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THE TOPOLOGICAL SPHERICAL SPACE FORM PROBLEM I

by

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This paper is intended as a first step towards the classification of closed piecewise linear of topological manifolds covered by S^{2n-1}. Alternatively we concern ourselves with free topological actions by a finite group on S^{2n-1}. Such a group necessarily has periodic cohomology, and the present state of knowledge can be summarised as follows:

(1) The finite group π has periodic cohomology if and only if every subgroup of order p^2 (any prime p) is cyclic.

(2) π acts freely and orthogonally on a sphere if and only if every subgroup of order pq (p, q primes not necessarily distinct) is cyclic.

(3) If π acts freely on a sphere, then Milnor [6] has proved that every subgroup of order $2p$ (p any prime) is cyclic.

The pq-condition is the only known sufficient condition for free topological actions. Moreover starting from any orthogonal action one can use the s-cobordism theorem to construct infinitely many free actions which are topologically distinct. Since there are only finitely many distinct orthogonal actions, the topological classification of the quotient spaces is a much deeper problem than the geometric. At present complete answers are known only for Z_2 and the cyclic groups of odd order, [16], [17] and [18]. Nothing new on existence has been produced in the last twelve years. But see T. Petrie, Bull. Amer. Math. Soc. 76 (1970), 1103–1106.

Our investigation starts from the work of R. G. Swan [11] on periodic resolutions for groups with periodic cohomology, together with the observation that the quotient complex Y of a topological realisation satisfies Poincaré duality. Indeed a minor strengthening of Swan's argument shows that Y is a simple Poincaré complex in the sense of [17]. This is contained in § 1 and § 2. Our main theorem (§ 3) shows that the structure group of the normal fibration of Y can be reduced from G to PL, and hence that we may try and replace Y by a closed manifold using the techniques of surgery. Although we concentrate on the piecewise linear case, since this is more familiar, our arguments are actually easier in the topological category. This is important from the point of view of classification, since L. Siebenmann has recently shown that surgery of topological manifolds is possible [7].

In a further paper we propose to classify the odd and 'mod 2' parts of the normal invariants, and to resolve some cf the surgical problems which arise. In particular we hope to recover Milnor's theorem about the non-existence of free dihedral actions on S^{2n-1}.

A certain amount of notation will be used without further comment.

$\hat{H}^*(\pi, A)$ denotes the Tate cohomology of a finite group π, obtained by splicing a 'positive' to a 'negative' resolution by means of the norm [2]. In particular $H_p(\pi, A) = \hat{H}^{-p-1}(\pi, A), p \geqq 1$.

$C^*(Y)$ & $C_*(Y)$ denote (co)chains with coefficients in the group ring of $\pi_1(Y)$, subject to the conventions introduced in [14] & [15].

G/PL is the classifying space for piecewise linear bundles fibre homotopically equivalent to a reducible fibration; G/Top is defined similarly, see [9], [4]. The homotopy groups may be read off from the table:

	$4k$	$4k+1$	$4k+2$	$4k+3$
$\pi_*(G/PL)$ or $\pi_*(G/Top)$	Z	0	Z_2	0

$Z_{(2)} = Z[\frac{1}{3}, \frac{1}{5}, \cdots, 1/p, \cdots]$, the integers localised at the prime 2, and Φ denotes Euler's function.

Finally our debt to the book of Wolf, [19] will be clear to any reader. Wherever possible our group theoretic notation conforms to his.

1. Groups with periodic cohomology

1.1. Definition. Let π be a finite group. For any prime p the cohomological p-period of π is the smallest integer q such that $\hat{H}^k(\pi, A)$ and $\hat{H}^{k+q}(\pi, A)$ have isomorphic p-primary components for all values of k and all coefficient modules A. The *cohomological period* of π is the lowest common multiple of the p-periods.

The following proposition is well known, see for example [2, XII 11.6].

1.2. Proposition. *π has finite p-period if and only if a Sylow subgroup π_p is cyclic or of generalised quaternion type.*

A complete classification exists for groups satisfying this Sylow condition for all p, which for future reference we reproduce below. The table is a modified version of one appearing in [19, p. 179]. In the two types of non-soluble groups (V and VI) $SL(2, p)$ denotes the group of 2×2

matrices of determinant 1 with entries in $\boldsymbol{F}_p$, the field of p elements, $p \geqq 5$. If ω generates $\boldsymbol{F}_p$, define an automorphism θ of $SL(2,p)$ by

$$\theta\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \quad \text{and} \quad \theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}$$

1.3)

Type	Generators	Relations	Conditions
I	A, B	$A^{m_1} = B^{m_2} = 1$ $BAB^{-1} = A^r$	$m_1 \geqq 1, m_2 \geqq 1$ $(m_2(r-1), m_1) = 1$ $r^{m_2} \equiv 1(m_1)$
II	A, B, R	As in I; also $R^2 = B^{m_2/2}$ $RAR^{-1} = A^{l_1}$, $RBR^{-1} = B^{l_2}$	As in I; also $l_1^2 \equiv r^{l_2-1} \equiv 1(m_1)$, $m_2 = 2^u v$, $u \geqq 2$, $l_2 \equiv -1(2^u)$ $l_2^2 \equiv 1(m_2)$.
III	A, B, P, Q	As in I; also $P^4 = 1$, $P^2 = Q^2 = (PQ)^2$ $AP = PA, AQ = QA$, $BPB^{-1} = Q, BQB^{-1} = PQ$	As in I; also $m_2 \equiv 1(2)$ $m_2 \equiv 0(3)$
IV	A, B, P, Q, R	As in III; also $R^2 = P^2, RPR^{-1} = QP$, $RQR^{-1} = Q^{-1}$ $RAR^{-1} = A^{l_1}$, $RBR^{-1} = B^{l_2}$	As in III; also $l_2^2 \equiv 1(m_2)$ $l_2 \equiv -1(3)$ $r^{l_2-1} \equiv l_1^2 \equiv 1(m_1)$
V		$K \times SL(2,p)$	K of type I; also $([K:1],$ $[SL(2,p):1]) = 1$
VI	π_1 of type V S	$[\pi:\pi_1] = 2$, $S^2 = -1_2 \in SL(2,p)$, S normalises both factors of π_1, $SB = BS$, $SAS^{-1} = A^{-1}$, $SLS^{-1} = \theta(L)$ for any $L \in SL(2,p)$.	

The next two technical results will be used frequently in the proof of the main theorem.

1.4. PROPOSITION. *Let π be a finite group with periodic cohomology. The* 2- *and* 3-*periods of π both divide* 4.

PROOF. The assertion about the 2-period is contained in [12, Theorem 1]. Theorem 2 of the same paper characterises the 3-period as

$$2[N_\pi(\pi_3) : C_\pi(\pi_3)],$$

where N and C stand for normaliser and centraliser respectively. Since $C_\pi(\pi_3)$ contains π_3, N/C has order prime to 3. However N/C is also contained in the automorphism group of π_3, which has order $2.3^{\nu-1}$. Hence $[N_\pi(\pi_3) : C_\pi(\pi_3)] \leqq 2$, and the 3-period divides 4.

In the cases when π operates freely and orthogonally on S^{2n-1}, it will be essential to compare the cohomological period with the degree over $\boldsymbol{R}$ of an irreducible fixed point free representation. With the notation of table 1.3 let d be the order of r in the multiplicative group of residues modulo m_1, of integers prime to m_1. Both the cohomological period and the representation degree are functions of d, but the latter may exceed the former by a factor of 2. This is the case, for example, with most groups of type IV [19, p. 208].

Groups of type III and IV are best studied through their generalised binary polyhedral subgroups, $\mathbf{T}_v^*$ and $\mathbf{O}_v^*$. $\mathbf{T}_v^*$ is obtained by putting $m_1 = 0$, $m_2 = 3^v$, and $\mathbf{O}_v^*$ by putting $l_1 = 0$, $l_2 = -1$ in addition. When $v = 1$ we obtain the usual binary tetrahedral and octahedral groups respectively.

1.5. PROPOSITION. (i) $\mathbf{T}_v^*$ *has both cohomological period and minimum representation degree equal to* 4

(ii) $\mathbf{O}_v^*$ *has cohomological period* 4, *and minimal representation degree* 8, *unless* $v = 1$, *when there is a representation of degree* 4.

PROOF. For the representation theory see [19, Chapter 7]. The assertion about the cohomological period follows from 1.4, since $[\mathbf{T}_v^* : 1] = 8.3^v$ and $[\mathbf{O}_v^* : 1] = 16.3^v$.

2. Definition and existence of polarised complexes

This section is a comparatively minor extension of the fundamental work of Swan in [11], although we follow the treatment given [15], rather than the original.

2.1. DEFINITION. Let π be a finite group and $n \geqq 4$. A (π, n)-*polarisation* of a finite dimensional complex Y consists of an isomorphism $\pi_1(Y, y_0) \cong \pi$ and a homotopy equivalence of the universal cover

$\tilde{Y} \xrightarrow{\sim} S^{n-1}$. Two polarised spaces Y_1 and Y_2 are equivalent if there is a homotopy equivalence $f: Y_1 \xrightarrow{\sim} Y_2$ which preserves the polarisations.

An argument in Cartan and Eilenberg [2, p. 357] shows that a necessary condition for such a Y to exist is that π has periodic cohomology. A generalisation of the Lefschetz fixed point theorem [10] shows that, if n is even, Y must be orientable, and n divisible by the cohomological period. Indeed once an orientation of S^{n-1} has been chosen, it follows that Poincaré duality for $\tilde{Y}$ implies Poincaré duality for Y.

2.2. THEOREM. *If π is a finite group with cohomological period s, the equivalence classes of (π, s)-polarised complexes Y correspond bijectively to generators g of $\hat{H}^s(\pi; \mathbf{Z})$.*

PROOF. By [11, Thm. 4.1] π has a periodic projective resolution of period s. This gives us an exact sequence

$$0 \to \mathbf{Z} \to P_{s-1} \to \cdots \to P_0 \to \mathbf{Z} \to 0,$$

defining a generator g_0 of $\mathrm{Ext}^s_\pi(\mathbf{Z}, \mathbf{Z}) = H^s(\pi; \mathbf{Z})$. If r is any integer prime to $[\pi : 1]$, let $[r, N]$ be the projective ideal of $\mathbf{Z}\pi$ generated by r and $N = \sum_{x\in\pi} x$. Since $[r, N] + \mathbf{Z} \cong \mathbf{Z}\pi + \mathbf{Z}$, we can modify the resolution above and obtain

$$0 \to \mathbf{Z} \to P_{s-1} + \mathbf{Z}\pi \to P_{s-2} + [r, N] \to \cdots \to P_0 \to \mathbf{Z} \to 0.$$

This replaces g_0 by rg_0, that is, we can realise any generator g of $H^s(\pi; \mathbf{Z})$, [11, Lemma 7.4]. By adding in elementary complexes $0 \to F \to F \to 0$, with F free on countably many generators, we may suppose each P_i to be free. By splicing together a large number of copies of the resolution we obtain a projective resolution of Z over π, chain homotopy equivalent to the chain complex of a $K(\pi, 1)$. Now apply [14, Thm. 4] to show there exists a $K(\pi, 1)$ whose chain complex in dimensions $\geqq 4$ is the given one. The $s-1$ skeleton Y of this space has chain complex equivalent to the one above, and is (π, s)-polarised. That equivalence classes of polarisations are in $(1-1)$ correspondence with the generators g follows from [13, Thm. 1.8].

The obstruction to replacing the complex in Theorem 2.2 by a finite complex is the Euler characteristic θ of the resolution in $\tilde{K}_0(Z\pi)$. This depends only on the homotopy type of Y, hence $\theta = \theta(g)$ depends only on the particular generator (k-invariant) in $H^s(\pi; Z)$. If g_0 is some fixed maximal generator of smallest dimension in $H^s(\pi; Z)$, all possible g in all possible dimensions are of the form rg_0^h. In the Mayer-Vietoris sequence of algebraic K-theory, associated with the Milnor square of rings

$$\begin{tikzcd} Z\pi \arrow[r] \arrow[d, "\varepsilon"'] & B = Z\pi/(N) \arrow[d] \\ Z \arrow[r] & Z_{[\pi:1]}, \end{tikzcd}$$

the unit r in $Z_{[\pi:1]}$ maps under $\partial : K_1 Z_{[\pi:1]} \to \tilde{K}_0 Z\pi$ to the class of $[r, N]$. Lemma 6.2 of [11] can be restated as

2.3. PROPOSITION

$$\theta(rg_0^h) = \partial r + h\theta(g_0).$$

In the case of the cyclic and generalised quaternion groups the existence of numerous free orthogonal actions on S^{2n-1} enables us to prove somewhat more.

2.4. COROLLARY. (i) *If* π *is cyclic,* $\theta(g) = 0$.

(ii) *If* π *is a generalised quaternion group* $\mathbf{Q}(2^v)$, $v \geqq 3$, $\theta(g)$ *belongs to the image of* ∂ *and has order at most* 2.

PROOF. The first assertion is trivial, since the lens spaces L^{2n-1} $(p; q, 1, \cdots, 1)$ cover all possible k-invariants $g \in H^{2n}(\boldsymbol{Z}_p; \boldsymbol{Z})$. If π is isomorphic to $\mathbf{Q}(2^v)$, π has cohomological period 4, and any free orthogonal action on S^3 defines a polarisation with minimal generator g_0, such that $\theta(g_0) = 0$. Let π have presentation $\{A, B : A^{2^{v-1}} = 1, A^{2^{v-2}} = B^2, BA = A^{-1}B\}$; there are 2^{v-2} distinct fixed point free unitary representations of degree 2, induced from the subgroup generated by A. The corresponding k-invariants are squares in the group of units $U(\boldsymbol{Z}_{2^v})$ and since $\partial(r^2) = 2\partial r$ (ii) follows from 2.3. This corollary is a special case of a more general result for groups with periodic cohomology, which can be proved using their hyperelementary subgroups.

There are two problems posed by the preceding results. (a) Are there any groups π for which $\theta(g)$ lies outside the image of ∂? And (b) is Image ∂ ever non trivial? A positive answer to (b) would provided examples of polarised complexes with non-vanishing finiteness obstruction. For example, over the group $\mathbf{Q}(8)$ Swan has shown [11, Lemma 6.3], that $[3, N]$ is projective, satisfies the equation $[3, N] + [3, N] = \boldsymbol{Z}\pi + \boldsymbol{Z}\pi$, but is not free. We have since shown that it is not. Problem (b) therefore turns on whether the module $[3, N]$ is stably free. Similar examples can be constructed for the higher order quaternion groups.

We next recall that a finite CW-complex Y^{n-1} is a *simple Poincaré complex* if it has a fundamental class $[Y]$ a representing cocycle η of which defines a *simple* chain homotopy equivalence of degree $-n+1$.

$$\eta_\cap : C^*Y \to C_*Y$$

In [17] such complexes were called finite Poincaré complexes; we shall call a complex Y finitely polarised, if it is both polarised, and a finite complex in this strong sense. So far we have shown that each generator $g \in \hat{H}^s(\pi; Z)$ determines a (π, s)-polarised complex $Y = Y_g$, unique up to homotopy type and with finiteness obstruction $\theta(g)$. We now suppose that $\theta(g) = 0$ and ask whether Y_g is homotopy equivalent to a simple Poincaré complex. Our strongest general result is

2.5. THEOREM. *If $\theta(g) = 0$, Y_{g^2} is homotopy equivalent to a simple Poincaré complex.*

PROOF. By assumption Y is finite, so the torsion $\tau(Y)$ (an element of $Wh(\pi)$) of the chain homotopy equivalence f between Y and its dual is defined. In dimension $r+1$ the algebraic mapping cone $C(f)$ of f takes the form

$$C^r Y + C_{s-r} Y \xrightarrow{\Delta_{r+1}} C^{r+1} Y + C_{s-r-1} Y,$$
$$\Delta_{r+1} = \begin{pmatrix} d^r & 0 \\ (-1)^{r+1} f & d_{s-r} \end{pmatrix}.$$

Let Y_1 denote the complex, whose chain complex is obtained by splicing the dual complex $C_*(DY)$ (C^*Y with a dimension shift) to the left hand end of $C_* Y$. $C_*(DY)$ and $C_* Y$ are chain homotopy equivalent, so Y_1 is homotopy equivalent to Y_{g^2}. By writing down the boundary operators of the enlarged algebraic mapping cone, one sees that, because s is even,

$$\tau(Y_1) = \tau(Y) + \tau(DY).$$

However τ is additive over homotopy equivalences, and $\tau(Y) = -\tau(DY)$. Therefore $\tau(Y_1) = 0$, and Y_1 is a finitely polarised complex for the group π. Observe that this purely algebraic argument does not depend on $C_*(DY)$ having a geometric realisation.

2.6. REMARK. In the case when π is cyclic it is known that $Wh(\pi)$ is torsion free, from which it follows that any (π, s)-polarised finite complex is finitely polarised. We ask whether the corresponding result holds in general.

2.7. THEOREM. *If π is a finite group with cohomological period s and $h = ([\pi : 1], \Phi[\pi : 1])$ then there exists a complex Y with finite $(\pi, 2sh)$ polarisation.*

PROOF. Let $\theta(g)$ be the finiteness obstruction for the resolution of 2.2. By replacing $\theta(g)$ by $\theta(rg)$ if necessary, we may suppose that the order of θ in $\tilde{K}_0 Z\pi$ is h [11, § 10]. By splicing together h copies of the original resolution we can construct a periodic *free* resolution of period sh by finitely generated modules. The argument for geometric realisation is the same as before, and since Y is now finite, Theorem 2.5 applies.

2.8. REMARK. The numerical factor h in the hypothesis of 2.7 can be improved by replacing the order of the group $[\pi : 1]$ by the *Artin exponent* $\mathfrak{A}(\pi)$.

This was introduced by T. Y. Lam in [5] as the smallest positive integer for which Artin's Theorem on induced representations holds. In general $\mathfrak{A}(\pi)$ divides $[\pi : 1]$ properly, for example, the argument of [5, Cor. 7.3] shows that, if π is of type I with $[\pi : 1] = m_1 m_2$ then $\mathfrak{A}(\pi) = m_2$.

As we have indicated in the introduction, the topological space form problem is particularly interesting for the non-abelian groups $\bar{\pi}$ of order pq, p and q distinct primes. $\bar{\pi}$ is of Type I, the number d introduced in 1 equals q, and so $[N_{\bar{\pi}}\langle A\rangle : C_{\bar{\pi}}\langle A\rangle] = q$. The p-period thus equals $2q$, and a similar argument shows the q-period to be 2. Since $(pq, \Phi(pq)) = q$, the smallest dimension for a finitely polarised complex given by Theorem 2.7 is $4q^2-1$.

3. Existence of normal invariants

From now on we assume that Y^{4n-1} is a finitely polarised complex, constructed as in 2, with fundamental group identified via the polarisation with the abstract group π. Poincaré duality implies that a regular neighbourhood of a stable embedding of Y^{4n-1} in S^{4n+k-1}, $k \gg 4n-1$, has boundary homotopically equivalent to the total space of an S^{k-1} fibration ν, called the Spivak fibration [8]. (The original published argument for the existence of ν is valid, because even though Y is not 1-connected, π acts trivially on the homology of the universal cover, see remark 2 on p. 89.) Given the relation to a regular neighbourhood it is clear that the Thom class of the mapping cylinder of the projection of ν, also a fibration, is spherical. Another way of putting this is to say that ν is reducible. Let the unique stable fibre homotopy equivalence class of ν be induced by a map $\varphi : Y \to BG$ the classifying space for stable spherical fibrations. Recall that $G = \varinjlim G_n$, where G_n denotes the monoid of self homotopy equivalences of S^{n-1}.

3.1. DEFINITION. There is a reducible *PL* (topological) vector bundle over Y if and only if $\varphi : Y \to BG$ factors through *BPL* (*BTop*). A homotopy class of such factorisations is called a *normal invariant* of the complex Y.

The main result of this section is that normal invariants exist. From the definition it is clear that the problem is one of reducing the structure monoid of a spherical fibration from G to *PL* or *Top*. In order to clarify this reduction we collect the technical tools into a number of preliminary propositions.

Let $\Omega_*^{\mathrm{Poin}}()$ denote the bordism theory defined by Poincaré complexes. An element of $\Omega_*^{\mathrm{Poin}}(X)$ is an equivalence class of pairs $[Y,f], f: Y \to X$ with Y a Poincaré complex. $[Y_1, f_1] \sim [Y_2, f_2]$ if there exists a Poincaré pair $(Z, Y_1 \cup Y_2)$ containing Y_1 and Y_2 as disjoint subcomplexes, and a map $F: Z \to X$ such that $F|Y_i = f_i : Y_i \to X$ $(i = 1, 2)$. As usual the group operation is given by disjoint union. Although $\Omega_*^{\mathrm{Poin}}()$ is not a homology theory, there is a generalised Hurewicz homomorphism

$$h' : \pi_n X \to \Omega_n^{\mathrm{Poin}}(X),$$

which interprets a homotopy class as a bordism class. The following proposition is due essentially to Sullivan [9, Theorem 7].

3.2. Proposition. *The generalised Hurewicz homomorphisms*

$$\text{(i)}\quad \pi_{2n}(G/PL) \to \Omega_{2n}^{\mathrm{Poin}}(G/PL),$$
$$\text{(ii)}\quad \pi_{2n}(G/Top) \to \Omega_{2n}^{\mathrm{Poin}}(G/Top)$$

are injective.

Proof. We give the proof of (i), that of (ii) is similar, given that the homotopy structures of G/PL and G/Top coincide except in dimension 4. To be precise δSq^2 defines a single non-vanishing k-invariant for the mod 2 homotopy type of G/PL, which does not occur for G/Top. In order to fix the ideas, suppose $n = 4k$, and consider the commutative diagram

$$\begin{tikzcd}
\mathbf{Z} \cong \pi_{4k}(G/PL) \arrow[rr, "h'"] \arrow[d, "\pi(\beta)"] & & \Omega_{4k}^{\mathrm{Poin}}(G/PL) \arrow[d, "\Omega(\beta)"] \\
\pi_{4k}(K(\mathbf{Z}_{(2)}, 4k)) \arrow[dr, "h"'] & & \Omega_{4k}^{\mathrm{Poin}}(K(\mathbf{Z}_{(2)}, 4k)) \arrow[dl, "S"] \\
& H_{4k}(K(\mathbf{Z}_{(2)}, 4k)) \arrow[d, equal] & \\
& \mathbf{Z}_{(2)} &
\end{tikzcd}$$

h is the usual Hurewicz isomorphism, and S is the Steenrod representation map, which sends the pair $[Y,f]$ to the homology class $f_*[Y]$. β is the composition

$$G/PL \xrightarrow[P_{(2)}]{} G/PL_{(2)} \xrightarrow[\text{proj.}]{} K(\mathbf{Z}_{(2)}, 4k),$$

where $G/PL_{(2)}$ is G/PL localised at 2, see [9, Theorem 4]. The left hand side of the diagram defines a monomorphism

$$\boldsymbol{Z} \to \boldsymbol{Z}_{(2)} \begin{cases} 1 \to 2 & k = 1 \\ 1 \to 1 & \text{otherwise.} \end{cases}$$

Therefore h is also a monomorphism.

The argument in dimensions $4k+2$ follows the same lines with Z_2 replacing $Z_{(2)}$.

Write $\bar{Y}$ for Y with an open $4n-1$ cell removed, and $\bar{v}$ for $v|\bar{Y}$.

3.3. Corollary. *If the monoid of $\bar{v}$ reduces from G to PL (or Top), so does the monoid of v. Furthermore all extensions of the factorised classifying map are homotopic.*

Proof. The obstruction to reducing the structure monoid over the last cell of Y determines an element in $\pi_{4n-2}(G/PL)$, which bounds in $\Omega^{\mathrm{Poin}}_{4n-2}(G/PL)$. By the preceding proposition the obstruction vanishes. Furthermore the obstruction to a homotopy between two different extensions lies in $H^{4n-1}(Y, \pi_{4n-1}) = 0$.

3.4. Proposition. *Let π be a cyclic, generalised quaternion, or soluble generalised binary polyhedral group.* (i) *If $\pi \cong Z_v$, $\mathbf{Q}(2^v)$, $\mathbf{T}^*_v$ or $\mathbf{O}^*_1$ the monoid of v reduces from G to PL (or Top).*

(ii) *If $\pi \cong \mathbf{O}^*_v$, $v \geqq 2$, the same is true, provided $n = 2k$.*

Proof. Without loss of generality, suppose that Y has only one $(4n-1)$-cell, so that $\bar{Y} = Y^{4n-2}$, the codimension 1 skeleton of Y. Because of the dimensional restriction in (ii) there exists a finite $(\pi, 4n)$-polarisation which is a manifold M^{4n-1}. M^{4n-1} is the quotient of S^{4n-1} by some fixed point free orthogonal representation. By induction over the cells one can construct maps $Y \underset{g}{\overset{f}{\rightleftarrows}} M$, which are inverse homotopy equivalences in codimension 1. This construction hinges on the fact that

$$\pi_i Y \cong \pi_i M = 0, \quad 2 \leqq i \leqq 4n-2.$$

Since M is a manifold v_M and $\bar{v}_M$ have structural group PL (or Top). Both in the absolute and relative cases the Spivak fibration is unique up to fibre homotopy equivalence [8, Corollary 3.4], and $f^*\bar{v}_M$ is reducible [8, Proposition 2.1]. It follows that the monoid of $\bar{v}_Y$ can be reduced to PL (or Top), and an application of 3.3 shows the same to be true for v_Y.

In the proof of the main theorem 3.3 implies the triviality of the top dimensional obstruction. The triviality of the remaining obstructions depends on the following Lemma, due to E. Thomas and reproved here by an elementary argument suggested by A. Dold.

3.5. PROPOSITION. *Let ξ be a fibration over Y with total space E and 1-connected fibre F, and $f : X \to Y$ a map such that*

(i) *for all k,*

$$H^k(Y; \pi_{k-1} F) \xrightarrow{f^*} H^k(X; \pi_{k-1} F)$$

is injective,

(ii) *for all k,*

$$H^k(Y; \pi_k F) \xrightarrow{f^*} H^k(X; \pi_k F)$$

is surjective, and

(iii) *$f^*\xi$ has a section s.*

Then $s \simeq f^*t$ for some section t of ξ.

PROOF. As usual f is homotopic to an inclusion. By (iii) the subbundle $f^*\xi$ over X admits a section and (i) and (ii) ensure that the relative groups $H^k(f; \pi_{k-1} F)$ vanish. Hence there is no obstruction to extending the section s over the whole of Y. This proves the proposition. The ground is now prepared for:

3.6. THEOREM. *If Y^{4n-1} is finitely polarised for the group π, Y^{4n-1} admits at least one PL (or Top) normal invariant, provided in case* IV, *that n is even and in case* VI *that $p \equiv \pm 3$ (mod 8).*

PROOF. It is necessary to consider the groups listed in (1.3) type by type. However the argument follows the fixed pattern of comparing Y with a covering space X for which normal invariants are known to exist. For clarity we formulate the

3.7. HYPOTHESIS. *Suppose that π either has odd order or contains a subgroup ρ which satisfies the following properties:*

(a) *$[e : 1]$ is divisible only by powers of 2 and 3,*

(b) *the restriction homomorphism $H^2(\pi; Z_2) \to H^2(\rho, Z_2)$ is an isomorphism, and*

(c) *the covering space X corresponding to the subgroup ρ admits normal invariants.*

The existence of a normal invariant for Y is equivalent to the existence of a cross section for the fibration ξ associated to the Spivak fibration ν with fibre G/PL. The obstructions to such a cross-section lie in $H^k(Y, \pi_{k-1}(G/PL))$ and by 3.3 we may neglect the top dimension. Furthermore, Y is the $(4n-1)$-skeleton of a $K(\pi, 1)$ and so the lower dimensional obstructions may be computed in the cohomology of π. A consequence of periodicity is that $\hat{H}^{\text{odd}}(\pi; \boldsymbol{Z}) = 0$; this can be deduced for example from [2, XII 10.1]. It follows that the only possible non-vanishing obstructions belong to the groups

$$\hat{H}^{4k-1}(\pi; \mathbf{Z}_2), \quad 1 \leqq k < n.$$

The theorem now follows immediately if π has odd order, and in general we see that it is now sufficient to show that the associated fibration with fibre $G/PL_{(2)}$ has a section. Let the map $f: X \to Y$ of 3.5 be the covering map associated with the inclusion of ρ in π. We check that conditions (i)–(iii) are satisfied. Given (c) condition (iii) is immediate.

Condition (ii) concerns two maps

$$f^* : \hat{H}^{4k}(\pi; \mathrm{Z}) \to \hat{H}^{4k}(\rho; \mathrm{Z})$$

and

$$f^* : \hat{H}^{4k+2}(\pi; \mathrm{Z}_2) \to \hat{H}^{4k+2}(\rho; \mathrm{Z}_2).$$

For any prime p the p-periodicity isomorphism is defined by 'cupping' with powers of $g_p \in \hat{H}^q(\pi; Z)$ of order the highest power of p dividing $[\pi : 1]$. The restriction of g_p to ρ has the same property. Now since by 1.4 the 2- and 3-periods of π and divide 4, and products are natural with respect to restriction, by (a) it is enough to consider

$$f^* : \hat{H}^0(\pi; \mathrm{Z}) \to \hat{H}^0(\rho; \mathrm{Z}) \quad \text{and} \quad f^* : \hat{H}^2(\pi; \mathrm{Z}_2) \to \hat{H}^2(\rho; \mathrm{Z}_2).$$

Since $\hat{H}^0(\sigma; Z) = Z/[\sigma\colon 1]\,Z$ for any subgroup σ of π, the first is surjective, (b) states the second is bijective. Condition (i) follows from the same reasoning applied to $\hat{H}^{-1}(\sigma; Z_2)$, which equals $\mathrm{Ker}(Z_2 \xrightarrow{[\sigma:1]} Z_2)$. To complete the proof of the theorem it remains to check 3.7 case by case. We observe that since ρ contains a 2-Sylow subgroup, $f^* : H^*(\pi; Z_2) \to H^*(\rho; Z_2)$ is always injective [2, XII, 10.1], and hence that for (b) it is enough to check the order of $H^*(\ ; Z_2)$. Also

$$H^2(\ ; Z_2) = \hat{H}^2(\ ; Z_2) \cong \hat{H}^{-2}(\ ; Z_2) = H_1(\ ; Z_2).$$

If π is of Type I in 1.3, let ρ be a 2-Sylow subgroup π_2. Either m_1 and m_2 are both odd, or m_1 is odd and m_2 even. In both cases 3.7 is satisfied, since π_2 is either cyclic or zero, and (c) follows from 3.4. Similar numerical restrictions dispose of Type II, for which π_2 is generalised quaternion. If π is of Type III (IV), π contains a generalised binary tetrahedral subgroup $\mathbf{T}_v^*$ (octahedral subgroup $\mathbf{O}_v^*$), which contains some π_2 and plays the part of ρ. It remains only to check part (b) of the hypothesis. This is a consequence of the explicit computation of the commutator quotients,

$$\mathbf{T}_v^*/[\mathbf{T}_v^*, \mathbf{T}_v^*] \cong \mathrm{Z}_3 \quad \text{and} \quad \mathbf{O}_v^*/[\mathbf{O}_v^*, \mathbf{O}_v^*] \cong \mathrm{Z}_2.$$

Before considering the non-soluble Types V and VI, it will be necessary to digress on the structure of $SL(2, p)$, the group of 2×2 matrices of determinant 1 with entries from the field $\boldsymbol{F}_p$. The unique element (-1_2) of order 2 in $SL(2, p)$ generates the centre; indeed $SL(2, p)$ is the central extension of Z_2 by the *linear fractional group* $SF(2, p)$. (Both groups are given in terms of explicit generators and relations in [3, Chaps. XII & XIII], where there is also a long discussion of the subgroup structure of $SF(2,p)$.) In particular, whatever the value of p, $SF(2,p)$ contains at least one subgroup isomorphic to the tetrahedral group $\mathbf{T}_1$ of order 12. In the central extension of Z_2, $\mathbf{T}_1$ lifts to the binary tetrahedral group $\mathbf{T}_1^*$,

$$
\begin{tikzcd}
O \arrow[r] & \boldsymbol{Z}_2 \arrow[r] & SL(2,p) \arrow[r] & SF(2,p) \arrow[r] & 1 \\
O \arrow[r] & \boldsymbol{Z}_2 \arrow[r] \arrow[u, equal] & \mathbf{T}_1{}^* \arrow[r] \arrow[u] & \mathbf{T}_1 \arrow[r] \arrow[u] & 1
\end{tikzcd}
\tag{3.8}
$$

If π is of type V, hypothesis 3.7 is satisfied with $\mathbf{T}_1^*$ playing the part of the subgroup ρ. A direct check shows that $H_1(\pi; \boldsymbol{Z}_2)$ is trivial, hence that (b) is satisfied. Finally, if π is of Type VI, replace $\mathbf{T}_1^*$ by $\mathbf{O}_1^* = \{\mathbf{T}_1^*, S\}$ and check that $H_1(\pi; Z_2) \cong \boldsymbol{Z}_2$. Note that $\mathbf{O}_1^*$ contains a Sylow 2-subgroup if $p \equiv \pm 3$ (8); otherwise, (b) fails.

The proof of the theorem is now complete.

Because we are primarily interested in the normal invariant as an aid to framed surgery, we have stated the preceding theorem in terms of finite polarisations. However since neither the finiteness nor the torsion obstruction enters the proof, the theorem holds for infinite polarised complexes.

The question of the existence of smooth normal invariants is much harder. Using K-theory it is possible to show that, for groups of Type I and at least one k-invariant, Y admits smooth normal invariants. This argument is outlined in [1, p. 83]. Observe however that 3.2 is unknown in the smooth case, so that we cannot pass immediately from one homotopy type to another.

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