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LOCAL CENTRAL LIMIT THEOREM FOR FIRST ENTRANCE OF A RANDOM WALK INTO A HALF SPACE

by

A. J. Stam

1. Introduction, notations

Throughout this paper the following assumptions apply. Let $\bar{X}_k = (X_{k1}, \dots, X_{kd})$, $k = 1, 2, \dots$, be independent strictly d -dimensional random vectors with common probability distribution F and characteristic function φ . (The bar distinguishes vectors from scalars and strict d -dimensionality means that the support of F is not contained in a hyperplane of dimension lower than d .) The second moments of the \bar{X}_i will be finite and the first moment vector $\bar{\mu}$ nonzero. We put $\bar{S}_n = \bar{X}_1 + \dots + \bar{X}_n$, $n = 1, 2, \dots$,

$$(1.1) \quad U(A) = \sum_{m=1}^{\infty} F^m(A),$$

where the exponent denotes convolution. The distribution function of X_{11} is F_1 .

We consider the first entrance of the random walk $\{\bar{S}_n\}$ into the half space $\{\bar{x} : a_1 x_1 + \dots + a_d x_d \geq t\}$, where $t > 0$. It is essential that the half line $\bar{x} = c\bar{\mu}$, $c > 0$, intersects the boundary of the half space. For convenience of notation we assume that the x_1 -axis of our coordinate system has been chosen in the direction of \bar{a} . This implies that we have to assume throughout this paper

$$(1.2) \quad \mu_1 > 0.$$

Now let $N(t) = \min \{n : S_{n1} \geq t\}$, and let R_t be the joint probability distribution of

$$Z_1(t) - t, Z_2(t), \dots, Z_d(t),$$

where $\bar{Z}(t) = \bar{S}_{N(t)}$. It will be shown in section 3 that R_t for $t \rightarrow \infty$ satisfies a local central limit theorem, if either F is nonarithmetic – i.e. $\{\bar{u} : \varphi(\bar{u}) = 1\} = \{0\}$ – or X_{1k} is arithmetic with span 1, $k = 1, \dots, d$. The approximating probability measure is the product of the well known limiting distribution of $Z_1(t) - t$ and a normal distribution for $Z_2(t), \dots, Z_d(t)$. The corresponding ‘marginal’ result for $Z_2(t), \dots, Z_d(t)$ also is derived.

We will need the strict ascending ladder process with respect to the x_1 -coordinate, i.e. the random walk $\bar{S}_{n_1}, \bar{S}_{n_2}, \dots$ in R_d , where n_1, n_2, \dots are the times at which a strict ascending ladder point occurs in the random walk $S_{11}, S_{21}, S_{31}, \dots$. We put

$$(1.3) \quad \bar{Y} = \bar{S}_{n_1}.$$

By Wald's identity for expectations we have, since $E\{n_1\} < \infty$ by (1.2),

$$(1.4) \quad \bar{v} \stackrel{\text{df}}{=} E\{\bar{Y}\} = \bar{\mu}E\{n_1\}.$$

By H_1 we denote the probability distribution of Y_1 .

Let E denote the covariance matrix of the random variables $X_{1j} - \mu_1^{-1}\mu_j X_{11}$, $j = 2, \dots, d$ and ε_{ij} the (i, j) -element of E^{-1} . We put

$$(1.5) \quad \begin{aligned} Z(x_1, \dots, x_d) \\ = \exp \left[-\frac{1}{2}\mu_1 x_1^{-1} \sum_{i=2}^d \sum_{j=2}^d \varepsilon_{ij}(x_i - \mu_1^{-1}\mu_i x_1)(x_j - \mu_1^{-1}\mu_j x_1) \right], \end{aligned}$$

$$(1.6) \quad L(x_1, \dots, x_d) = \mu_1^{-1}(2\pi)^{-\rho}(\text{Det } E)^{-\frac{1}{2}}Z(x_1, \dots, x_d),$$

where

$$(1.7) \quad \rho = \frac{1}{2}(d-1).$$

If x_1 is kept fixed, $\mu_1^{\rho+1}x_1^{-\rho}L(x_1, x_2, \dots, x_d)$ considered as a function of x_2, \dots, x_d , is a $(d-1)$ -dimensional normal probability density. By C_d we denote the class of continuous functions on R_d with compact support. The indicator function of a set A is written I_A .

Proofs are based on the results obtained in Stam [1].

2. Preliminary lemmas

LEMMA 2.1. *If F is nonarithmetic and $E|X_{11}|^\rho < \infty$, then for $g \in C_d$*

$$(2.1) \quad \lim_{x_1 \rightarrow \infty} \left\{ x_1^\rho \int g(\bar{z} - \bar{x})U(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int g(\bar{z})d\bar{z} \right\} = 0,$$

uniformly in x_2, \dots, x_d .

This is theorem 3.1 of Stam [1], II. We also need theorem 3.2 of the same paper:

LEMMA 2.2. *If there is a Cartesian coordinate system such that the components of \bar{X}_1 in this system are arithmetic with span 1 and their joint characteristic function ζ satisfies the condition: $\zeta(\bar{u}) = 1$ if u_1, \dots, u_d are integer multiples of 2π and $|\zeta(\bar{u})| < 1$ elsewhere and if $E|X_{11}|^\rho < \infty$, then*

$$\lim_{x_1 \rightarrow \infty} \{x_1^\rho U(\{\bar{x}\}) - \mu_1^\rho L(\bar{x})\} = 0,$$

uniformly in x_2, \dots, x_d , if \bar{x} is restricted to lattice points of U .

LEMMA 2.3. *If F satisfies the conditions of lemma 2.1 and $g(\bar{x}) = I_{[a,b)}(x_1)g_1(\bar{x})$ with $g_1 \in C_d$, then (2.1) holds for g .*

PROOF. We may write $g = h + h_1$ with $h \in C_d$ and $|h_1| \leq h_2 \in C_d$. Then

$$(2.2) \quad \begin{aligned} & \left| x_1^\rho \int g(\bar{z} - \bar{x})U(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int g(\bar{z})d\bar{z} \right| \leq \\ & \left| x_1^\rho \int h(\bar{z} - \bar{x})U(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int h(\bar{z})d\bar{z} \right| + \\ & \left| x_1^\rho \int h_2(\bar{z} - \bar{x})U(d\bar{z}) \right| + \mu_1^\rho L(\bar{x}) \int h_2(\bar{z})d\bar{z}. \end{aligned}$$

Since $L(\bar{x})$ is bounded, we may choose h, h_1 and h_2 so that

$$(2.3) \quad \mu_1^\rho L(\bar{x}) \int h_2(\bar{z})dz < \varepsilon/4.$$

Then

$$(2.4) \quad \begin{aligned} & \left| x_1^\rho \int h_2(\bar{z} - \bar{x})U(d\bar{z}) \right| \leq \mu_1^\rho L(\bar{x}) \int h_2(\bar{z})dz \\ & + \left| x_1^\rho \int h_2(\bar{z} - \bar{x})U(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int h_2(\bar{z})d\bar{z} \right| \end{aligned}$$

and the lemma follows from (2.2), (2.3), (2.4) and lemma 2.1.

LEMMA 2.4. *The random variables Y_1, \dots, Y_d of (1.3) have finite second moments. If $\mu_j = 0, j \geq 2$,*

$$(2.6) \quad \text{cov}(Y_j, Y_k) = E\{n_1\} \text{cov}(X_{1j}, X_{1k}), \quad j, k = 2, \dots, d.$$

See theorems 1.2, 1.4, 1.5 of Nevels [2].

LEMMA 2.5. *The covariance matrix of the random variables $Y_j - v_1^{-1}v_j Y_1, j = 2, \dots, d$, is $E\{n_1\} \cdot E$, where E is defined as in section 1.*

PROOF. By (1.4) we have $v_1^{-1}v_j = \mu_1^{-1}\mu_j$. So

$$Y_j - v_1^{-1}v_j Y_1 = \sum_{k=1}^{n_1} W_{kj},$$

where $W_{kj} = X_{kj} - \mu_1^{-1}\mu_j X_{k1}$ has expectation zero. The lemma follows from lemma 2.5 by considering the random walk with steps $(X_{k1}, W_{k2}, \dots, W_{kd})$.

LEMMA 2.6. *If $E|X_{11}|^\lambda < \infty$, where $\lambda > 0$, then $E|Y_1|^\lambda < \infty$.*

PROOF. See Nevels [2], theorem 1.1.

3. Local limit theorems for R_t

THEOREM 3.1. *If F is nonarithmetic and $E|X_{11}|^\rho < \infty$, we have for $g \in C_d$*

$$\lim_{t \rightarrow \infty} t^\rho \left| \int g(x_1, x_2 - a_2, \dots, x_d - a_d) R_t(d\bar{x}) - \int g(x_1, x_2 - a_2, \dots, x_d - a_d) \beta(x_1) q_t(x_2, \dots, x_d) d\bar{x} \right| = 0,$$

uniformly in a_2, \dots, a_d . Here

$$(3.1) \quad \beta(x_1) = 0, \quad x_1 \leq 0, \quad \beta(x_1) = v_1^{-1} \{1 - H_1(x_1)\}, \quad x_1 > 0,$$

and q_t is the $(d-1)$ -dimensional normal density with covariance matrix $\mu_1^{-1} t E$ and means $\mu_1^{-1} \mu_j t, j = 2, \dots, d$.

PROOF. First we assume that $X_{11} \geq 0$ with probability 1. Since $g \in C_d$, it is sufficient to show that

$$(3.2) \quad \lim_{t \rightarrow \infty} \left| t^\rho \int g(x_1, x_2 - a_2, \dots, x_d - a_d) R_t(d\bar{x}) - t^\rho q_t(a_2, \dots, a_d) \int \beta(x_1) g(\bar{x}) d\bar{x} \right| = 0,$$

uniformly in a_2, \dots, a_d . We have

$$(3.3) \quad \begin{aligned} & t^\rho \int g(x_1, x_2 - a_2, \dots, x_d - a_d) R_t(d\bar{x}) \\ &= t^\rho \int I_{[t, \infty)}(x_1) g(x_1 - t, x_2 - a_2, \dots, x_d - a_d) F(d\bar{x}) \\ & \quad + t^\rho \sum_{m=1}^{\infty} \int \int I_{(-\infty, t)}(x_1) I_{[t, \infty)}(x_1 + \xi_1) g(x_1 + \xi_1 - t, x_2 + \xi_2 - a_2, \dots, \\ & \quad \quad \quad x_d + \xi_d - a_d) F^m(d\bar{x}) F(d\bar{\xi}). \end{aligned}$$

Here the first term tends to zero for $t \rightarrow \infty$, uniformly in a_2, \dots, a_d , since $E|X_{11}|^\rho < \infty$. The second term may be written

$$(3.4) \quad T_2 = \int \Lambda(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}),$$

where $\bar{a} = (0, a_2, \dots, a_d)$ and

$$(3.4a) \quad \Lambda(\bar{\xi}, t, \bar{a}) = t^\rho \int I_{[-\xi_1, 0)}(x_1 - t) g(x_1 + \xi_1 - t, x_2 + \xi_2 - a_2, \dots, x_d + \xi_d - a_d) U(d\bar{x}).$$

By lemma 2.3, applied to the function $I_{[-\xi_1, 0)}(x_1) g(\bar{x} + \bar{\xi})$ with $\bar{\xi}$ fixed,

$$(3.5) \quad A(\bar{\xi}, t, \bar{a}) = \eta(\bar{\xi}, t, \bar{a}) + \mu_1^\rho L(t, a_2, \dots, a_d) \int_{I_{[-\xi_1, 0]}(z_1)} g(\bar{z} + \bar{\xi}) d\bar{z},$$

where $\lim_{t \rightarrow \infty} \eta(\bar{\xi}, t, \bar{a}) = 0$, uniformly in a_2, \dots, a_d for fixed $\bar{\xi}$. Equivalently

$$(3.6) \quad \lim_{t \rightarrow \infty} \zeta(\bar{\xi}, t) = 0,$$

for fixed $\bar{\xi}$, where $\zeta(\bar{\xi}, t) = \sup_{\bar{a}} \eta(\bar{\xi}, t, \bar{a})$. We now write

$$(3.7) \quad \begin{aligned} T_2 &= T_3 + T_4, \\ T_3 &= \int_{I_{[\frac{1}{2}t, \infty)}(\xi_1)} A(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}), \\ T_4 &= \int_{I_{[0, \frac{1}{2}t)}(\xi_1)} A(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}). \end{aligned}$$

Since $\int g(\bar{z} - \bar{y}) U(d\bar{z})$ is bounded in \bar{y} , we have by (3.4a) and the assumption that $E|X_{11}|^\rho < \infty$,

$$(3.8) \quad T_3 \leq c_1 t^\rho \{1 - F_1(\frac{1}{2}t)\} \rightarrow 0.$$

To T_4 we now apply (3.5) and (3.6) with the Lebesgue dominated convergence theorem. It is noted that L is bounded by a constant and that

$$t^\rho I_{[0, \frac{1}{2}t)}(\xi_1) \leq 2^\rho (t - \xi_1)^\rho, \quad 0 \leq \xi < \frac{1}{2}t.$$

So (3.4a) and lemma 2.3 show that $I_{[0, \frac{1}{2}t)}(\xi_1) A(\bar{\xi}, t, \bar{a})$ and therefore also $I_{[0, \frac{1}{2}t)}(\xi_1) \zeta(\bar{\xi}, t)$ is bounded by a constant. So

$$(3.9) \quad \lim_{t \rightarrow \infty} \left[T_4 - \int_{I_{[0, \frac{1}{2}t)}(\xi_1)} \gamma(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}) \right] = 0,$$

uniformly in a_2, \dots, a_d , where $\gamma(\bar{\xi}, t, \bar{a})$ is the second term on the right in (3.5). Since $\gamma(\bar{\xi}, t, \bar{a})$ is bounded by a constant, (3.9) implies

$$(3.10) \quad \lim_{t \rightarrow \infty} \left[T_4 - \int \gamma(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}) \right] = 0,$$

uniformly in a_2, \dots, a_d . Now

$$\begin{aligned} \gamma(\bar{\xi}, t, \bar{a}) &= \mu_1^\rho L(t, a_2, \dots, a_d) \int_{I_{[0, \xi_1]}(y_1)} g(\bar{y}) d\bar{y}, \\ \int \gamma(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}) &= \mu_1^\rho L(t, a_2, \dots, a_d) \int \{1 - F_1(y_1)\} g(\bar{y}) d\bar{y}. \end{aligned}$$

So by (1.5) and (1.6)

$$(3.11) \quad \int \gamma(\bar{\xi}, t, \bar{a}) F(d\bar{\xi}) = t^\rho q_t(a_2, \dots, a_d) \int \beta(y_1) g(\bar{y}) d\bar{y},$$

and (3.2) follows from (3.3), (3.4), (3.7), (3.8), (3.10) and (3.11).

If $P\{X_{11} < 0\} > 0$, we apply the part of the theorem proved above, to the random walk arising by sampling the \bar{S}_n -process at the strict ladder times of the process $\{S_{n1}\}$. It is noted that the first entrance of $\{\bar{S}_n\}$ into the half space $\{x_1 \geq t\}$ necessarily is a ladder point of $\{S_{n1}\}$. The theorem now follows by lemma 2.5 and (1.4). Lemma 2.6 guarantees that the condition on the absolute moment of order ρ of the x_1 -component is satisfied.

THEOREM 3.2. *If F is nonarithmetic and $E|X_{11}|^\rho < \infty$, we have for $h \in C_{d-1}$*

$$\lim_{t \rightarrow \infty} t^\rho \left| \int h(x_2 - a_2, \dots, x_d - a_d) R_t(d\bar{x}) - \int h(x_2 - a_2, \dots, x_d - a_d) q_t(x_2, \dots, x_d) dx_2 \cdots dx_d \right| = 0,$$

uniformly in a_2, \dots, a_d . Here q_t is the same as in theorem 3.1.

PROOF. Since $h \in C_{d-1}$, it is sufficient to show that, uniformly in a_2, \dots, a_d ,

$$(3.12) \quad \lim_{t \rightarrow \infty} t^\rho \left| \int h(x_2 - a_2, \dots, x_d - a_d) R_t(d\bar{x}) - q_t(a_2, \dots, a_d) \times \int h(x_2, \dots, x_d) dx_2 \cdots dx_d \right| = 0.$$

First we assume that $X_{11} \geq 0$. We then start the proof of (3.2) anew at (3.3), where for $g(x_1, \dots, x_d)$ we now take $h(x_2, \dots, x_d)$. We obtain (3.4), (3.5), (3.6), since lemma 2.3 applies to the function $I_{[-\xi_1, 0]}(\xi_1) h(x_2 + \xi_2, \dots, x_d + \xi_d)$ with ξ fixed. To obtain (3.8) and (3.9) we have to take into account the factor $I_{[-\xi_1, 0]}(x_1 - t)$ in (3.4a). This means that in the integral in (3.4a) the variable x_1 is restricted to the interval $[t - \xi_1, t)$. We then have in T_3

$$(3.13) \quad A(\bar{\xi}, t, \bar{a}) \leq t^\rho \int_{[0, t)} I_{[0, t)}(x_1) |h(x_2 + \xi_2 - a_2, \dots, x_d + \xi_d - a_d)| U(d\bar{x}).$$

By lemma 2.3, for $m \geq 1$,

$$\int I_{[m, m+1)}(x_1) |h(x_2 + \xi_2 - a_2, \dots, x_d + \xi_d - a_d)| U(d\bar{x}) \leq c_2 m^{-\rho},$$

so

$$(3.14) \quad A(\bar{\xi}, t, \bar{a}) \leq t^\rho \{c_0 + c_2 \sum_{m=1}^{[t+1]} m^{-\rho}\}.$$

Therefore $T_3 \rightarrow 0$, uniformly, since $E|X_{11}|^\rho < \infty$. For $\rho = \frac{1}{2}$ and $\rho = 1$ we have to appeal to the existence of first and second moments. To apply the Lebesgue dominated convergence theorem to T_4 we note that the

second term on the right in (3.5) is bounded by $c_3|\xi_1|$ with c_3 a constant. In the same way as (3.14) we obtain

$$I_{[0, \frac{1}{2}t)}(\xi_1)A(\bar{\xi}, t, \bar{a}) \leq c_4 t^\rho \sum_{[t-\xi_1]^{[t+1]}} m^{-\rho} \leq c_5 |\xi_1|.$$

So $|\zeta(\bar{\xi}, t)| \leq c_6 |\xi_1|$ and (3.9) follows by the existence of first moments. The relation (3.10) also follows and (3.11) is replaced by

$$\int \gamma(\bar{\xi}, t, \bar{a})F(d\bar{\xi}) = t^\rho q_t(a_2, \dots, a_d) \int h(y_2, \dots, y_d) dy_2 \dots dy_d.$$

The relation (3.12) now follows from the counterparts of (3.3), (3.4), (3.7), (3.8), (3.10) and (3.11), if $X_{11} \geq 0$. The proof is concluded in the same way as the proof of theorem 3.1.

THEOREM 3.3. *Let F satisfy the conditions of lemma 2.2. For $t > 0$ let $\bar{a}(t)$ be a d -vector such that $0 \leq a_1(t) \leq K$ and $t + \bar{a}(t)$ belongs to the F -lattice. Then*

$$\lim_{t \rightarrow \infty} t^\rho |R_t\{\bar{a}(t)\} - v_1^{-1} H_1(E_t) q_t(a_2(t), \dots, a_d(t))| = 0,$$

uniformly in $\bar{a}(t)$ for fixed K . Here E_t denotes the open interval $(a_1(t), \infty)$ and q_t the same normal density as in theorem 3.1.

COROLLARY. *If X_{11}, \dots, X_{1d} are integer valued such that $\varphi(\bar{u}) = 1$ if u_1, \dots, u_d are integer multiples of 2π and $|\varphi(\bar{u})| < 1$ elsewhere, and if $E|X_{11}|^\rho < \infty$, then*

$$\lim_{h \rightarrow \infty} h^\rho |R_h(\bar{k}) - v_1^{-1} H_1((k_1, \infty)) q_h(k_2, \dots, k_d)| = 0,$$

uniformly in k_2, \dots, k_d , if h, k_1, \dots, k_d are integers with $h > 0, k_1 \geq 0$.

PROOF. First assume $X_{11} \geq 0$ with probability 1. We have

$$t^\rho R_t\{\bar{a}(t)\} = t^\rho P\{\bar{S}_1 = t + \bar{a}(t)\} + T_2,$$

where the first term is dealt with by the existence of EX_{11}^ρ

$$T_2 = t^\rho \sum_{m=1}^{\infty} P\{S_{m1} < t, \bar{S}_{m+1} = t + \bar{a}(t)\},$$

$$T_2 = t^\rho \sum_{m=1}^{\infty} \sum_{\bar{\xi}} P\{X_{m+1} = \bar{\xi}\} P\{S_{m1} < t, \bar{S}_m = t + \bar{a}(t) - \bar{\xi}\},$$

where $\bar{\xi}$ runs through points of the F -lattice. Because of the second factor we may write

$$T_2 = t^\rho \sum_{\xi_1 > a_1(t)} F(\{\bar{\xi}\}) U(\{t + \bar{a}(t) - \bar{\xi}\}).$$

By lemma 2.2 we have for fixed $\bar{\xi}$

$$t^\rho U\{t + \bar{a}(t) - \bar{\xi}\} = \mu_1^\rho L(t, a_2(t), \dots, a_d(t)) + \eta,$$

where $\eta \rightarrow 0$ as $t \rightarrow \infty$, uniformly in $\bar{a}(t)$ if $0 \leq a_1(t) \leq K$, if $\bar{\xi}$ is kept fixed. The proof now proceeds in the same way as with theorem 3.1. We write $T_2 = T_3 + T_4$ where the sum is taken over the sets $\{\xi_1 \geq \frac{1}{2}t\}$ and $\{a_1(t) < \xi_1 < \frac{1}{2}t\}$, respectively. Handling of T_3 and T_4 requires the same estimations as in the proof of theorem 3.1.

The lattice counterpart of theorem 3.2 is restricted to integer valued X_{11}, \dots, X_{1d} , since under the more general assumptions of theorem 3.3 the lattice description of $Z_2(t), \dots, Z_d(t)$ is difficult.

THEOREM 3.4. *If X_{11}, \dots, X_{1d} are integer valued, such that $\varphi(\bar{u}) = 1$ if u_1, \dots, u_d are integer multiples of 2π and $|\varphi(\bar{u})| < 1$ elsewhere, and if $E|X_{11}|^\rho < \infty$, then*

$$(3.15) \quad \lim_{h \rightarrow \infty} h^\rho |P\{Z_2(h) = k_2, \dots, Z_d(h) = k_d\} - q_h(k_2, \dots, k_d)| = 0,$$

uniformly in k_2, \dots, k_d . Here h, k_2, \dots, k_d are integers and q_t is the same normal density as in theorem 3.1.

PROOF. First take $P\{X_{11} \geq 0\} = 1$. We have

$$\begin{aligned} h^\rho P\{Z_2(h) = k_2, \dots, Z_d(h) = k_d\} \\ = h^\rho P\{X_{11} \geq h, X_{12} = k_2, \dots, X_{1d} = k_d\} + T_2, \end{aligned}$$

where the first term tends to zero uniformly in (k_2, \dots, k_d) as $h \rightarrow \infty$ since $E|X_{11}|^\rho < \infty$ and

$$\begin{aligned} T_2 &= h^\rho \sum_{m=1}^{\infty} P\{S_{m1} < h, S_{m+1,1} \geq h, S_{m+1,r} = k_r, \quad r = 2, \dots, d\} \\ &= h^\rho \sum_{m=1}^{\infty} \sum' \sum'' F^m\{i_1, \dots, i_d\} F\{j_1, \dots, j_d\}, \end{aligned}$$

where \sum' and \sum'' are subject to the restrictions $i_1 < h$, $i_1 + j_1 \geq h$, $i_r + j_r = k_r$, $r = 2, \dots, d$. So

$$(3.16) \quad T_2 = h^\rho \sum_{j_1, \dots, j_d} F\{j_1, \dots, j_d\} \sum_{i_1=h-j_1}^{h-1} U\{i_1, k_2 - j_2, \dots, k_d - j_d\}.$$

By lemma 2.2 we have for fixed j_1, \dots, j_d and $h - j_1 \leq i_1 < h - 1$

$$U\{i_1, k_2 - j_2, \dots, k_d - j_d\} = \mu_1^\rho L(h, k_2, \dots, k_d) + \eta,$$

with $\lim_{h \rightarrow \infty} \eta = 0$, uniformly in k_2, \dots, k_d .

The relation (3.15) now follows with (1.5) and (1.6) if passing to the limit in (3.16) under the sum over j_1, \dots, j_d is justified. This is done by the same methods as in the proof of theorem 3.2.

If $P\{X_{11} < 0\} > 0$ we consider the random walk at the ladder times of the process $\{S_{n1}\}$.

Summary

Let $\bar{X}_1, \bar{X}_2, \dots$ be independent strictly d -dimensional random vectors, with common distribution, with finite second moments and positive x_1 -component of the first-moment vector. Let $\bar{S}_n = \bar{X}_1 + \dots + \bar{X}_n$, $n = 1, 2, \dots$, $N(t) = \min\{n: S_{n1} \geq t\}$ and $\bar{Z}(t) = \bar{S}_{N(t)}$.

If $E|X_{11}|^\rho < \infty$, where $\rho = \frac{1}{2}(d-1)$, the joint distribution of $Z_1(t) - t$, $Z_2(t), \dots, Z_d(t)$ satisfies a local central limit theorem for $t \rightarrow \infty$. The approximating probability measure is the product of the well known limiting distribution for $Z_1(t) - t$ and a normal distribution for $Z_2(t), \dots, Z_d(t)$. The difference is $o(t^{-\rho})$ as in a local central limit theorem for sums of independent $(d-1)$ -vectors.

The theorem is stated and proved for nonarithmetic F and for F restricted to a (rotated) cubic lattice with span 1. A special case of the global version was proved by the author in Zeitschr. für Wahrsch. th. u. verw. Geb. 10 (1968), 81–86.

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