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RENEWAL THEORY IN r DIMENSIONS II

by

A. J. Stam

1. Introduction

This paper is a direct continuation of Stam [6], which will be cited as I. The notation and definitions of I will be taken over without reference. The same holds for the assumptions of I, section 1: strict d-dimensionality, finite second moments and nonzero first moment vector.

We now assume

$$\mu_1 > 0. \tag{1.1}$$

The restriction of U_F to the strip $\{\bar{x} : t \leqq x \ \leqq t+a\}$ is a finite measure with variation tending to $\mu_1^{-1}a$ as $t \to \infty$ if X_{11} is nonarithmetic. It will be shown that this measure satisfies a local central limit theorem for $t \to \infty$, if $E|X_{11}|^\rho < \infty$ and F is nonarithmetic. The limit theorem (theorem 3.1) has the usual form applying to the n-fold convolution of a probability measure, with n replaced by $\mu_1^{-1}t$. See e.g. Spitzer [4], Ch. II.7 and Stone [8]. For arithmetic F a similar result holds (theorem 3.2).

We might have considered any strip $\{\bar{x} : t \leqq (\bar{c}, \bar{x}) \leqq t+a\}$ with the unit vector $\bar{c}$ such that $(\bar{\mu}, \bar{c}) > 0$. What is done here is choosing a coordinate system with positive x_1-axis in the direction of $\bar{c}$.

The global version of the limit theorem, with $\mu_2 = \cdots = \mu_d = 0$, was proved in Stam [5]. Theorems 5.3 and 5.4 of I are special cases of the local theorems, viz. $\mu_2 = \cdots = \mu_d = 0$, $x_2 = \cdots = x_d = 0$.

Proofs follow the same lines as in I, with the complication that limits for $x_1 \to \infty$ have to be uniform with respect to $x_2, \cdots, x_d$.

Section 4 contains some results on the order of decrease of $U_F(A+\bar{x})$ as $|\bar{x}| \to \infty$ if certain moments of F exist.

The following notation is used throughout this paper. Let E be the covariance matrix of the random variables $X_{1j} - \mu_1^{-1}\mu_j X_{11}$, $j = 2, \cdots, d$, and ε_{ih} the (i, h)-element of E^{-1}. We put

$$Z(\bar{x}) = \exp\left[-\tfrac{1}{2}\mu_1 x_1^{-1} \sum_{i=2}^{d} \sum_{j=2}^{d} \varepsilon_{ij}(x_i - \mu_1^{-1}\mu_i x_1)(x_j - \mu_1^{-1}\mu_j x_1)\right], \tag{1.2}$$

$$L(\bar{x}) = \mu_1^{-1}(2\pi)^{-\rho}(\text{Det } E)^{-\frac{1}{2}} Z(\bar{x}), \tag{1.3}$$

so that $\mu_1^{\rho+1} x_1^{-\rho} L(\bar{x})$ for fixed $x_1 > 0$ is a gaussian probability density on R_{d-1}.

By $\mathscr{C}_d$ we denote the class of continuous functions on R_d with compact support.

2. Preliminary lemmas

LEMMA 2.1. *If for every $g \in K_d$*

$$\lim_{|\bar{x}|\to\infty} \int g(\bar{y}-\bar{x})\{W_G(d\bar{y}) - W_H(d\bar{y})\} = 0, \tag{2.1}$$

uniformly in the direction of $\bar{x}$, then the same is true for every $g \in \mathscr{C}_d$. The class K_d is defined in I, definition 2.3.

PROOF. It is sufficient to show that to any $g \in \mathscr{C}_d$ and any $\varepsilon > 0$ there is $g_\varepsilon \in K_d$ with

$$\int |g(\bar{y}-\bar{x}) - g_\varepsilon(\bar{y}-\bar{x})| W(d\bar{y}) < \tfrac{1}{2}\varepsilon, \tag{2.2}$$

uniformly in $\bar{x}$, where $W = W_G + W_H$. The relation (2.1) then follows by the inequality

$$\left| \int g(\bar{y}-\bar{x})\{W_G(d\bar{y}) - W_H(d\bar{y})\} \right| \leq$$
$$\left| \int g_\varepsilon(\bar{y}-\bar{x})\{W_G(d\bar{y}) - W_H(d\bar{y})\} \right| + \int |g(\bar{y}-\bar{x}) - g_\varepsilon(\bar{y}-\bar{x})| W(d\bar{y}).$$

To prove (2.2) we take a probability density $h \in K_d$ and put

$$h_a(\bar{z}) = ah(a\bar{z}), \qquad a > 0, \tag{2.3}$$

$$g_a(\bar{z}) = \int g(\bar{z}-\bar{x}) h_a(\bar{x}) d\bar{x}. \tag{2.4}$$

Then $h_a \in K_d$ and $g_a \in K_d$. We have

$$\int |g(\bar{y}-\bar{x}) - g_a(\bar{y}-\bar{x})| W(d\bar{y}) \leq \iint |g(\bar{y}-\bar{x}) - g(\bar{y}-\bar{x}-\bar{t})| W(d\bar{y}) h_a(\bar{t}) d\bar{t}.$$

Since $g \in \mathscr{C}_d$, we have by I, lemma 2.4

$$\begin{aligned} \int |g(\bar{y}-\bar{x}) - g_a(\bar{y}-\bar{x})| W(d\bar{y}) &\leq C_1 \int_{|\bar{t}|\geq\delta} h_a(\bar{t}) d\bar{t} \\ &+ \int_{|\bar{t}|\leq\delta} \int_{D+\bar{x}} |g(\bar{y}-\bar{x}) - g(\bar{y}-\bar{x}-\bar{t})| W(d\bar{y}) h_a(\bar{t}) d\bar{t}, \end{aligned} \tag{2.5}$$

where $0 < \delta < 1$ and the bounded set D is taken so that $g(\bar{z}) = 0$, $g(\bar{z}-\bar{t}) = 0$ for $\bar{z} \notin D$ and all $\bar{t}$ with $|\bar{t}| \leq 1$. Since g is uniformly continuous, we first may take δ so small that the second term on the right in (2.5) is smaller than $\frac{1}{4}\varepsilon$, and then a so large that by (2.3) the first term is smaller than $\frac{1}{4}\varepsilon$.

LEMMA 2.2. *If F is gaussian, the density w_F of W_F satisfies*

$$\lim_{x_1\to\infty} |w_F(\bar{x})-L(\bar{x})| = 0, \tag{2.6}$$

uniformly in $x_2, \cdots, x_d$.

COROLLARY. *Under the conditions of lemma 2.2*

$$\lim_{x_1\to\infty} \left\{\int g(\bar{z}-\bar{x})W_F(d\bar{z})-L(\bar{x})\int g(\bar{z})d\bar{z}\right\} = 0, \tag{2.7}$$

uniformly in $x_2, \cdots, x_d$, if $g \in \mathscr{C}_d$.

PROOF. By I, lemma 2.2, it is sufficient that (2.6) holds uniformly in a cone $C_\theta = \{\bar{x} : x_1 \geqq 0, |x_j-\mu_1^{-1}\mu_j x_1| \leqq \theta\, x_1, j \geqq 2\}$. Let

$$Y_{m1} = S_{m1}, \; Y_{mk} = S_{mk}-\mu_1^{-1}\mu_k S_{m1}, \; k = 2, \cdots, d,$$

with $\bar{S}_m = \bar{X}_1 + \cdots + \bar{X}_m$. Then the density f_m of F^m and the joint density q_m of $Y_{m1}, \cdots, Y_{md}$ are connected by

$$f_m(\bar{x}) = q_m(x_1, x_2-\mu_1^{-1}\mu_2 x_1, \cdots, x_d-\mu_1^{-1}\mu_d x_1). \tag{2.8}$$

Let P be the covariance matrix of $Y_{11}, \cdots, Y_{1d}$, and π_{ij} be the (i,j)-element of P^{-1}. Put

$$\eta = \pi_{11}^{-1}\sum_{j=2}^{d}\pi_{ij}(x_j-\mu_1^{-1}\mu_j x_1). \tag{2.9}$$

Since $E\{Y_{m1}\} = m\mu_1$, $E\{Y_{mk}\} = 0$, $k \geqq 2$, the relation (2.8) gives

$$f_m(\bar{x}) = (2\pi m)^{-\frac{1}{2}d}(\text{Det}\, P)^{-\frac{1}{2}} \exp\left[-\frac{\pi_{11}}{2m}(x_1-m\mu_1+\eta)^2\right]$$
$$\cdot \exp\left[\frac{\pi_{11}}{2m}\eta^2-\frac{1}{2m}\sum_{i=2}^{d}\sum_{j=2}^{d}\pi_{ij}(x_i-\mu_1^{-1}\mu_i x_1)(x_j-\mu_1^{-1}\mu_j x_1)\right],$$
$$f_m(\bar{x}) = (2\pi m)^{-\frac{1}{2}d}(\text{Det}\, P)^{-\frac{1}{2}}Z_m(\bar{x}) \exp\left[-\frac{\pi_{11}}{2m}(x_1-m\mu_1+\eta)^2\right]$$

where

$$Z_m(\bar{x}) = \exp\left[-\frac{1}{2m}\sum_{i=2}^{d}\sum_{j=2}^{d}\varepsilon_{ij}(x_i-\mu_1^{-1}\mu_i x_1)(x_j-\mu_1^{-1}\mu_j x_1)\right]. \tag{2.10}$$

Since π_{11} Det P = Det E,

$$f_m(\bar{x}) = (2\pi m)^{-\rho}(\text{Det}\, E)^{-\frac{1}{2}}p^{(m)}(x_1+\eta)Z_m(\bar{x}),$$

where $p^{(m)}$ is the m-fold convolution of the normal density with mean μ_1 and variance π_{11}^{-1}. It is noted that $\pi_{11} > 0$ since P is nonsingular. So

$$w_F(\bar{x}) = (2\pi)^{-\rho}(\text{Det } E)^{-\frac{1}{2}} \sum_{m=1}^{\infty} p^{(m)}(x_1+\eta)Z_m(\bar{x}). \tag{2.11}$$

In defining the cone C_θ we take θ so small that

$$|\eta| \leqq \tfrac{1}{2}x_1, \qquad \bar{x} \in C_\theta. \tag{2.12}$$

We divide C_θ into $C_\theta R_A$ and $C_\theta R_A^c$ with

$$R_A = \{\bar{x} : A^2|x_1| \leqq \sum_{j=2}^{d} (x_j - \mu_1^{-1}\mu_j x_1)^2\}. \tag{2.12a}$$

Put $\lambda = (2\pi)^{-\rho}(\text{Det } E)^{-\frac{1}{2}}$. Since E is nonsingular we have for $\bar{x} \in C_\theta R_A$

$$\begin{aligned} w_F(\bar{x}) &\leqq \lambda \sum_{m=1}^{\infty} p^{(m)}(x_1+\eta) \exp(-c_1 A^2 x_1 m^{-1}) \\ &\leqq \lambda \sum_{m=1}^{\infty} p^{(m)}(x_1+\eta) \exp\{-\tfrac{2}{3}c_1 A^2 m^{-1}(x_1+\eta)\}, \end{aligned} \tag{2.13}$$

$$L(x) \leqq \mu_1^{-1}\lambda \exp(-c_1\mu_1 A^2), \tag{2.14}$$

with $c_1 > 0$. Moreover, by the inequality $|\exp(-\alpha)-\exp(-\beta)| \leqq |\alpha-\beta|$ $\alpha \geqq 0$, $\beta \geqq 0$, we have for $\bar{x} \in C_\theta R_A^c$

$$\begin{aligned} |w_F(\bar{x})-L(\bar{x})| &\leqq \lambda \sum_{m=1}^{\infty} p^{(m)}(x_1+\eta)|Z_m(\bar{x})-Z(\bar{x})| + \lambda|h(x_1+\eta)-\mu_1^{-1}|Z(\bar{x}) \\ &\leqq cA^2 \sum_{m=1}^{\infty} p^{(m)}(x_1+\eta)\left|\frac{x_1}{m}-\mu_1\right| + \lambda|h(x_1+\eta)-\mu_1^{-1}|, \end{aligned} \tag{2.15}$$

where

$$h(x) = \sum_{m=1}^{\infty} p^{(m)}(x).$$

For given $\varepsilon > 0$ by (2.12), (2.13), (2.14) we may take A so large that $|w_F(\bar{x})-L(\bar{x})| < \varepsilon$ for $x_1 \geqq c_3$ and $\bar{x} \in C_\theta R_A$, since

$$\lim_{A\to\infty} \sum_{m=1}^{\infty} p^{(m)}(z) \exp(-\tfrac{2}{3}c_1 A^2 m^{-1} z) = 0,$$

uniformly in z for $z \geqq c_3 > 0$. For this A the right-hand side of (2.15) then tends to zero as $x_1 \to \infty$, uniformly in $C_\theta R_A^c$. For the second term we apply the renewal theorem for densities. The first term is more complicated. It is noted that $|\eta| \leqq c_4|x_1|^{\frac{1}{2}}$, where c_4 depends on A. We may define the family of random variables M_z, $z > 0$, with

$$P\{M_z = m\} = p^{(m)}(z)/h(z), \qquad m = 1, 2, \cdots.$$

Then $z^{-1}M_z \to \mu^{-1}$ in quadratic mean as $z \to \infty$. We refer to Kalma [1], [2]. A similar technique is used in the proof of theorem 5.3 in I. A direct proof proceeds by dividing the sum over m into three parts:

$$\left|\frac{x_1}{m} - \mu_1\right| < \varepsilon, \quad \frac{x_1}{m} - \mu_1 \leqq -\varepsilon \quad \text{and} \quad \frac{x_1}{m} - \mu_1 \geqq \varepsilon.$$

The corollary follows from (2.6) and the fact that

$$\lim_{x_1 \to \infty} \{L(\bar{x}+\bar{z}) - L(\bar{x})\} = 0,$$

uniformly in $x_2, \cdots, x_d$ and uniformly with respect to $\bar{z}$ in bounded sets.

LEMMA 2.3. *Let* $\{x(t, \bar{\tau}), (t, \bar{\tau}) \in E \subset R_k\}$ be a family of positive random variables such that

$$\lim_{t \to \infty} E[\{x(t, \bar{\tau}) - c\}^2] = 0, \tag{2.16}$$

uniformly in $\bar{\tau}$, where c is a positive constant. Then for any θ and any $\varepsilon > 0$

$$\lim_{t \to \infty} P\{|x^\theta(t, \bar{\tau}) - c^\theta| \geqq \varepsilon\} = 0, \tag{2.17}$$

uniformly in $\bar{\tau}$. *If moreover to any* $\delta > 0$ *there are* $K(\delta)$ *and* $T(\delta)$ *with*

$$E[x^\theta(t, \bar{\tau}) I\{x^\theta(t, \bar{\tau}) \geqq K(\delta)\}] < \delta \tag{2.18}$$

for $t \geqq T(\delta)$ *and every* $\bar{\tau}$, *we have*

$$\lim_{t \to \infty} E\{x^\theta(t, \bar{\tau})\} = c^\theta, \tag{2.19}$$

uniformly in $\bar{\tau}$.

REMARK. A sufficient condition for (2.18) is the existence of $s > 1$ with

$$E\{x^{s\theta}(t, \bar{\tau})\} \leqq M < \infty, (t, \bar{\tau}) \in E.$$

See Loève [3], § 11.4.

PROOF. The relation (2.17) follows from (2.16) for $\theta = 1$ by Chebychev's inequality and then for any real θ since

$$\{|x^\theta(t, \bar{\tau}) - c^\theta| \geqq \varepsilon\} \subset \{|x(t, \bar{\tau}) - c| \geqq \eta\},$$

for some positive η independent of t and $\bar{\tau}$.

Now let B be the distribution function of $x^\theta(t, \bar{\tau})$. Then

$$E|x^\theta(t,\bar{\tau})-c^\theta| \leqq \int_{c^\theta-\eta}^{c^\theta+\eta} |x-c^\theta|B(dx)$$
$$+\left\{\int_0^{c^\theta-\eta}+\int_{c^\theta+\eta}^{K}+\int_K^\infty\right\}(xB(dx))+c^\theta P\{|x^\theta(t,\bar{\tau})-c^\theta| \geqq \eta\}$$
$$\leqq \eta+(K+2c^\theta)P\{|x^\theta(t,\bar{\tau})-c^\theta| \geqq \eta\}+\int_K^\infty xB(dx).$$

We now prove (2.19) by first taking $\eta = \varepsilon/3$, then $K = K(\frac{1}{3}\varepsilon)$ as in (2.18) and finally applying (2.17).

LEMMA 2.4. *If F is nonarithmetic and $g \in \mathscr{C}_d$,*

$$\lim_{x_1\to\infty} \left|\int g(\bar{z}-\bar{x})W_F(d\bar{z})-L(\bar{x})\int g(\bar{z})d\bar{z}\right| = 0,$$

uniformly in $x_2, \cdots, x_d$.

PROOF. From lemma 2.2 (corollary), lemma 2.1 and I, theorem 3.2.

LEMMA 2.5. *Let a Cartesian coordinate system exist, such that the components $Z_1, \cdots, Z_d$ of $\bar{X}_1$ in this system have joint characteristic function ζ with $\zeta(\bar{u}) = 1$ if $u_1, \cdots, u_d$ are integer multiples of 2π and $|\zeta(\bar{u})| < 1$ elsewhere. Then*

$$\lim_{x_1\to\infty} \{W_F(\{\bar{x}\})-L(\bar{x})\} = 0,$$

uniformly in $x_2, \cdots, x_d$ if $\bar{x}$ is restricted to lattice points of F.

PROOF. From lemma 2.2 and I, theorem 3.4. The rotation of the F-lattice is a consequence of our choice of coordinates.

LEMMA 2.6. *For fixed nonnegative integer k with $E|X_{11}|^k < \infty$, let*

$$V_F(A) = \sum_{m=1}^\infty m^{\rho-k}F^m(A). \tag{2.20}$$

Then, if F is nonarithmetic, we have for $g \in \mathscr{C}_d$,

$$\lim_{x_1\to\infty} \left\{x_1^k \int g(\bar{z}-\bar{x})V_F(d\bar{z})-\mu_1^k L(\bar{x})\int g(\bar{z})d\bar{z}\right\} = 0, \tag{2.21}$$

uniformly in $x_2, \cdots, x_d$.

PROOF. We will show that

$$\lim_{x_1\to\infty} \left\{\int z_1^k g(\bar{z}-\bar{x})V_F(d\bar{z})-\mu_1 L(x)\int g(\bar{z})d\bar{z}\right\} = 0, \tag{2.22}$$

uniformly in $x_2, \cdots, x_d$. The relation (2.21) then follows by the inequality

$$\left| \int (z_1^k - x_1^k) g(\bar{z}-\bar{x}) V_F(d\bar{z}) \right| \leqq C_1 x_1^{-1} \int z_1^k |g(\bar{z}-\bar{x})| V_F(d\bar{z}),$$

which is a consequence of the fact that $g \in \mathscr{C}_d$.

In the same way as in the proof of I, theorem 5.3.

$$\int z_1^k g(\bar{z}-\bar{x}) V_F(d\bar{z}) = \Phi(\bar{x}) + \int g(\bar{z}-\bar{x}) W_F Q^k(d\bar{z}), \tag{2.23}$$

with $\lim_{|\bar{x}|\to\infty} \Phi(\bar{x}) = 0$, uniformly in the direction of $\bar{x}$, and

$$Q(E) = \int_E x_1 F(d\bar{x}). \tag{2.24}$$

Since Q is a finite signed measure, we may write $Q^k = K' + K''$, where K' is restricted to a bounded set and the variation of K'' is so small that in

$$\begin{aligned}\int g(\bar{z}-\bar{x}) W_F Q^k(d\bar{z}) - \mu_1^k L(\bar{x}) \int g(\bar{z}) d\bar{z} \\ = \left\{ \int g(\bar{z}-\bar{x}) W_F Q^k(d\bar{z}) - \int g(\bar{z}-\bar{x}) W_F K'(d\bar{z}) \right\} \\ + \left\{ \int g(\bar{z}-\bar{x}) W_F K'(d\bar{z}) - L(\bar{x}) K'(R_d) \int g(\bar{z})\, d\bar{z} \right\} \\ + L(\bar{x}) \int g(\bar{z})\, d\bar{z} \{K'(R_d) - \mu_1^k\}\end{aligned}$$

the first and third term on the right are smaller than $\frac{1}{3}\varepsilon$. For the first term we apply I, lemma 2.4. The second term is written

$$\begin{aligned}\int \left\{ \int g(\bar{z}+\bar{\zeta}-\bar{x}) W_F(d\bar{z}) - L(\bar{x}-\bar{\zeta}) \int g(\bar{y})\, d\bar{y} \right\} K'(d\bar{\zeta}) \\ + \int \left\{ L(\bar{x}-\bar{\zeta}) - L(\bar{x}) \right\} K'(d\bar{\zeta}) \cdot \int g(\bar{y})\, d\bar{y}.\end{aligned}$$

Here the first term tends to zero as $x_1 \to \infty$, uniformly in $x_2, \cdots, x_d$, by lemma 2.4, since K' is restricted to a bounded set. The same holds for the second term by (1.3). One should distinguish the sets R_A and R_A^c defined by (2.12a).

3. Local limit theorems for U_F

THEOREM 3.1. *If F is nonarithmetic and $E|X_{11}|^\rho < \infty$,*

$$\lim_{x_1\to\infty} \left\{ x_1^\rho \int g(\bar{z}-\bar{x}) U_F(d\bar{z}) - \mu_1^\rho L(\bar{x}) \int g(\bar{z})\, d\bar{z} \right\} = 0, \tag{3.1}$$

for $g \in \mathscr{C}_d$, uniformly in $x_2, \cdots, x_d$.

PROOF. When d is odd, the theorem coincides with lemma 2.6 for $k = \rho$.

Now assume that d is even, $d \geqq 6$. It is no restriction to assume that $g \geqq 0$. First we intend to show

(I) The relation (3.1) holds uniformly in the set R_A^c, with R_A^c defined by (2.12a).

Putting

$$\alpha = \int g(\bar{z})\, d\bar{z}, \tag{3.2}$$

$$q(m, \bar{x}) = \int g(\bar{z}-\bar{x}) F^m(d\bar{z}), \qquad m = 1, 2, \cdots, \tag{3.3}$$

we have by lemma 2.6 with $k = 0, 1, 2$,

$$\sum_{m=1}^{\infty} m^{\rho} q(m, \bar{x}) = \alpha L(\bar{x}) + \varepsilon_0(\bar{x}), \tag{3.4}$$

$$x_1 \sum_{m=1}^{\infty} m^{\rho-1} q(m, \bar{x}) = \mu_1 \alpha L(\bar{x}) + \varepsilon_1(\bar{x}), \tag{3.5}$$

$$x_1^2 \sum_{m=1}^{\infty} m^{\rho-2} q(m, \bar{x}) = \mu_1^2 \alpha L(\bar{x}) + \varepsilon_2(\bar{x}), \tag{3.6}$$

with

$$\lim_{x_1 \to \infty} \varepsilon_i(\bar{x}) = 0, \qquad i = 0, 1, 2, \tag{3.7}$$

uniformly in $x_2, \cdots, x_d$. Consider the family of positive integer valued random variables $\{M(\bar{x}), \bar{x} \in R_A^c, x_1 > 0\}$:

$$P\{M(\bar{x}) = m\} = \frac{x_1^2 m^{\rho-2} q(m, \bar{x})}{\mu_1^2 \alpha L(\bar{x}) + \varepsilon_2(\bar{x})}, \qquad m = 1, 2, \cdots. \tag{3.8}$$

Expectation with respect to the distribution (3.8) will be denoted by E_1. From (3.4)–(3.7) and the inequality

$$L(\bar{x}) \geqq C_1(A) > 0, \qquad \bar{x} \in R_A^c, \tag{3.9}$$

it follows that

$$\lim_{x_1 \to \infty} E_1[x_1^{-1} M(\bar{x}) - \mu_1^{-1}]^2 = 0, \tag{3.10}$$

uniformly in $x_2, \cdots, x_d$. By lemma 2.3 and (3.9) this implies

$$\lim_{x_1 \to \infty} E_1\{x_1^{\rho-2} M^{2-\rho}(\bar{x})\} = \mu_1^{\rho-2}, \tag{3.11}$$

uniformly in R_A^c – and hence the desired result (I) – if to every $\delta > 0$ there are $J(\delta)$ and $T(\delta)$ with

(3.12) $$x_1^\rho \sum_{m=1}^{[x_1/J(\delta)]} q(m, \bar{x}) < \delta$$

for all $\bar{x} \in R_A^c$ with $x_1 \geqq T(\delta)$. In the same way as I, (5.8), we derive

(3.13) $$\int |z_1|^\rho h(\bar{z}) F^m(d\bar{z}) \leqq m^\rho \int h(\bar{z}) F^{m-1} R(d\bar{z})$$

for $h \geqq 0$, where

(3.14) $$R(E) = \int_E |x_1|^\rho F(d\bar{x}).$$

So, since $g \in \mathscr{C}_d$, it is sufficient for (3.12) that

(3.15) $$\sum_{m=1}^{[x_1/J(\delta)]} m^\rho \int g(\bar{z}-\bar{x}) F^{m-1} R(d\bar{z}) < \delta.$$

The first term in (3.15) tends to zero as $x_1 \to \infty$, uniformly in R_A^c, since R is a finite measure. From (3.10), (2.17), (3.8) and (3.9) we have for $J > \mu_1$,

(3.16) $$\lim_{x_1 \to \infty} E[x_1^{-2} M^2(\bar{x}) I\{x_1^{-1} M(\bar{x}) < J^{-1}\}] = 0,$$
$$\lim_{x_1 \to \infty} \sum_{m=1}^{[x_1/J]} m^\rho \int g(\bar{z}-\bar{x}) F^m(d\bar{z}) = 0,$$

both uniformly in R_A^c. For $J > \mu_1$, and $\bar{x} \in R_A^c$

$$\sum_{m=2}^{[x_1/J]} m^\rho \int g(\bar{z}-\bar{x}) F^{m-1} R(d\bar{z})$$
$$\leqq 2^\rho \int \left\{ \sum_{m=1}^{[x_1/J]} m^\rho \int g(\bar{z}+\bar{\zeta}-\bar{x}) F^m(d\bar{z}) \right\} R(d\bar{\zeta}) \leqq 2^\rho \int \eta(\zeta_1 - x_1) R(d\bar{\zeta}),$$

where η is a bounded funtcion by I, lemma 2.4, and $\lim_{t\to\infty} \eta(t) = 0$ by (3.16). This proves (3.15) and therefore (I).

Now we will prove

(II). To any $\varepsilon > 0$ and $A > 0$ there is $\xi(\varepsilon, A)$ with

(3.17) $$x_1^\rho \int g(\bar{z}-\bar{x}) U_F(d\bar{z}) < \varepsilon + c_1 \exp(-c_0 A^2),$$

for all $\bar{x} \in R_A$ with $x_1 \geqq \xi(\varepsilon, A)$, where R_A is given by (2.12a) and c_0, c_1 do not depend on A or ε.

By (3.13), since $g \in \mathscr{C}_d$,

(3.18) $$x_1^\rho \int g(\bar{z}-\bar{x}) U_F(d\bar{z}) \leqq C_2 \sum_{m=1}^{\infty} m^\rho \int (g(\bar{z}-\bar{x}) F^{m-1} R(d\bar{z})$$
$$\leqq C_2 \int g(\bar{z}-\bar{x}) R(d\bar{z}) + 2^\rho C_2 \int g(\bar{z}-\bar{x}) W_F R(d\bar{z}).$$

Here the first term tends to zero as $x_1 \to \infty$, uniformly in $x_2, \cdots, x_d$ and by lemma 2.4 the second term is majorized by

$$2^\rho C_2 \alpha \int L(\bar{x}-\bar{\zeta})R(d\bar{\zeta})+2^\rho C_2 \int \theta(x_1-\zeta_1)R(d\bar{\zeta}) \tag{3.19}$$

where θ is a bounded function by I, lemma 2.4, and $\lim_{t\to\infty} \theta(t) = 0$. So the second term in (3.19) tends to zero as $x_1 \to \infty$. The inequality (3.17) now follows by considering the first term of (3.19), using the definition of $L(\bar{x})$ and writing $R = R'+R''$ where the measure R' has total variation smaller than $\frac{1}{2}\varepsilon$ and R'' is restricted to a bounded set.

The theorem now follows from (I), (II) and the definition of $L(\bar{x})$.

For $d = 2$ and $d = 4$ the proof of (I) remains unchanged up to and including (3.10). The relation (3.11) now follows from the remark to lemma 2.3, since $0 < 2-\rho < 2$. The proof of (II) holds for $d = 4$ but not for $d = 2$ since (3.13) is derived by Minkowski's inequality with exponent ρ.

For $d = 2$ we have

$$x_1^{\frac{1}{2}} \sum_{m=1}^{[x_1]} \int g(\bar{z}-\bar{x})F^m(d\bar{z}) \leqq x_1 \sum_{m=1}^{[x_1]} m^{-\frac{1}{2}} \int g(\bar{z}-\bar{x})F^m(d\bar{z}),$$

$$x_1^{\frac{1}{2}} \sum_{m=[x_1]+1}^{\infty} \int g(\bar{z}-\bar{x})F^m(d\bar{z}) \leqq \sum_{m=[x_1]+1}^{\infty} m^{\frac{1}{2}} \int g(\bar{z}-\bar{x})F^m(d\bar{z}),$$

so

$$x_1^{\frac{1}{2}} \int g(\bar{z}-\bar{x})U_F(d\bar{z}) \leqq \int g(\bar{z}-\bar{x})W_F(dz)+x_1 \sum_{m=1}^{\infty} m^{-\frac{1}{2}} \int g(\bar{z}-\bar{x})F^m(d\bar{z}),$$

and (II) now follows by lemma 2.4 and lemma 2.6 with $k = 1$.

THEOREM 3.2. *Under the lattice conditions of lemma* 2.5, *if* $E|X_{11}|^\rho < \infty$,

$$\lim_{x_1\to\infty} \{x_1^\rho U_F(\{\bar{x}\})-\mu_1^\rho L(\bar{x})\} = 0, \tag{3.20}$$

uniformly in $x_2, \cdots, x_d$, *if* $\bar{x}$ *is restricted to lattice points of F.*

PROOF. From lemma 2.5, by methods similar to those used in the proof of theorem 3.1. We need the following version of (2.21):

$$\lim_{x_1\to\infty} \{x_1^k V_F(\{\bar{x}\})-\mu_1^k L(\bar{x})\} = 0,$$

uniformly in $x_2, \cdots, x_d$, if $\bar{x}$ is restricted to lattice points of F.

It is noted that a corresponding theorem for densities may be derived by similar techniques, if $\varphi \in L_1$. A similar remark hold for the results of I.

4. Existence of moments of order unequal to ρ

Theorems 5.1 and 5.2 of I give some information about the order of decrease of $U_F(A+\bar{x})$ if $\bar{x} \to \infty$ in a direction different from $\bar{\mu}$. The following supplementary result will be derived by direct appeal to one-dimensional renewal theory. Second moments need not be finite.

THEOREM 4.1. *Let* $S = \{\bar{x} : (\bar{x}, \mu) \leqq \theta|\bar{x}||\bar{\mu}|\}$, *with* $-1 \leqq \theta < 1$. *Then, if* $E|X_{11}|^p < \infty$, *where* $p > 1$, *we have*

$$\int_S (1+|\bar{x}|)^{p-2} U_F(d\bar{x}) < \infty. \tag{4.1}$$

PROOF. It is sufficient to show that (1) holds with S replaced by

$$C = \{\bar{x} : (\bar{x}, \bar{\alpha}) \geqq \gamma|\bar{x}|\},$$

where $\gamma \in (0, 1)$ and the unit vector $\bar{\alpha}$ are such that $\bar{\mu} \notin C$. We may choose our coordinate system in such a way that $\mu_1 > 0$ and

$$C \subset K = \{\bar{x} : x_1 \leqq 0,\ x_2^2 + \cdots + x_d^2 \leqq \beta x_1^2\},$$

where β is a positive constant. By applying the inequality $|\bar{x}| \geqq |x_1|$ if $p < 2$ and $|\bar{x}| \leqq |x_1|\sqrt{1+\beta}$ for $\bar{x} \in K$ if $p \geqq 2$, we find

$$\int_K (1+|\bar{x}|)^{p-2} U_F(d\bar{x}) \leqq \int_{x_1 \leqq 0} (1+c|x_1|)^{p-2} U_F(d\bar{x}),$$

with $c = 1$ or $c = \sqrt{1+\beta}$. Let $U_1 = \sum_1^\infty F_1^m$ be the one-dimensional renewal measure belonging to the probability distribution F_1 of X_{11}. Then

$$\int_{x_1 \leqq 0} (1+c|x_1|)^{p-2} U_F(dx) = \int_{x_1 \leqq 0} (1+c|x_1|)^{p-2} U_1(dx_1) < \infty.$$

(See Stone and Wainger [9], Stam [7].)

Let u_F denote the density of U_F, if present. The density version of theorem 3.1 says that if $E|X_{11}|^\rho < \infty$,

$$\lim_{x_1 \to \infty} x_1^\rho \{\mu_F(\bar{x}) - \mu_1^{-1} q(\bar{x})\} = 0, \tag{4.2}$$

uniformly in $x_2, \cdots, x_d$. Here $q(\bar{x})$ for fixed x_1 is a gaussian probability density in $x_2, \cdots, x_d$ with covariance matrix proportional to x_1. The form of (4.2) suggests that ρ may be replaced by p if $E|X_{11}|^p < \infty$, and that a similar remark might apply to (3.1).

For $p > \rho$ this is not true. As an example take $X_{11}, \cdots, X_{1d}$ independent, X_{11} negative exponential with parameter 1 and X_{1j} gaussian with zero expectation and unit variance, $j = 2, \cdots, d$. For $x_2 = \cdots = x_d = 0$ we then should have

$$\lim_{x\to\infty} x^p \left[\sum_{m=1}^{\infty} \frac{x^{m-1}}{(m-1)!} e^{-x} m^{-\rho} - x^{-\rho}\right] = 0$$

for any $p > 0$. Take $d = 5$. We have

$$\sum_{m=1}^{\infty} \frac{x^{m-1}}{(m-1)!} e^{-x} m^{-2} = \sum_{n=0}^{\infty} \left[\frac{x^n}{(n+2)!} + \frac{x^n}{n!(n+1)^2(n+2)}\right] e^{-x}.$$

Here the first term between brackets gives rise to x^{-2} plus exponential terms and the second term to a contribution of order x^{-3} by the law of large numbers for the Poisson distribution with parameter tending to ∞.

For $p < \rho$ we would obtain $x_1^p \mu_F(\bar{x}) \to 0$ and this is correct for $2 < p < \rho$.

THEOREM 4.2. *If* $E|X_{11}|^p < \infty$, *where* $2 < p < \rho$, *and F has finite second moments, we have for bounded A, uniformly in* $x_2, \cdots, x_d$,

$$\lim_{x_1\to\infty} x_1^p U_F(A+\bar{x}) = 0. \tag{4.3}$$

PROOF. By the boundedness of A and by (3.13) with ρ replaced by p we have

$$x_1^p F^m(A+\bar{x}) \leqq c \int_{A+\bar{x}} z_1^p F^m(d\bar{z}) \leqq cm^p F^{m-1} R(A+\bar{x}),$$

where R is defined by (3.14) with ρ replaced by p. So

$$x_1^p U_F(A+\bar{x}) \leqq cR(A+\bar{x}) + cHR(A+\bar{x}), \tag{4.4}$$

where $H = \sum_1^\infty (m+1)^p F^m$. Since $p < \rho$ we have

$$\lim_{|\bar{y}|\to\infty} H(A+\bar{y}) = 0.$$

(See the proof of (3.8) in I.) Since R is a finite measure, (4.3) follows from (4.4).

Summary

Let $\bar{X}_1, \bar{X}_2, \cdots$ be strictly d-dimensional random vectors with common distribution F, with finite second moments and with $\mu_1 = EX_{11} > 0$. Let $U(A) = \sum_1^\infty F^m(A)$, where F^m is the m-fold convolution of F. The restriction of U to the strip $\{\bar{x} : t \leqq x_1 \leqq t+a\}$ is a finite measure with variation tending to $\mu_1^{-1}a$ if F is nonarithmetic. For $t \to \infty$ this measure satisfies a central limit theorem. The paper derives the local form of this limit theorem. A version of it for purely arithmetic F also is given. The global form was proved by the author in Zeitschrift für Wahrsch. th. u. verw. Geb., 10 (1968), 81–86. The paper is a continuation of Comp. Math. 21 (1969), 383–399.

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