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A PROPERTY OF THE SORGENFREY LINE

by

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1. Introduction

In 1948, R. Sorgenfrey [7] gave an example of a paracompact space S, now generally known as the *Sorgenfrey line*, such that $S \times S$ is not paracompact or even normal. This space S consists of the set of real numbers, topologized by taking as a base all half-open intervals of the form $[a, b)$ with $a < b$. It is known, and easy to verify, that S is perfect, where we call a space *perfect* if every open subset is an F_σ-subset [3, 4]). The purpose of our note is to extend that result to S^ω, the product of countably many copies of S:

THEOREM 1.1. *The space S^ω is perfect.*

The above theorem will be applied in [5]. It also seems to provide the first example of a space X for which X^ω is perfect, but which is not semi-stratifiable (see section 3).

2. Proof of Theorem 1.1.

It will suffice to prove Proposition 2.1 and Lemma 2.3 below. Proposition 2.1 may have some independent interest.

PROPOSITION 2.1. *If $X_1, X_2, \cdots$ are topological spaces, and if $\prod_{i=1}^n X_i$ is perfect for all $n \in N$, then $\prod_{i=1}^\infty X_i$ is perfect.*

PROOF. Let U be an open subset of $\prod_{i=1}^\infty X_i$. The definition of the product topology implies that

$$U = \bigcup_{n=1}^{\infty} (U_n \times \prod_{i=n+1}^{\infty} X_i),$$

[1] Partially supported by an N.S.F. contract.
[2] Partially supported by an N.S.F. contract.
[3] There seems to be no established terminology for this concept, and our choice of the term 'perfect' was suggested by the established term 'perfectly normal'. There is, of course, no connection between this use of the word 'perfect' and its use in the terms 'perfect map' and 'perfect subset'.
[4] Other known properties of S are summarized in [5; Example 3.4].

with U_n open in $\prod_{i=1}^{n} X_i$ for all $n \in N$. By assumption, $U_n = \bigcup_{k=1}^{\infty} A_{k,n}$, where $A_{k,n}$ is closed in $\prod_{i=1}^{n} X_i$. But then

$$U = \bigcup_{k,n=1}^{\infty} (A_{k,n} \times \prod_{i=n+1}^{\infty} X_i),$$

so U is an F_σ in $\prod_{i=1}^{\infty} X_i$. That completes the proof.

It follows immediately from Proposition 2.1 that X^ω is perfect whenever X^n is perfect for all $n \in N$. We do not know, however, whether X^ω must be perfect whenever X^2 is perfect.

It should be remarked that there are analogues of Proposition 2.1 in which the property of being perfect is combined with other topological properties, such as normality [3; Theorem 2], paracompactness [6; Theorem 4.9], or the Lindelöf property [5; Proposition 2.1.(c)].

LEMMA 2.2. *If X is a perfect topological space, and if Y metrizable, then $X \times Y$ is perfect.*

PROOF. This was shown in the proof of [4; Proposition 5].

LEMMA 2.3. *The product S^n is perfect for every $n \in N$.*

PROOF. By induction. The lemma is clear for $n = 0$ if we define S^0 to a one-point space, so let us assume the lemma for n and let us prove it for $n+1$.

Denote S^{n+1} by X, so that $X = \prod_{i=1}^{n+1} X_i$ with $X_i = S$ for all i. For each $m \leqq n+1$, let $X(m) = \prod_{i=1}^{n+1} X_i(m)$, where $X_i(m) = S$ if $i \neq m$ and $X_m(m) = R$ (the real line with the usual topology). Then $X(m)$ is homeomorphic to $S^n \times R$ for every $m \leqq n+1$, so our inductive hypothesis and Lemma 2.2 imply that each $X(m)$ is perfect.

Let us now show that any open subset U of X is an F_σ in X. For each $m \leqq n+1$, let $U(m)$ be the interior of U as a subset of $X(m)$, and let $U^* = \bigcup_{m=1}^{n+1} U(m)$. Since $X(m)$ is perfect, $U(m)$ is an F_σ in $X(m)$ and hence also in X (whose topology is finer than that of $X(m)$), so U^* is also an F_σ in X. Thus it only remains to show that the set $A = U - U^*$ is an F_σ in X.

For each $x \in X$, let $\{W_j(x) : j \in N\}$ denote the base for the neighborhoods of x in X defined by

$$W_j(x) = \left\{ y \in X : x_i \leqq y_i < x_i + \frac{1}{j} \text{ for all } i \leqq n+1 \right\}.$$

For each $j \in N$, let

$$A_j = \{x \in A : W_j(x) \subset U\}.$$

Then $A = \bigcup_{j=1}^{\infty} A_j$, so that we need only show that each A_j is closed in X.

Suppose $x \notin A_j$, and let us show that x is not in the closure of A_j in X. For each $L \subset \{1, \cdots, n+1\}$, let

$$A_{j,L}(x) = \{y \in A_j : x_i = y_i \text{ iff } i \in L\}.$$

Then $A_j = \bigcup \{A_{j,L}(x) : L \subset \{1, \cdots, n+1\}\}$, so it will suffice to find, for each L, a neighborhood of x in X disjoint from $A_{j,L}(x)$.

If the neighborhood $W_j(x)$ of x is disjoint from $A_{j,L}(x)$, we are through. If not, pick some point z in $W_j(x) \cap A_{j,L}(x)$. Then the set

$$V = W_j(x) \cap \{y \in X : y_i < z_i \text{ if } i \notin L\}$$

is a neighborhood of x in X, and we need only show that V is disjoint from $A_{j,L}(x)$.

Suppose not, so that there is some $y \in V \cap A_{j,L}(x)$. Then $y \in W_j(x)$ and $y \neq x$ (because $y \in A_j$ while $x \notin A_j$), so there is an $m \leqq n+1$ for which $y_m > x_m$. Clearly $m \notin L$. Let

$$W = W_j(y) \cap \{u \in X : y_m < u_m\}.$$

Then W is open in $X(m)$ and $W \subset W_j(y) \subset U$ (since $y \in A_j$), so by definition of $U(m)$ we have $W \subset U(m) \subset U^*$. Furthermore $z \in W$, because

$$\begin{aligned} &z_i = x_i = y_i && \text{if } i \in L, \\ &y_i < z_i < x_i + \frac{1}{j} \leqq y_i + \frac{1}{j} && \text{if } i \notin L. \end{aligned}$$

Hence $z \in U^*$, contradicting the choice of

$$z \in A_{j,L}(x) \subset A = U - U^*.$$

That completes the proof.

3. Semi-stratifiable spaces

A topological space X is called *semi-stratifiable* [1] if each open U in X can be expressed as $U = \bigcup_{n=1}^{\infty} A_n(U)$, with each $A_n(U)$ closed in X, so that $A_n(U) \subset A_n(U')$ whenever $U \subset U'$. A first-countable space X is semi-stratifiable if and only if it is semi-metrizable [1; Remark 1.3].

Clearly every semi-stratifiable space is perfect, and it is easily checked that semi-stratifiability is preserved by countable products. Thus X^{ω} is perfect whenever X is semi-stratifiable. Theorem 1.1 implies that the converse of this last result is false, since the Sorgenfrey line S is not semi-stratifiable by [2].

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