COMPOSITIO MATHEMATICA

V. W. BRYANT

The convexity of the subset space of a metric space

Compositio Mathematica, tome 22, nº 4 (1970), p. 383-385
<http://www.numdam.org/item?id=CM_1970__22_4_383_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 4, 1970, pag. 383-385
Wolters-Noordhoff Publishing
Printed in the Netherlands

THE CONVEXITY OF THE SUBSET SPACE OF A METRIC SPACE

by

V. W. Bryant

Let (X, d) be a metric space and let d^* be the Hausdorff metric on the set X^* consisting of the non-empty closed bounded subsets of X. In this paper we consider what conditions on (X, d) will ensure that (X^*, d^*) is metrically convex.

DEFINITION. (i) (X, d) is (metrically) *convex* if for any two distinct points $x, y \in X$ there exists $z \in X$, distinct from them both, with

$$d(x, y) = d(x, z)+d(z, y).$$

(ii) (X, d) is a (metric) *segment space* if for any $x, y \in X$ there exists an isometry $f: [0, d(x, y)] \to (X, d)$ with $f(0) = x$ and $f(d(x, y)) = y$.

It is clear that every segment space is convex, and it has been shown that the two concepts coincide in a complete space (see, for example, [1; p. 41]). Now, if $A \subseteq X$ and $0 \leqq \delta$, then the set A_δ is defined by

$$A_\delta = \{x \in X : \exists a \in A \quad \text{with} \quad d(x, a) \leqq \delta\}.$$

We note that in any metric space $\bar{A} = \bigcap_{n=1}^{\infty} A_{1/n}$, and that in any segment space $A_{\gamma+\delta} = (A_\gamma)_\delta$ for $0 \leqq \gamma, \delta$. The Hausdorff metric d^* is defined on X^* by

$$d^*(A, B) = \inf\{0 \leqq \delta : B \subseteq A_\delta \quad \text{and} \quad A \subseteq B_\delta\} \qquad (A, B \in X^*).$$

It is known that if (X, d) is compact (resp. complete), then (X^*, d^*) is compact (resp. complete), proofs of these results being found in [3; p. 38] and [2; p. 29 IV].

We are now ready to investigate the convexity of (X^*, d^*). Since $d^*(\{x\}, \{y\}) = d(x, y)$ for $x, y \in X$, it is clear that the convexity of (X^*, d^*) implies the convexity of (X, d). The theorem below shows when the converse implication holds.

THEOREM. *If (X, d) is a compact convex metric space, then so too is (X^*, d^*).*

PROOF. In view of the above remarks we need only show that (X^*, d^*) is convex. Let $A, B \in X^*$ with $d^*(A, B) = \delta > 0$. Then by the compactness of (X, d) (and hence of A, B) $\overline{A_\delta} = A_\delta$ and $\overline{B_\delta} = B_\delta$. Since (X, d) is a segment space it follows that

$$A \subseteq \bigcap_{n=1}^{\infty} B_{\delta+1/n} = \bigcap_{n=1}^{\infty} (B_\delta)_{1/n} = \overline{B_\delta} = B_\delta$$

and similarly $B \subseteq A_\delta$. Let $C = A_{\delta/2} \cap B_{\delta/2}$. Then we show that $C \neq \phi$ (whence $C \in X^*$) and that $d^*(A, C) = d^*(C, B) = \frac{1}{2}d^*(A, B)$. If $a \in A \subseteq B_\delta$, then $d(a, b) \leqq \delta$ for some $b \in B$ and there exists $c \in C$ with $d(a, c) = d(c, b) = \frac{1}{2}d(a, b) \leqq \frac{1}{2}\delta$. Thus $c \in A_{\delta/2} \cap B_{\delta/2} = C$ and $a \in C_{\delta/2}$. This shows that $C \neq \phi$ and $A \subseteq C_{\delta/2}$, and similarly $B \subseteq C_{\delta/2}$. Thus $C \in X^*$, $d^*(A, C) \leqq \delta/2$ and $d^*(C, B) \leqq \delta/2$. But

$$\delta = d^*(A, B) \leqq d^*(A, C) + d^*(C, B) \leqq \tfrac{1}{2}\delta + \tfrac{1}{2}\delta = \delta$$

and so $d^*(A, C) = d^*(C, B) = \frac{1}{2}d^*(A, B)$ as required. Hence (X^*, d^*) is a convex space and the theorem is proved.

We now give examples to show that neither the total-boundedness nor the completeness of a convex space (X, d) necessarily implies the convexity of (X^*, d^*).

1. Let X be the subspace of 3-dimensional Euclidean Space given by

$$X = \{(x, y, z) : x^2+y^2 < 1, |z| \leqq 1\} \cup \{(x, y, z) : x^2+y^2 = 1, \\ x \text{ rational}, z = -1\} \cup \{(x, y, z) : x^2+y^2 = 1, x \text{ irrat}, z = 1\}.$$

Then (X, d) is a totally bounded segment space. However, by considering

$$A = \{(x, y, z) \in X : x^2+y^2 = 1, z = -1\} \in X^*$$
$$B = \{(x, y, z) \in X : x^2+y^2 = 1, z = 1\} \in X^*$$

we see that (X^*, d^*) is not convex.

2. Let X be the normed vector space of all real null sequences with metric d induced by the usual norm. Then (X, d) is a complete segment space and we show that (X^*, d^*) is not convex. For if

$A = \{\{x_n\} \in X : x_n \neq 0$ for an odd no. of n, and $x_n = 1+1/n$ whenever $x_n \neq 0\} \in X^*$

$B = \{\{x_n\} \in X : x_n \neq 0$ for an even no. of n, and $x_n = 1+1/n$ whenever $x_n \neq 0\} \in X^*$,

then $d^*(A, B) = 1$ and $\overline{A_{\frac{1}{2}}} \cap \overline{B_{\frac{1}{2}}} = A\frac{1}{2} \cap B\frac{1}{2} = \phi$. Thus there exists no $C \in X^*$ with $d^*(A, C) = d^*(C, B) = \frac{1}{2}d^*(A, B)$. It follows that (X^*, d^*) is neither a segment space nor a convex space.

Finally we give an example to show that the existence of a segment from $\{x\}$ to $\{y\}$ in (X^*, d^*) does not imply the existence of a segment from x to y in (X, d).

3. Let X be the subset of R^2 given by $X = \{(x, y) : |x| < 1, 0 \leqq y \leqq 1$ and (x is rational if and only if y is)$\} \cup \{(-1, 0), (1, 0)\}$, and define a metric d on X by

$$d((x, y), (x', y')) = |y-y'| \cdot (1-\max(|x|, |x'|))+|x-x'|, \qquad ((x, y), (x', y') \in X).$$

Then $d^*(\{(-1, 0)\}, \{(1, 0)\}) = 2$ and the mapping $f : [0, 2] \to (X^*, d^*)$ given by $f(\lambda) = \{(x, y) \in X : x = \lambda-1\}$ for $\lambda \in [0, 2]$ is an isometry with $f(0) = \{(-1, 0)\}$ and $f(2) = \{(1, 0)\}$. This is therefore a segment between $\{(-1, 0)\}$ and $\{(1, 0)\}$ in (X^*, d^*). However, there exists no segment between $(-1, 0)$ and $(1, 0)$ in (X, d).

REFERENCES

L. M. Blumenthal
[1] Distance Geometry, Oxford, 1953.

C. Kuratowski
[2] Topologie I (2nd edition), Warsaw, 1948.

C. Kuratowski
[3] Topologie II, Warsaw, 1950.

(Oblatum 16–III–1970)

Dr. V. W. Bryant,
Department of Pure Mathematics,
The University of Sheffield,
England.