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**THE RADIUS OF UNIVALENCE AND STARLIKENESS
 OF SOME CLASSES OF REGULAR FUNCTIONS**

by

Emanuel G. Calys

1.

The writing of this paper has been motivated by recent results of A. E. Livingston [7] and S. D. Bernardi [2].

Let S denote the class of functions $f(z) = z + \sum_2^\infty a_n z^n$ which are regular and univalent in $E\{z : |z| < 1\}$ and which map E onto domains $D(f)$. We denote by S^* and K the subclasses of S where $D(f)$ are, respectively, starlike with respect to the origin, and convex. Let P denote the class of functions $p(z)$ which are regular and satisfy $p(0) = 1$, $\operatorname{Re}(p(z)) > 0$, for z in E . In [1] the following theorem was proven.

THEOREM A. *Let $f(z) = z + \sum_2^\infty a_n z^n$ be a member of S^* . Then*

$$(1.1) \quad F(z) = (c+1)z^{-c} \int_0^z t^{c-1} f(t) dt = z + \sum_2^\infty \left(\frac{c+1}{c+n} \right) a_n z^n$$

is also a member of the same class for $c = 1, 2, 3, \dots$.

Theorem A represents a generalization of the corresponding theorem by R. J. Libera [6] for the case $c = 1$. Solving the relation (1.1) for the inverse function $f(z)$, we have

$$(1.2) \quad f(z) = \left(\frac{1}{1+c} \right) z^{1-c} [z^c F(z)]'.$$

In [2] S. D. Bernardi proved that if $F(z) \in S^*$, then $f(z)$, defined by (1.2), is univalent and starlike for $|z| < r_0$, where

$$r_0 = [-2 + (3 + c^2)^{\frac{1}{2}}] / (c - 1) \text{ for } c = 2, 3, 4, \dots$$

This result is sharp. For $c = 1$, $r_0 = \frac{1}{2}$ and this result is due to A. E. Livingston [7].

In this paper we determine the radius of univalence and starlikeness of functions $f(z) = z + a_2 z^2 + \dots$ which are regular in E and satisfy

$$(1.3) \quad F(z) = \frac{2}{z} \int_0^z \frac{f(t)g(t)}{t} dt,$$

where $F(z) \in S^*$ and (i) $g(z) \in K$, (ii) $g(z) \in S$ and (iii) $g(z)/z \in P$. We shall employ the same techniques used in [7].

2.

THEOREM 1. *If $f(z)$ is regular in E and satisfies (1.3), where $F(z) \in S^*$ and $g(z) \in K$, then $f(z)$ is univalent and starlike for $|z| < 2 - \sqrt{3}$. This result is sharp.*

PROOF. Since F is in S^* , $\operatorname{Re} \{zF'(z)/F(z)\} > 0$ for all z in E . Hence there exists w , regular in E , such that $|w(z)| \leq 1$ for z in E and such that

$$\frac{f(z)g(z) - \int_0^z \frac{f(t)g(t)dt}{t}}{\int_0^z \frac{f(t)g(t)dt}{t}} = \frac{zF'(z)}{F(z)} = \frac{1 - zw(z)}{1 + zw(z)}.$$

Thus

$$f(z)g(z) = \frac{2}{1 + zw(z)} \int_0^z \frac{f(t)g(t)dt}{t},$$

and

$$\frac{zf'(z)}{f(z)} = \frac{2 - zw(z) - z^2w'(z)}{1 + zw(z)} - \frac{zg'(z)}{g(z)}.$$

Therefore

$$(2.1) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \operatorname{Re} \left\{ \frac{2 - zw(z) - z^2w'(z)}{1 + zw(z)} \right\} - \left| \frac{zg'(z)}{g(z)} \right|.$$

Thus $f(z)$ will be univalent and starlike for those values of z for which the right-hand side of (2.1) is positive. Since $g(z)$ is in K ,

$$\left| \frac{zg'(z)}{g(z)} \right| \leq \frac{1}{1 - |z|} \quad [4, \text{p. 13}].$$

Therefore the right-hand side of (2.1) will be positive if

$$(2.2) \quad \operatorname{Re} \left\{ \frac{2 - zw(z) - z^2w'(z)}{1 + zw(z)} \right\} - \frac{1}{1 - |z|} > 0.$$

Condition (2.2) is equivalent to

$$(2.3) \quad \operatorname{Re} \{ [1 + (|z| - 2)zw(z) + (|z| - 1)z^2w'(z) - 2|z|][1 + \overline{zw(z)}] \} > 0,$$

or

$$(2.4) \quad \operatorname{Re} \{ (1 - |z|)z^2w'(z)[1 + \overline{zw(z)}] \} < \{ 1 - (1 + |z|) \operatorname{Re} (zw(z)) - 2|z| + (-2 + |z|)|z|^2|w(z)|^2 \}.$$

Using the well known result

$$|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2} \tag{3, p. 18}$$

and the fact that $\operatorname{Re}(zw(z)) \leq |z||w(z)|$, we see that (2.4) will be satisfied if

$$(2.5) \quad (1 - |z|)|z|^2 \frac{(1 - |w(z)|)^2}{1 - |z|^2} (1 + |z||w(z)|) < \{1 - (1 + |z|)|w(z)||z| - 2|z| + (|z| - 2)|z|^2|w(z)|^2\}.$$

Since $|w(z)| \leq 1$, $(1 + |z||w(z)|)/(1 + |z|) \leq 1$ and (2.5) will be satisfied if

$$|z|^2(1 - |w(z)|^2) < 1 - 2|z| - (1 + |z|)|z||w(z)| + (|z| - 2)|z|^2|w(z)|^2$$

which is equivalent to

$$(2.6) \quad (|z|^2 + 2|z|) + (1 - |z|)|z|^2|w(z)|^2 + (1 + |z|)|z||w(z)| < 1.$$

Hence, it suffices to show that (2.6) holds for all functions w , regular in E and satisfying $|w(z)| \leq 1$, provided $|z| < 2 - \sqrt{3}$.

In (2.6) put $a = |z|$, $x = |w(z)|$ and consider the function

$$p(x) = a^2 + 2a + a(1 + a)x + a^2(1 - a)x^2.$$

Clearly, $p(x)$ is increasing in $[0, 1]$ and $p(1) = 3a + 3a^2 - a^3$ is less than one for $0 \leq a < 2 - \sqrt{3}$. Condition (2.2) is thus seen to be satisfied if $|z| < 2 - \sqrt{3}$. Hence $f(z)$ is univalent and starlike for $|z| < 2 - \sqrt{3}$.

To see that the result is sharp, let $F(z) = z/(1 - z)^2$ and $g(z) = z/(1 + z)$. Then $F(z)$ is in S^* , $g(z)$ is in K and $f(z) = (z^2 + z)/(1 - z)^3$. Thus $f'(z) = (z^2 + 4z + 1)/(1 - z)^4$ and $f'(-2 + \sqrt{3}) = 0$. Hence $f(z)$ is not univalent in $|z| < r$ if $r > 2 - \sqrt{3}$.

THEOREM 2. *If $f(z)$ is regular in E and satisfies (1.3), where $F(z) \in S^*$ and $g(z) \in S$, then $f(z)$ is univalent and starlike for $|z| < \frac{1}{5}$. This result is sharp.*

The proof of this theorem is similar to that of Theorem 1. The only essential difference is the estimate

$$\left| \frac{zg'(z)}{g(z)} \right| \leq \frac{1 + |z|}{1 - |z|} \tag{4, p. 5}.$$

To see that the result is sharp, let $F(z) = z/(1 - z)^2$ and $g(z) = z/(1 + z)^2$. Then $f(z)g(z) = z^2/(1 - z)^3$ and we have

$$\frac{zf'(z)}{f(z)} = \frac{2 + z}{1 - z} - \frac{zg'(z)}{g(z)} = \frac{2 + z}{1 - z} - \frac{1 - z}{1 + z} = \frac{1 + 5z}{1 - z^2} = 0$$

for $z = -\frac{1}{5}$. Thus $f(z)$ is not starlike in $|z| < r$ if $r > \frac{1}{5}$.

REMARK. The above example shows that we cannot improve on the result of Theorem 2 if instead of $g(z)$ in S we assume $g(z)$ in S^* .

THEOREM 3. *If $f(z)$ is regular in E and satisfies (1.3), where $F(z) \in S^*$ and $g(z)/z \in P$, then $f(z)$ is univalent and starlike for $|z| < (5 - \sqrt{17})/4$. This result is sharp.*

PROOF. Let $h(z) = g(z)/z$. Then

$$\frac{zf'(z)}{f(z)} = \frac{1 - 2zw(z) - z^2w'(z)}{1 + zw(z)} - \frac{zh'(z)}{h(z)},$$

where $w(z)$ is regular in E and $|w(z)| \leq 1$. Using the estimate

$$\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2|z|}{1 - |z|^2} \quad [5]$$

and the techniques of Theorem 1, the result follows.

To see that the result is sharp, let $F(z) = z/(1-z)^2$ and $g(z) = z(1-z)/(1+z)$. Then $f(z)h(z) = z/(1-z)^3$, $f(z) = (z^2+z)/(1-z)^4$ and $f'(z) = (2z^2+5z+1)/(1-z)^5 = 0$ for $z = (-5 + \sqrt{17})/4$. Hence $f(z)$ is not univalent in any disk $|z| < r$ if $r + (5 - \sqrt{17})/4$.

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