \leqq * d_n$ does not hold. This particular property can be easily shown in the following way. Let A be any infinite regressive isol and let a_n be any regressive function that ranges over a set in A. Set $B = A+1$ and $d_n = a_n$. Then B will also be a regressive isol. In addition, by [4, Proposition 5]

$$\Sigma_B d_n \in \Lambda_R.$$

On the other hand it is easy to see that the relation $B \leqq * d_n$ would imply that $d_n \leqq * d_{n+1}$ and it would then be an easy consequence of this fact that d_n were a recursive function. Thus $\Sigma_B d_n$ would be a regressive isol and yet $B \leqq * d_n$ would not be true.

Many of the interesting cases where infinite series $\Sigma_A e_n$ play a role are when e_n is a recursive function. The reason for this is that the canonical extension of an increasing recursive function when evaluated at a regressive isol is representable as a particular infinite series of this form; refer to (4.3). The occurrence of infinite series of the form $\Sigma_A e_n$ with $A \leqq * e_n$, in the study of properties of regressive isols is also interesting. Here are two theorems that can be obtained, and we state these without proofs. The second theorem is an unpublished result of Judy Gersting.

THEOREM B. Let $f(x, y)$ be a recursive and combinatorial function of x and y. Let $D_f(X, Y)$ denote the Myhill-Nerode canonical extension of $f(x, y)$ to Λ^2 [cf. [13]]. Let $A, B \in \Lambda_R$ with $A+B \in \Lambda_R$. Then

$$D_f(A, B) \in \Lambda_R,$$

and there is a function d_n such that $A+B \leqq * d_n$ and,

$$D_f(A, B) = \Sigma_{A+B} d_n.$$

THEOREM C. Let A be a regressive isol and $A \leqq * e_n$. Let B be an isol such that $B \leqq \Sigma_A e_n$. Then B will also be a regressive isol, and there will exist a function u_n such that

$$B = \Sigma_A u_n \text{ with } A \leqq * u_n \text{ and } (\forall n)[u_n \leqq e_n].$$

In Theorem B, the function d_n appearing there need not be recursive, even though $f(x, y)$ is a recursive function of x and y. If in the hypothesis of Theorem C we assume that the function e_n appearing there is recursive, then it is still possible that there would be no recursive function u_n that would satisfy the conclusion of the theorem.

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