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Compositio Mathematica, tome 23, nº 1 (1971), p. 79-85

<http://www.numdam.org/item?id=CM_1971__23_1_79_0>

COMPOSITIO MATHEMATICA, Vol. 23, Fasc. 1, 1971, pag. 79–85
Wolters-Noordhoff Publishing
Printed in the Netherlands

SEMIRINGS WITH DESCENDING CHAIN CONDITION AND WITHOUT NILPOTENT ELEMENTS

by

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1. Introduction

Several decades ago, Artin, Nesbitt, and Thrall [1] published their classic work on rings with descending chain condition on left ideals. In recent years Herstein [5], Divinsky [3], Kertész [6], Szaśz [8], and others have compiled these and other results in their books or articles. In this paper the author extends to a class of semirings some of these results in ring theory.

2. Definitions

A semiring is a non-empty set R on which two associative binary operations, addition and multiplication, are defined such that the multiplication distributes over the addition from both sides and such that there exists $e \in R$ with $x+e = e+x = x$ and $ex = xe = e$ for all $x \in R$. We call e the *zero* of R and denote it by 0.

A semiring R is *left semisubtractive* if for each x, $y \in R$ there exists $z \in R$ with $z+x = y$ or $x = z+y$.

A *left semi-ideal* of a semiring R is a non-empty subset A of R such that for each $x, y \in A$ and $r \in R$ it is true $x+y$, $rx \in A$. A left semi-ideal A of R is a *left l-ideal* if $x, y+x \in A$ imply $y \in A$ and is a *left r-ideal* if x, $x+y \in A$ imply $y \in A$. Similarly one defines these concepts using 'right' instead of 'left'. A subset that is both a left and right semi-ideal is called a *semi-ideal*. Similarly one defines *l-ideal* and *r-ideal*. If a subset is a left [right] l-ideal and left [right] r-ideal, it is called a *left* [*right*] *ideal*. An *ideal* is a subset that is both a left and right ideal.

A semiring R satisfies the *descending chain condition of left l-ideals* (abbreviated DCC) if for each sequence $R \supseteq L_1 \supseteq L_2 \supseteq \cdots$ of left l-ideals there is a positive integer n such that $L_n = L_{n+1} = L_{n+2} = \cdots$. This is clearly equivalent to the property that each non-empty set of left l-ideals of R contains a minimal member.

The definitions of *nilpotent* and *idempotent* elements of a semiring are the same as in ring theory. The element 0 will be excluded when considering nilpotent or idempotent elements.

The *zeroid* of a semiring R, as introduced by Bourne and Zassenhaus [2], is $\{x \in R | z+x = z \text{ or } x+z = z \text{ for some } z \in R\}$. A semiring R has *right additive cancellation* if $x+z = y+z$ for $x, y, z \in R$ implies $x = y$. It follows that a left semisubtractive semiring with zero as its zeroid has right additive cancellation.

CONVENTION. We will let R denote a left semisubtractive semiring with DCC, with zero as its zeroid, and without nilpotent elements.

3. Preliminary results

PROPOSITION 1. *Each nonzero left l-ideal A of R contains an idempotent x with $A = Rx$.*

PROOF. By DCC, A contains a minimal nonzero left l-ideal B. For each $c \neq 0$ in B, Bc is a nonzero left semi-ideal of R in B. Let Bc^* be the left l-ideal of R generated by Bc (see [7]). Since $Bc^* \subseteq B$, $Bc^* = B$. Hence $c \in Bc^*$ which means $xc = c+yc$ for some $x, y \in B$. By left semisubtractivity, $b+x = y$ or $x = b+y$ fpr some $b \in B$. If $x = b+y$, then $bc = c$. If $b+x = y$, then $0 = c+bc$ and hence $c = c+b(c+bc) = c+bc+b^2c = b^2c$. In either case there exists $e \in B$ such that $c = ec$.

For some $d \in B$, $d+e^2 = e$ or $e^2 = d+e$. If $J = \{x \in B | xc = 0\}$, then J is a left ideal of R, so that $J = (0)$. If $d+e^2 = e$, then $ec+dc = e^2c+dc = ec$ and hence $d \in J$. If $e^2 = d+e$, again $d = 0$. Therefore A contains an idempotent e. We now show A has an idempotent x such that, if $y \in A$ with $yx = 0$, then $y = 0$. For each idempotent $e \in A$, let $M_e = \{x \in A | xe = 0\}$ which is a left ideal of R. Choose idempotent x of A such that M_x is minimal, and suppose $M_x \neq (0)$. Now M_x has an idempotent g; note that $gx = 0$. For some $h \in A$, $h+xg = g+x$ or $xg = h+g+x$. In the first case, from $hx+xgx = gx+x^2$ we get $hx = x$ and similarly $hg = gh = g$. Thus $h^2+xg = h^2+hxg = hg+hx = g+x = h+xg$, so that $h^2 = h$. Clearly $M_h \subseteq M_x$; since $gx = 0$ and $gh = g \neq 0$, $M_h \neq M_x$, a contradiction. For the other case, we have $hg+g = gh+g = hx+x = ghx = 0$. Hence for $k = hxg+g+x$, we have $k^2 = k \in A$. For $z \in M_k$, $zk = 0$ and hence $zg+zx = zhxhg+zhxg+zg+zx = zhxhg$, so that $zx = zgx+zx^2 = zhxhgx = 0$, meaning $z \in M_x$. Thus $M_k \subseteq M_x$ but $M_x \neq M_k$, a contradiction. Therefore $M_x = (0)$.

Finally we show $y = yx$ for each $y \in A$ and that $A = Rx$. If $y \in A$, then $z+y = yx$ or $y = z+yx$ for some $z \in A$. If $z+y = yx$, then $yx = yx^2 = zx+yx$ and hence $zx = 0$ meaning $z = 0$. The other case gives the same result. Therefore $y = yx$ for all $y \in A$. Since $Rx \subseteq A = Ax \subseteq Rx$, $A = Rx$. This completes the proof.

Observe from Proposition 1 that each non-zero left l-ideal of R has a right identity. Also, if e is an idempotent of R, then Re [eR] is a left [right] ideal. Clearly, Re is a left semi-ideal. If $xe, y+xe = ze \in Re$, then $ye+xe = ye+xe^2 = ze^2 = ze = y+xe$, so that $y = ye \in Re$ and Re is a left l-ideal, and similarly Re is a left r-ideal. Consequently any left l-ideal is a left ideal by Proposition 1.

THEOREM 2. *If A is a non-zero ideal of R, then A contains an idempotent element e such that $A = eR$ and such that e is the identity of A.*

PROOF. By Proposition 1, A contains an idempotent element e such that $A = Re$. Let $B = \{x \in A | ex = 0\}$. Now B is a right ideal of R. Since e is a right identity of A, $Be = B$. Since $B^2 = (Be)B = B(eB) = (0)$, we have $B = (0)$. Letting $y \in A$, there exists $z \in A$ such that $z+y = ey$ or $y = z+ey$. If $z+y = ey$, then $ey = e^2y = ez+ey$, so that $ez = 0$ and $z = 0$. By the other case $z = 0$ also. Thus $y = ey$ for each $y \in A$, so that e is the identity of A, and $A = eR$. This completes the proof.

COROLLARY. *The semiring R contains an identity* 1.

For $a, b \in R$, $(a+b)(1+1)$ is $a+b+a+b$ and also $a+a+b+b$. Thus $a+b+a = a+a+b$. For some $y \in R$, $y+a+b = b+a$ or $a+b = y+b+a$. In the first case $a+a+b = a+b+a = a+y+a+b$, so that $a = a+y$ and $y = 0$. Similarly $y = 0$ in the other case. Consequently R is a hemiring, that is, a semiring with commutative addition.

The *center* of R is the set $C = \{x \in R | yx = xy$ for every $y \in R\}$. The following proposition is analogous to a theorem in ring theory [4].

PROPOSITION 3. *Each idempotent element e of R is in C if and only if e is the identity for some non-zero ideal of R.*

We now are able to prove that any left ideal of R has DCC.

THEOREM 4. *If A is a left ideal of R, then any left semi-ideal* [*ideal*] *of A is also a left semi-ideal* [*ideal*] *of R.*

PROOF. The proof is the same as the proof of the analogous ring theory theorem.

COROLLARY. *Any left ideal of R has DCC.*

It is to be observed from Theorem 4 that, if B is a right semi-ideal [ideal] of an ideal A, then B is a right semi-ideal [ideal] of R. This fact will be useful to us later in this paper.

4. Central idempotent elements

An idempotent of a hemiring is *central* if it belongs to the center of the hemiring. Further, an idempotent is *semiprimitive* if it is central and if it cannot be expressed as $u+v$ where u and v are central idempotents with $uv = 0$. The concepts of *orthogonal* and *pairwise orthogonal* idempotents in hemirings are defined analogously as in rings. At this point two characterizations of semiprimitives can be given.

PROPOSITION 5. *A central idempotent e of R is semiprimitive if and only if there does not exist a central idempotent $u \neq e$ such that $eu = u$ (that is, e is the only central idempotent of R in eR).*

PROOF. Let e be semiprimitive and suppose there is a central idempotent $u \neq e$ such that $eu = u$. For some $v \in R, v+u = e$ or $u = v+e$. If $v+u = e$, then $vu+u = vu+u^2 = eu = u$ and hence $uv = vu = 0$. Thus $v+u = v^2+u$, so that $v^2 = v$. Clearly $v \neq 0$ and $v \in C$. Consequently, v is a central idempotent. Since $v+u = e$ and $uv = 0$ we have a contradiction to e being semiprimitive. If $u = v+e$, then $ev+e = v+e$, so that $ev = v$. Also $uv = 0$, so that $0 = (v+e)v = v^2+v$ and $0 = u^3v = v^4 + 3v^3+3v^2+v = v^4+v$ which implies $v^2 = v^4$. Since $v^2 \in C$, v^2 is a central idempotent with $e = v^2+u$, a contradiction. The converse follows easily from the contrapositive.

Before giving the second characterization, two definitions are necessary. A hemiring is *simple* if the only ideals it contains are (0) and itself. An ideal of a hemiring is *simple* if it is simple as a hemiring.

PROPOSITION 6. *A central idempotent e of R is semiprimitive if and only if Re is simple.*

PROOF. If e is semiprimitive, then it is the only central idempotent of Re by Proposition 5. Let J be a non-zero ideal of Re. By the observation before Theorem 2, Re is an ideal, so that J is an ideal of R by Theorem 4. Thus $J = Ru$, where u is a central idempotent. Since $u \in J \subseteq Re$, $u = e$. Hence, $J = Re$ and Re is simple. The converse is proved the same as in ring theory.

THEOREM 7. *Every central idempotent e of R which is not semiprimitive is a sum of a finite number of pairwise orthogonal semiprimitive idempotents.*

PROOF. The ideal Re contains semiprimitive idempotents. Suppose u and v are distinct semiprimitive idempotents of R in Re. By Proposition 6, Ru and Rv are simple ideals. If $uv \neq 0$, *then* $Ru = Ru \cap Rv = Rv$. By Proposition 5, $u = v$ which is a contradiction. Hence $uv = 0$.

Let M be the set of all semiprimitive idempotents of R in Re. The elements of M are pairwise orthogonal. Consider any finite sum of elements of M, say $\Sigma u_i = u$. Clearly $u^2 = u = ue = eu$. For some $x \in Re$, $x+u = e$ or $u = x+e$. If $x+u = e$, then $ux = 0$ and as well $xu = 0$; with this $x = x^2$. Clearly $x \in C$, so that Rx is an ideal in Re. If $u = x+e$, then $ex = x$. Hence $x+e = x^2+2x+e$ and $x^2+x = 0$. Also $ux = xu = 0$, so that $x^2 = x^2+x^4+x^3 = x^4+x(x+x^2) = x^4$. Since $x^2 \in C$, Rx^2 is an ideal in Re. Considering the set N of all these Rx or Rx^2, as the case might be, choose a minimal member of N. If it is not (0), then it is equal to Rf, where f is a central idempotent of Re such that $e = f+\Sigma v_i$, $v_i \in M$, or it is equal to Rf^2, where f^2 is a central idempotent of Re such that $f+e = \Sigma w_i$, $w_i \in M$. Considering the first case we observe that, by the corollary to Theorem 4, Rf contains a minimal non-zero ideal K which is also an ideal of R. By Theorem 2, $K = Rv$ where v is a central idempotent of R. Since K is simple, v is semiprimitive, and hence $v \in M$.

Suppose $v = v_j$ for some j. Hence $v_j \in Rf$ which implies $v_j = xf$ for some $x \in R$. Since $e = f+\Sigma v_i$, $xv^j = xfv_j+x(\Sigma v_i)v_j = v_j+xv_j$, so that $v_j = 0$ which is a contradiction. Therefore $v \neq v_i$ for every i.

Take $w = v+\Sigma v_i$; then $w = y+e$ or $y+w = e$ for some $y \in Re$. As before $w^2 = w = we = ew$, and $w \in C$. Suppose $w = y+e$; then as before $y^4 = y^2$, $y^2 \in C$, and hence Ry^2 is an ideal of Re. Thus $v+\Sigma v_i = y+f+\Sigma v_i$, so that $v = y+f$. Since Rf is an ideal, $y \in Rf$. Therefore $Ry^2 \subseteq Rf$. Assume $f \in Ry^2$; then $f = ry^2$ for some $r \in R$. Since $vy = 0$, $v = vf = vry^2 = 0$, a contradiction. Thus $f \notin Ry^2$ and $Ry^2 \neq Rf$, a contradiction.

Suppose then that $y+w = e$; then as before $y^2 = y$, $y \in C$, and hence Ry is an ideal in Re, and as well $y+v+\Sigma v_i = f+\Sigma v_i$ and $y+v = f$, so that $Ry \subseteq Rf$. As well $Ry \neq Rf$, a contradiction. Consequently, for this case the minimal member of N has to be (0).

Consider now the second case; again Rf^2 contains a non-zero ideal of the form Rv where v is a semiprimitive idempotent and hence in M. If $v = w_j$ for some j, then $w_j \in Rf^2$ and hence $w_j = xf^2$ for some $x \in R$. Thence $xw_j = x(\Sigma w_i)w_j = xfw_j+xew_j = xfw_j+xw_j$ and $xfw_j = 0$. Thus $0 = f(xfw_j) = w_j^2 = w_j$, a contradiction. Therefore $v \neq w_i$ for every i.

Take $w = v+\Sigma v_i$; then $w = y+e$ or $y+w = e$ for some $y \in Re$. As before $w^2 = w = we = ew$, and $w \in C$. Suppose $w = y+e$; then as before $y^4 = y^2$, $y^2 \in C$, and hence Ry^2 is an ideal of Re. Thus $v+\Sigma v_i = y+f+\Sigma v_i$, so that $v = y+f$. Since Rf is an ideal, $y \in Rf$. Therefore $Ry^2 \subseteq Rf$. Assume $f \in Ry^2$; then $f = ry^2$ for some $r \in R$. Since $vy = 0$, $v = vf = vry^2 = 0$, a contradiction. Thus $f \notin Ry^2$ and $Ry^2 \neq Rf$, a contradiction.

Suppose then that $y+w = e$; then as before $y^2 = y$, $y \in C$, and hence Ry is an ideal in Re, and as well $y+v+\Sigma v_i = f+\Sigma v_i$ and $y+v = f$, so that $Ry \subseteq Rf$. As well $Ry \neq Rf$, a contradiction. Consequently, for this case the minimal member of N has to be (0).

Consider now the second case; again Rf^2 contains a non-zero ideal of the form Rv where v is a semiprimitive idempotent and hence in M. If $v = w_j$ for some j, then $w_j \in Rf^2$ and hence $w_j = xf^2$ for some $x \in$ R. Thence $xw_j = x(\Sigma w_i)w_j = xfw_j + xew_j = xfw_j + xw_j$ and $xfw_j = 0$. Thus $0 = f(xfw_j) = w_j^2 = w_j$, a contradiction. Therefore $v \neq w_i$ for every i.

Take $w = v+\Sigma w_i$; then $w = y+e$ or $y+w = e$ for some $y \in Re$. As before $w^2 = w$, $we = w$, and $w \in C$. Suppose $w = y+e$; then $y^4 = y^2$, $y^2 \in C$, and Ry^2 is an ideal of Re. Since $f^2+f = 0$, $v+\Sigma w_i = e+y = f^2+f+e+y = f^2+\Sigma w_i+y$ and hence $v = f^2+y$. Since Rf^2 is an ideal, $y \in Rf^2$. Therefore $Ry^2 \subseteq Rf^2$. By assuming $f^2 \in Ry^2, f^2 = ry^2$, $r \in R$, and thus, since $vy = 0$, $v = vf^2 = vry^2 = 0$, a contradiction. Therefore $Ry^2 \neq Rf^2$, a contradiction.

If $y+w = e$, then $y^2 = y$, $y \in C$, and Ry is an ideal in Re. As well $f+y+v+\Sigma w_i = f+y+w = f+e = \Sigma w_i$ and $y+v = f^2+f+y+v = f^2$, so that $y \in Rf^2$. Thus $Ry \subseteq Rf^2$ and $Ry \neq Rf^2$, a contradiction. Consequently, for this case the minimal member of N has to be (0). Therefore e is a finite sum of semiprimitive idempotents, as we wanted to prove.

5. Direct sums and a structure theorem

The concept of *direct sum* in hemirings is the same as in ring theory. Hence we have the following theorem which is proved the same as in ring theory:

PROPOSITION 8. *If $A_1, \cdots, A_m$ are distinct simple ideals of R and if $A = A_1 + \cdots + A_m$, then A is their direct sum.*

We conclude the section with the main theorem of the paper. It is a generalization to hemirings of a well-known structure theorem discussed by Artin, Nesbitt, and Thrall [1].

THEOREM 9. *The hemiring R has only a finite number of non-zero simple ideals and is their direct sum.*

PROOF. By the corollary to Theorem 2, R contains an identity 1 which is a central idempotent. If 1 is semiprimitive, then R is simple by Proposition 6 and the proof is complete. Assume 1 is not semiprimitive. By Theorem 7, $1 = \Sigma e_i$ where the e_i are pairwise orthogonal semiprimitive idempotents. Since $R = R\cdot 1 = R(\Sigma e_i) \subseteq \Sigma Re_i \subseteq R$, $R = \Sigma Re_i$. By Proposition 6, each Re_i is a simple ideal. By Proposition 8, R is the direct sum of Re_i.

Let I be a non-zero simple ideal of R. If $RIR = (0)$, then $I^3 = (0)$, a contradiction. Hence $I = RIR$. Thus $I = RIR \subseteq I(\Sigma Re_i) \subseteq \Sigma IRe_i \subseteq I$, so that $I = \Sigma IRe_i$. Some $IRe_i \neq (0)$ since $I \neq (0)$; hence $I \cap Re_i \neq (0)$ which implies $I = I \cap Re_i = Re_i$. Therefore, R has only a finite number of simple ideals and the proof is complete.

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(Oblatum: 9–III–70)

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