COMPOSITIO MATHEMATICA

WOLFGANG M. SCHMIDT

Irregularities of distribution. VI

Compositio Mathematica, tome 24, nº 1 (1972), p. 63-74
<http://www.numdam.org/item?id=CM_1972__24_1_63_0>

© Foundation Compositio Mathematica, 1972, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http://http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

COMPOSITIO MATHEMATICA, Vol. 24, Fasc. 1, 1972, pag. 63-74
Wolters-Noordhoff Publishing
Printed in the Netherlands

IRREGULARITIES OF DISTRIBUTION. VI

by

Wolfgang M. Schmidt [1]

1. Introduction

We are interested in the distribution of an arbitrary sequence of numbers in an interval. We are thus returning to questions investigated in the first part [10] of the present series. However, the present paper can be read independently.

Let U be the unit interval consisting of numbers ξ with $0 < \xi \leqq 1$, and let $\omega = \{\xi_1, \xi_2, \cdots\}$ be a sequence of numbers in this interval. Given an α in U and a positive integer n, we write $Z(n, \alpha)$ for the number of integers i with $1 \leqq i \leqq n$ and $0 \leqq \xi_i < \alpha$. We put

$$D(n, \alpha) = |Z(n, \alpha) - n\alpha|.$$

The sequence ω is called *uniformly distributed* if $D(n) = o(n)$, where $D(n)$ is the supremum of $D(n, \alpha)$ over all numbers α in U. Answering a question of Van der Corput [3], Mrs. Van Aardenne-Ehrenfest [1] showed that $D(n)$ cannot remain bounded. Later [2] she proved that there are infinitely many integers n with $D(n) > c_1 \log\log n/\log\log\log n$ where c_1 is a positive absolute constant, and K. F. Roth [9] improved this to $D(n) > c_2(\log n)^{\frac{1}{2}}$.

For $\kappa \geqq 0$ let $S(\kappa)$ be the set of all numbers α in U with

$$D(n, \alpha) \leqq \kappa \qquad (n = 1, 2, \cdots).$$

Further let $S(\infty)$ be the union of the sets $S(\kappa)$, i.e. the set of numbers α in U for which $D(n, \alpha)$ remains bounded as a function of n. Erdös [4, 5] asked whether $S(\infty)$ was necessarily a proper subset of U. This question was answered in the affirmative by the author in the first paper [10] of this series, where among other things it was shown that $S(\infty)$ has Lebesgue measure zero. In the present paper we shall show that $S(\infty)$ is at most a countable set.

Recall that a number γ is a *limit point* of a set S if there is a sequence of distinct elements of S which converge to γ. The *derivative* $S^{(1)}$ of S

[1] Supported in part by Air Force Office of Scientific Research grant AF-AFOSR-69-1712.

consists of all the limit points of S. The higher derivatives are defined inductively by $S^{(d)} = (S^{(d-1)})^{(1)}$ ($d = 2, 3, \cdots$). Our main theorem is as follows.

THEOREM. *Suppose $d > 4\kappa$. Then $S^{(d)}(\kappa)$ is empty.*

No special importance attaches to the quantity 4κ which could be somewhat reduced at the cost of further complications. But at the end of this paper we shall exhibit a sequence[2] for which $S^{(d)}(d)$ is not empty for $d = 1, 2, \cdots$, and hence 4κ may not be replaced by $\kappa - \varepsilon$ where $\varepsilon > 0$. One shows easily by induction on d that a set S of real numbers for which $S^{(d)}$ is empty is at most countable and is nowhere dense. We therefore obtain the

COROLLARY. *The sets $S(\kappa)$ are at most countable and they are nowhere dense. The set $S(\infty)$ is at most countable.*

Let θ be irrational and let $\omega = \omega(\theta)$ be the sequence $\{\theta\}, \{2\theta\}, \cdots$ where $\{\ \}$ denotes fractional parts. One can easily show (see Hecke [6], § 6) that the numbers $\{k\theta\}$ where k is an integer belong to $S(\infty)$. In answer to a question by Erdös and Szüsz, it was shown by Kesten [7] that the numbers $\{k\theta\}$ are the only elements of $S(\infty)$. Hence in this case the set $S(\infty)$ is known and is countable.

Now let $\boldsymbol{I}$ be a subinterval of U of the type $\alpha < \xi \leqq \beta$ and put $D(n, \boldsymbol{I}) = |Z(n, \beta) - Z(n, \alpha) - n(\beta - \alpha)|$. If $\omega = \omega(\theta)$, then $D(n, \boldsymbol{I})$ is bounded as a function of n if (Ostrowski [8]) and only if (Kesten [7]) $\boldsymbol{I}$ has length $l(\boldsymbol{I}) = \beta - \alpha = \{k\theta\}$ where k is an integer. Hence in this example there are *continuum many intervals* $\boldsymbol{I}$ for which $D(n, \boldsymbol{I})$ remains bounded.

2. A proposition which implies the theorem

In what follows, U^0 will be the open interval $0 < \xi < 1$. All the numbers $\alpha, \beta, \gamma, \delta, \theta, \eta, \lambda, \mu, \alpha_i, \beta_i, \cdots$ will be in U^0. A *neighborhood* of a number α will by definition be an open interval containing α which is contained in U^0. It will be convenient to extend the definition of the derivatives of a set S by putting $S^{(0)} = S$. By $I, J, \cdots$ we shall denote intervals of the type $a < n \leqq b$ where the end points are integers with $0 \leqq a < b$. Such an interval of length $l(I) = b - a$ contains precisely $l(I)$ integers.

The sequence ω will be fixed throughout. For α in U^0 we put

$$f(n, \alpha) = Z(n, \alpha) - n\alpha,$$

[2] In fact it is Van der Corput's sequence, as constructed in [3].

so that $D(n, \alpha) = |f(n, \alpha)|$. We write

$$g^{+}(I, \alpha) = \max_{n \in I} f(n, \alpha), \ g^{-}(I, \alpha) = \min_{n \in I} f(n, \alpha)$$

and

$$h(I, \alpha) = g^{+}(I, \alpha) - g^{-}(I, \alpha).$$

PROPOSITION. *Suppose $d \geqq 0$ and $\varepsilon > 0$. Let R be a set whose d-th derivative $R^{(d)}$ has a non-empty intersection with U^0.*

Then there are $w = 2^d$ elements $\lambda_1, \cdots, \lambda_w$ of R with neighborhoods $\boldsymbol{L}_1, \cdots, \boldsymbol{L}_w$ and a number p such that

$$w^{-1} \sum_{j=1}^{w} h(I, \mu_j) > \tfrac{1}{2}(d+1) - \varepsilon \tag{1}$$

for every interval I with $l(I) \geqq p$ and every $\mu_1 \in \boldsymbol{L}_1, \cdots, \mu_w \in \boldsymbol{L}_w$.

Applying this with $\mu_1 = \lambda_1, \cdots, \mu_t = \lambda_t$ we see that there is a λ_j with $h(I, \lambda_j) > \frac{1}{2}(d+1) - \varepsilon$. There are integers m, n with $f(m, \lambda_j) - f(n, \lambda_j) > \frac{1}{2}(d+1) - \varepsilon$, hence with

$$\max(D(m, \lambda_j), D(n, \lambda_j)) = \max(|f(m, \lambda_j)|, |f(n, \lambda_j)|) > \tfrac{1}{4}(d+1) - \tfrac{1}{2}\varepsilon.$$

This shows that $\lambda_j \notin S(\frac{1}{4}(d+1) - \varepsilon)$.

Now take $R = S(\frac{1}{4}(d+1) - \varepsilon)$. The assumption that an element α of U^0 lies in $R^{(d)}$ leads to the contradiction that $\lambda_j \in R$ and $\lambda_j \notin R$. Hence $R^{(d)} = S^{(d)}(\frac{1}{4}(d+1) - \varepsilon)$ is empty except for the possible elements 0 and 1. At any rate $S^{(d+1)}(\frac{1}{4}(d+1) - \varepsilon)$ is empty for $d \geqq 0$, and hence $S^{(d)}(\frac{1}{4}d - \varepsilon)$ is empty for $d \geqq 1$. It follows that $S^{(d)}(\kappa)$ is empty for $d > 4\kappa$.

Hence our proposition implies the theorem. The proposition will be proved by induction on d. Its generality is necessary to carry out this inductive proof.

3. The case $d = 0$

When $d = 0$ the hypotheses of the proposition are satisfied if R consists of a single element α in U^0. In this case the conclusion must hold with $w = 2^0 = 1$ and with $\lambda_1 = \alpha$. Hence when $d = 0$ the proposition may be reformulated as follows.

LEMMA 1. *Suppose α is in U^0 and $\varepsilon > 0$. There is a neighborhood $\boldsymbol{A}$ of α and a number p such that*

$$h(I, \beta) > \tfrac{1}{2} - \varepsilon \tag{2}$$

for every β in $\boldsymbol{A}$ and every interval I with $l(I) \geqq p$.

For α in U^0 put

$$c(\alpha) = \begin{cases} 0 \text{ if } \alpha \text{ is irrational,} \\ 1/z \text{ if } \alpha = y/z \text{ with coprime positive integers } y, z. \end{cases}$$

Since $0 \leqq c(\alpha) \leqq \frac{1}{2}$, Lemma 1 is a consequence of

LEMMA 2. *The inequality* (2) *in Lemma* 1 *may be replaced by*

$$h(I, \beta) > 1-c(\alpha)-\varepsilon.$$

PROOF. If $\alpha = y/z$, then given any real ψ there are integers m, n with $1 \leqq n \leqq z$ and $|n\alpha-m-\psi| \leqq c(\alpha)/2$. Now suppose that α is irrational. Kronecker's Theorem implies that for every ψ there are positive integers m, n with $|n\alpha-m-\psi| < \varepsilon/8$. Find particular solutions m, n for $\psi = 0$, $\varepsilon/4$, $2\varepsilon/4, \cdots, [4\varepsilon^{-1}]\varepsilon/4$ (where [ ] denotes the integer part), and denote the maximum of the numbers n so obtained by p. Then for every ψ there will be integers m, n with $1 \leqq n \leqq p$ and with $|n\alpha-m-\psi| < \varepsilon/4$. Hence for *every* α in U^0 there is a $p = p(\alpha, \varepsilon)$ such that for every ψ there are integers m, n with

$$1 \leqq n \leqq p \text{ and } |n\alpha-m-\psi| < \tfrac{1}{2}c(\alpha)+\tfrac{1}{4}\varepsilon.$$

Let A be the neighborhood of α consisting of numbers β in U^0 with $|\beta-\alpha|p < \varepsilon/4$. For every β in A and every ψ there are integers m, n with $1 \leqq n \leqq p$ and $|n\beta-m-\psi| < \frac{1}{2}c(\alpha)+\frac{1}{2}\varepsilon$. Since this is true for every ψ, there will also be integers m, n with $1 \leqq n \leqq p$ and $0 < n\beta-m-\psi < c(\alpha)+\varepsilon$. It is clear that the interval $1 \leqq n \leqq p$ may be replaced by any interval I with $l(I) \geqq p$. Thus for every such interval I and every ψ and for every β in A there are integers m, n with

$$n \in I \text{ and } 0 < n\beta-m-\psi < c(\alpha)+\varepsilon.$$

Now choose integers m, n with

$$n \in I \text{ and } 0 < n\beta-m+g^-(I, \beta) < c(\alpha)+\varepsilon.$$

We have $Z(n, \beta) = f(n, \beta)+n\beta \geqq g^-(I, \beta)+n\beta > m$, whence $Z(n, \beta) \geqq m+1$. This implies that

$$\begin{aligned} h(I, \beta) &= g^+(I, \beta)-g^-(I, \beta) \\ &\geqq Z(n, \beta)-n\beta-g^-(I, \beta) \\ &> m+1-n\beta+(n\beta-m-c(\alpha)-\varepsilon) \\ &= 1-c(\alpha)-\varepsilon. \end{aligned}$$

4. Variations on Lemma 2

Write

$$f(n, \alpha, \beta) = f(n, \beta)-f(n, \alpha) = Z(n, \beta)-Z(n, \alpha)-n(\beta-\alpha).$$

LEMMA 3. *Suppose* $\varepsilon > 0$, $q \geqq 1$ *and*

$$0 < |\alpha-\beta| < \varepsilon/(8q). \tag{3}$$

Then there is a p and there are neighborhoods $\boldsymbol{A}$ *of* α *and* $\boldsymbol{B}$ *of* β *such that for every* $\gamma \in \boldsymbol{A}$ *and* $\delta \in \boldsymbol{B}$ *and for every interval I with* $l(I) \geqq p$ *there are two subintervals J and J′ with* $l(J) = l(J') = q$ *such that*

$$f(n, \gamma, \delta)-f(n', \gamma, \delta) > 1-\varepsilon \tag{4}$$

for every $n \in J$ *and every* $n' \in J'$.

PROOF. We may assume that $\alpha < \beta$. Put $p_0 = [(\beta-\alpha)^{-1}]$. Every number in U has a distance less than $\varepsilon/8$ from at least one of the numbers $\beta-\alpha$, $2(\beta-\alpha)$, $\cdots$, $p_0(\beta-\alpha)$. Thus for every ψ there are integers m, n with $1 \leqq n \leqq p_0$ and $|n(\beta-\alpha)-m-\psi| < \varepsilon/8$. Let $\boldsymbol{A}, \boldsymbol{B}$ be disjoint neighborhoods of α, β such that elements γ of $\boldsymbol{A}$ and δ of $\boldsymbol{B}$ satisfy

$$16|\gamma-\alpha| \max(q, p_0) < \varepsilon \text{ and } 16|\delta-\beta| \max(q, p_0) < \varepsilon, \tag{5}$$

respectively. For every $\gamma \in \boldsymbol{A}$ and $\delta \in \boldsymbol{B}$ and for every ψ there are integers m, n with $1 \leqq n \leqq p_0$ and $|n(\delta-\gamma)-m-\psi| < \varepsilon/4$, and similarly there are numbers m, n with $1 \leqq n \leqq p_0$ and $0 < n(\delta-\gamma)-m-\psi < \varepsilon/2$. Here the interval $1 \leqq n \leqq p_0$ may be replaced by any interval I_0 with $l(I_0) \geqq p_0$.

Now suppose that $\gamma \in \boldsymbol{A}$, $\delta \in \boldsymbol{B}$ and $l(I_0) \geqq p_0$. Let n'_0 be the integer in I_0 with

$$f(n'_0, \gamma, \delta) = \min_{n \in I_0} f(n, \gamma, \delta).$$

Choose integers m, n_0 with

$$n_0 \in I_0 \text{ and } 0 < n_0(\delta-\gamma)-m+f(n'_0, \gamma, \delta) < \varepsilon/2.$$

We have

$$Z(n_0, \delta)-Z(n_0, \gamma) = f(n_0, \gamma, \delta)+n_0(\delta-\gamma) \geqq f(n'_0, \gamma, \delta)+n_0(\delta-\gamma) > m,$$

whence $Z(n_0, \delta)-Z(n_0, \gamma) \geqq m+1$. This implies that

$$\begin{aligned} f(n_0, \gamma, \delta)-f(n'_0, \gamma, \delta) &\geqq m+1-n_0(\delta-\gamma)-f(n'_0, \gamma, \delta) \\ &> m+1-m-\tfrac{1}{2}\varepsilon \\ &= 1-\tfrac{1}{2}\varepsilon. \end{aligned} \tag{6}$$

Since $\alpha < \beta$ and since $\boldsymbol{A}, \boldsymbol{B}$ are disjoint, any elements $\gamma \in \boldsymbol{A}$ and $\delta \in \boldsymbol{B}$ have $\gamma < \delta$. Moreover by (3) and (5) they satisfy

$$0 < q(\delta-\gamma) < \varepsilon/4. \tag{7}$$

Put $p = p_0+2q$ and let I be an interval with $l(I) \geqq p$. The interval I_0 obtained from I by removing intervals of length q from both ends has

$l(I_0) \geqq p_0$. Hence for every $\gamma \in \boldsymbol{A}$ and $\delta \in \boldsymbol{B}$ there are integers n_0, n_0' in I_0 with (6). Let J and J' be the intervals

$$n_0 < n \leqq n_0+q \quad\text{and}\quad n_0'-q < n' \leqq n_0',$$

respectively. For every n in J and every n' in J' one has

$$f(n, \gamma, \delta)-f(n_0, \gamma, \delta) \geqq -(\delta-\gamma)(n-n_0) \geqq -q(\delta-\gamma) > -\varepsilon/4,$$
$$f(n', \gamma, \delta)-f(n_0', \gamma, \delta) < \varepsilon/4$$

by (7). These inequalities in conjunction with (6) yield (4). Since J and J' have length q and are contained in I, the lemma follows.

Write

$$g^+(J, \alpha, \beta) = \max_{n\in J} f(n, \alpha, \beta),\ g^-(J, \alpha, \beta) = \min_{n\in J} f(n, \alpha, \beta).$$

The statement in Lemma 3 that (4) holds for $n \in J$ and $n' \in J'$ may now be expressed by

$$g^-(J, \gamma, \delta)-g^+(J', \gamma, \delta) > 1-\varepsilon.$$

We shall need the function

$$h(J, J', \alpha, \beta) = \max\,(g^-(J, \alpha, \beta)-g^+(J', \alpha, \beta), g^-(J', \alpha, \beta)-g^+(J, \alpha, \beta)).$$

LEMMA 4. *Suppose $\theta_1, \cdots, \theta_t$ belong to the derivative $R^{(1)}$ of some set R. Let $\boldsymbol{D}_1, \cdots, \boldsymbol{D}_t$ be neighborhoods of $\theta_1, \cdots, \theta_t$, respectively. Suppose $\varepsilon > 0$ and $q \geqq 1$.*

Then there is an r and there are elements $\alpha_1, \beta_1, \cdots, \alpha_t, \beta_t$ of R with neighborhoods $\boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_t, \boldsymbol{B}_t$ satisfying $\boldsymbol{A}_i \subseteq \boldsymbol{D}_i$, $\boldsymbol{B}_i \subseteq \boldsymbol{D}_i$ $(i = 1, \cdots, t)$ and with the following property. For $\gamma_1 \in \boldsymbol{A}_1$, $\delta_1 \in \boldsymbol{B}_1, \cdots,$ $\gamma_t \in \boldsymbol{A}_t$, $\delta_t \in \boldsymbol{B}_t$ and for intervals I, I' with $l(I) \geqq r$, $l(I') \geqq r$ there are subintervals $J \subseteq I$ and $J' \subseteq I'$ with

$$l(J) = l(J') = q$$

and with

$$h(J, J', \gamma_i, \delta_i) > 1-\varepsilon \qquad (i = 1, \cdots, t). \tag{8}$$

We shall apply this lemma only in the special case when $I = I'$. The general formulation is necessary to carry out a proof by induction on t.

PROOF. Suppose at first that $t = 1$. Since θ_1 is a limit point of R, there are elements α_1, β_1 of R which belong to $\boldsymbol{D}_1$ and which have

$$0 < |\alpha_1-\beta_1| < \varepsilon/(8q).$$

By Lemma 3 there is a p and there are neighborhoods $\boldsymbol{A}_1, \boldsymbol{B}_1$ of α_1, β_1 such that for every $\gamma_1 \in \boldsymbol{A}_1$, $\delta_1 \in \boldsymbol{B}_1$ and for every I with $l(I) \geqq p$ there are subintervals J_1, J_2 of length q with

$$g^-(J_1, \gamma_1, \delta_1) - g^+(J_2, \gamma_1, \delta_1) > 1-\varepsilon. \tag{9}$$

We may shrink the neighborhoods $\boldsymbol{A}_1, \boldsymbol{B}_1$, if necessary, to get $\boldsymbol{A}_1 \subseteq \boldsymbol{D}_1$, $\boldsymbol{B}_1 \subseteq \boldsymbol{D}_1$. If an interval I' also has length $l(I') \geqq p$, then I' has subintervals J_1', J_2' of length q with

$$g^-(J_1', \gamma_1, \delta_1) - g^+(J_2', \gamma_1, \delta_1) > 1-\varepsilon. \tag{10}$$

By adding (9) and (10) we see that either

$$g^-(J_1, \gamma_1, \delta_1) - g^+(J_2', \gamma_1, \delta_1) > 1-\varepsilon$$

or

$$g^-(J_1', \gamma_1, \delta_1) - g^+(J_2, \gamma_1, \delta_1) > 1-\varepsilon.$$

In the first case we take $J = J_1$, $J' = J_2'$, and in the second case we take $J = J_2$, $J' = J_1'$. The inequality (8) is then true for $i = 1$. Hence when $t = 1$, Lemma 4 is true with $r = r^{(1)} = p$.

The induction from $t-1$ to t goes as follows. Construct $\alpha_1, \beta_1, \cdots, \alpha_{t-1}, \beta_{t-1}, \boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_{t-1}, \boldsymbol{B}_{t-1}$ and $r^{(t-1)}$ such that (8) holds (under the conditions stated in the lemma) for $i = 1, \cdots, t-1$. By the case $t = 1$ we can find α_t, β_t in $\boldsymbol{R}$ with neighborhoods $\boldsymbol{A}_t, \boldsymbol{B}_t$ contained in $\boldsymbol{D}_t$ and a number $\bar{r}^{(1)}$ such that for every $\gamma_t \in \boldsymbol{A}_t$, $\delta_t \in \boldsymbol{B}_t$ and for intervals I, I' with $l(I) \geqq \bar{r}^{(1)}$, $l(I') \geqq \bar{r}^{(1)}$ there are subintervals $I_0 \subseteq I$, $I_0' \subseteq I'$ with $l(I_0) = l(I_0') = r^{(t-1)}$ such that

$$h(I_0, I_0', \gamma_t, \delta_t) > 1-\varepsilon. \tag{11}$$

By our construction of $\alpha_1, \beta_1, \cdots, \alpha_{t-1}, \beta_{t-1}, \boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_{t-1}, \boldsymbol{B}_{t-1}$ and $r^{(t-1)}$ there are subintervals $J \subseteq I_0$, $J' \subseteq I_0'$ with $l(J) = l(J') = q$ such that (8) holds for $i = 1, \cdots, t-1$. Now in view of (11) and since $h(J, J', \gamma_t, \delta_t) \geqq h(I_0, I_0', \gamma_t, \delta_t)$, the inequality (8) holds for $i = 1, \cdots, t$. This shows that Lemma 4 is true with $r = \bar{r}^{(1)}$.

5. An inequality

LEMMA 5. *Suppose α, β belong to U^0 and suppose that J, J' are subintervals of an interval I. Then*

$$h(I, \alpha) + h(I, \beta) \geqq h(J, J', \alpha, \beta) + \tfrac{1}{2}(h(J, \alpha) + h(J, \beta) + h(J', \alpha) + h(J', \beta)). \tag{12}$$

PROOF. We may assume without loss of generality that

$$h(J, J', \alpha, \beta) = g^-(J, \alpha, \beta) - g^+(J', \alpha, \beta).$$

Then we have $f(n, \alpha, \beta) - f(n', \alpha, \beta) \geqq h(J, J', \alpha, \beta)$, i.e.

$$f(n, \beta) - f(n, \alpha) - f(n', \beta) + f(n', \alpha) \geqq h(J, J', \alpha, \beta) \tag{13}$$

for every $n \in J$ and every $n' \in J'$. Let $m_\alpha, n_\alpha, m_\beta, n_\beta$ be integers in J with

$$f(m_\alpha, \alpha) = g^+(J, \alpha), \quad f(n_\alpha, \alpha) = g^-(J, \alpha),$$
$$f(m_\beta, \beta) = g^+(J, \beta), \quad f(n_\beta, \beta) = g^-(J, \beta).$$

Then

$$f(m_\alpha, \alpha) - f(n_\alpha, \alpha) = h(J, \alpha), \tag{14}$$

$$f(m_\beta, \beta) - f(n_\beta, \beta) = h(J, \beta). \tag{15}$$

Similarly, there are elements $m'_\alpha, n'_\alpha, m'_\beta, n'_\beta$ of J' such that

$$f(m'_\alpha, \alpha) - f(n'_\alpha, \alpha) = h(J', \alpha), \tag{16}$$

$$f(m'_\beta, \beta) - f(n'_\beta, \beta) = h(J', \beta). \tag{17}$$

Applying (13) with $n = m_\alpha$, $n' = m'_\beta$ we obtain

$$f(m_\alpha, \beta) - f(m_\alpha, \alpha) - f(m'_\beta, \beta) + f(m'_\beta, \alpha) \geqq h(J, J', \alpha, \beta).$$

Applying (13) with $n = n_\beta$, $n' = n'_\alpha$ we obtain

$$f(n_\beta, \beta) - f(n_\beta, \alpha) - f(n'_\alpha, \beta) + f(n'_\alpha, \alpha) \geqq h(J, J', \alpha, \beta).$$

Adding these two inequalities and the four equations (14), (15), (16), (17) we get

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \geqq 2h(J, J', \alpha, \beta) + h(J, \alpha) + h(J, \beta) + h(J', \alpha) + h(J', \beta),$$

where

$$\varphi_1 = f(m'_\alpha, \alpha) - f(n_\alpha, \alpha), \quad \varphi_2 = f(m'_\beta, \alpha) - f(n_\beta, \alpha),$$
$$\varphi_3 = f(m_\beta, \beta) - f(n'_\beta, \beta), \quad \varphi_4 = f(m_\alpha, \beta) - f(n'_\alpha, \beta).$$

Since $h(I, \alpha) \geqq \varphi_1$, $h(I, \alpha) \geqq \varphi_2$, $h(I, \beta) \geqq \varphi_3$, $h(I, \beta) \geqq \varphi_4$, the lemma follows.

6. Proof of the proposition

Lemma 1 shows the truth of the proposition when $d = 0$. From here on we shall have $d \geqq 1$, and we shall assume the truth of the proposition for $d-1$ and proceed to prove it for d.

By this assumption we see that if $\varepsilon > 0$ and if $R^{(d)}$ and U^0 have a non-empty intersection, then there are $t = 2^{d-1}$ elements $\theta_1, \cdots, \theta_t$ of $R^{(1)}$ with neighborhoods $\boldsymbol{D}_1, \cdots, \boldsymbol{D}_t$ and a number $p^{(d-1)}$ such that

$$t^{-1} \sum_{j=1}^{t} h(I, \eta_j) > \tfrac{1}{2}d - \tfrac{1}{2}\varepsilon \tag{18}$$

for $\eta_1 \in \boldsymbol{D}_1, \cdots, \eta_t \in \boldsymbol{D}_t$ and every interval I with $l(I) \geqq p^{(d-1)}$. We now apply Lemma 4 with these particular $\theta_1, \cdots, \theta_t, \boldsymbol{D}_1, \cdots, \boldsymbol{D}_t$ and with

$q = p^{(d-1)}$. We construct elements $\alpha_1, \beta_1, \cdots, \alpha_t, \beta_t$ of R with neighborhoods $\boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_t, \boldsymbol{B}_t$ and

$$r = r(\theta_1, \cdots, \theta_t; \boldsymbol{D}_1, \cdots, \boldsymbol{D}_t; p^{(d-1)}) \tag{19}$$

with the properties enunciated in that lemma.

Now suppose that $l(I) = r$ and let $\gamma_1, \delta_1, \cdots, \gamma_t, \delta_t$ be elements of $\boldsymbol{A}_1, \boldsymbol{B}_1, \cdots, \boldsymbol{A}_t, \boldsymbol{B}_t$, respectively. There are subintervals J, J' of I with $l(J) = l(J') = p^{(d-1)}$ such that

$$h(J, J', \gamma_i, \delta_i) > 1-\varepsilon \qquad (i = 1, \cdots, t).$$

Hence by Lemma 5 we have

$$\begin{aligned} h(I, \gamma_i)+h(I, \delta_i) > (1-\varepsilon)+\tfrac{1}{2}(h(J, \gamma_i)+h(J, \delta_i) \\ +h(J', \gamma_i)+h(J', \delta_i)) \qquad (i = 1, \cdots, t). \end{aligned} \tag{20}$$

Now γ_j lies in $\boldsymbol{D}_j$ since $\gamma_j \in \boldsymbol{A}_j$ and $\boldsymbol{A}_j \subseteq \boldsymbol{D}_j$ $(j = 1, \cdots, t)$. We therefore may apply (18) with $\eta_1 = \gamma_1, \cdots, \eta_t = \gamma_t$, and we obtain

$$\sum_{j=1}^{t} h(J, \gamma_j) > t(\tfrac{1}{2}d-\tfrac{1}{2}\varepsilon).$$

More generally, each of the four quantities

$$\chi_1 = \sum_{j=1}^{t} h(J, \gamma_j), \chi_2 = \sum_{j=1}^{t} h(J, \delta_j), \chi_3 = \sum_{j=1}^{t} h(J', \gamma_j), \chi_4 = \sum_{j=1}^{t} h(J', \delta_j)$$

exceeds $t(\frac{1}{2}d-\frac{1}{2}\varepsilon)$. Taking the sum of the inequalities (20) with $i = 1, \cdots, t$ and dividing by $2t$ we obtain

$$\begin{aligned} (2t)^{-1}(\sum_{i=1}^{t} h(I, \gamma_i)+\sum_{i=1}^{t} h(I, \delta_i)) > (\tfrac{1}{2}-\tfrac{1}{2}\varepsilon)+(4t)^{-1}(\chi_1+\chi_2+\chi_3+\chi_4) \\ > \tfrac{1}{2}(d+1)-\varepsilon. \end{aligned} \tag{21}$$

The $w = 2t = 2 \cdot 2^{d-1} = 2^d$ quantities $\lambda_1 = \alpha_1, \cdots, \lambda_t = \alpha_t$, $\lambda_{t+1} = \beta_1, \cdots, \lambda_{2t} = \beta_t$ and their respective neighborhoods $\boldsymbol{L}_1 = \boldsymbol{A}_1, \cdots, \boldsymbol{L}_t = \boldsymbol{A}_t, \boldsymbol{L}_{t+1} = \boldsymbol{B}_1, \cdots, \boldsymbol{L}_{2t} = \boldsymbol{B}_t$ and $p = r$ where r is given by (19) have the desired properties stated in the proposition. Namely, (21) shows that (1) is true for every interval I with $l(I) \geqq p$ and arbitrary elements $\mu_1, \cdots, \mu_w$ in $\boldsymbol{L}_1, \cdots, \boldsymbol{L}_w$.

7. An example

Let R_0 be the set consisting of 0, and for integers $d \geqq 1$ let R_d be the set consisting of 0 and of the numbers

$$2^{-g_1}+ \cdots +2^{-g_t} \tag{22}$$

where $t, g_1, \cdots, g_t$ are integers with

$$1 \leqq t \leqq d \text{ and } 1 \leqq g_1 < g_2 < \cdots < g_t. \tag{23}$$

LEMMA 6. *For every* $d \geqq 1$,

$$R_d^{(1)} = R_{d-1}.$$

PROOF. It is clear that $R_1^{(1)} = R_0$. We now proceed by induction on d and assume that $d \geqq 2$ and that $R_{d-1}^{(1)} = R_{d-2}$. Since the relation $R_{d-1} \subseteq R_d^{(1)}$ is rather obvious, it will remain for us to show that $R_d^{(1)} \subseteq R_{d-1}$.

Let ξ be the limit of a sequence of distinct numbers $\eta(1), \eta(2), \cdots$ of R_d; we have to show that ξ lies in R_{d-1}. We clearly may assume that none of the numbers $\eta(n)$ is 0. Let $t(n), g_1(n), \cdots, g_{t(n)}(n)$ be the numbers $t, g_1, \cdots, g_t$ in (22) which belong to $\eta(n)$. In view of (23) there are only finitely many numbers in R_d for which g_t lies under a given upper bound, and hence $g_{t(n)}(n)$ must tend to infinity. Therefore ξ is also the limit of the sequence $\hat{\eta}(1), \hat{\eta}(2), \cdots$ where

$$\hat{\eta}(n) = \eta(n) - 2^{-g_{t(n)}(n)}.$$

The numbers $\hat{\eta}(n)$ lie in R_{d-1}. If infinitely many of them are equal, then their limit ξ is in R_{d-1}. If infinitely many among them are distinct, then we know by induction that their limit ξ is in R_{d-2}, hence a fortiori in R_{d-1}.

We now construct a sequence $\omega_0 = \{\xi_1, \xi_2, \cdots\}$ as follows. We put $\xi_1 = 0$, and if $k \geqq 0$ and if $\xi_1, \cdots, \xi_{2^k}$ have already been constructed, then we define $\xi_{2^k+1}, \cdots, \xi_{2^{k+1}}$ by

$$\xi_{2^k+t} = \xi_t + \frac{1}{2^{k+1}} \qquad (t = 1, \cdots, 2^k). \tag{24}$$

Thus $\omega_0 = \{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \cdots\}$. In what follows, *the sets* $S(\kappa)$ *will be defined in terms of this sequence* ω_0.

LEMMA 7. *For every integer* $d \geqq 0$,

$$R_d \subseteq S(d).$$

Repeated application of Lemma 6 shows that $R_d^{(d)}$ consists of 0, and we obtain the

COROLLARY. *The sets* $S^{(d)}(d)$ *are non-empty for* $d = 0, 1, 2, \cdots$.

PROOF OF LEMMA 7. The assertion is true for $d = 0$ since $S(0)$ contains 0. Assuming the truth of the lemma for $d-1$ we now proceed to prove it for d. It will suffice to show that every element η of R_d of the type

$$\eta = 2^{-g_1} + \cdots + 2^{-g_d}$$

lies in $S(d)$. Put $\hat{\eta} = 2^{-g_1} + \cdots + 2^{-g_{d-1}}$. We know by our inductive hypothesis that

$$(25) \qquad |Z(n, \hat{\eta}) - n\hat{\eta}| \leqq d-1 \qquad (n = 1, 2, \cdots).$$

The first 2^{g_d} elements of ω_0 are the numbers $j2^{-g_d}$ $(j = 0, 1, \cdots, 2^{g_d}-1)$ in some order. Hence there is precisely one t_0 with $1 \leqq t_0 \leqq 2^{g_d}$ and $\xi_{t_0} = \hat{\eta}$. The other elements ξ_t with $1 \leqq t \leqq 2^{g_d}$ lie outside the interval $\boldsymbol{I}$ given by

$$\hat{\eta} \leqq \xi < \eta = \hat{\eta} + 2^{-g_d}.$$

Now if $t' = t + m2^{g_d}$ where $1 \leqq t \leqq 2^{g_d}$ and where m is a nonnegative integer, then

$$\xi_t \leqq \xi_{t'} < \xi_t + 2^{-g_d-1} + 2^{-g_d-2} + \cdots = \xi_t + 2^{-g_d}$$

by repeated application of (24). Therefore ξ_t lies in $\boldsymbol{I}$ precisely if $t \equiv t_0 \pmod{2^{g_d}}$. This implies that

$$n2^{-g_d} - 1 < Z(n, \eta) - Z(n, \hat{\eta}) < n2^{-g_d} + 1 \qquad (n = 1, 2, \cdots),$$

hence that

$$|Z(n, \eta) - Z(n, \hat{\eta}) - n(\eta - \hat{\eta})| = |Z(n, \eta) - Z(n, \hat{\eta}) - n2^{-g_d}| < 1 \qquad (n = 1, 2, \cdots).$$

Combining this inequality with (25) we obtain $|Z(n, \eta) - n\eta| < d$, which shows that η lies in $S(d)$.

REFERENCES

T. Van Aardenne-Ehrenfest

[1] Proof of the impossibility of a just distribution, etc. Indag. Math. 7 (1945), 71–76.

T. Van Aardenne-Ehrenfest

[2] On the impossibility of a just distribution. Ibid. 11 (1949), 264–269.

J. G. Van der Corput

[3] Verteilungsfunktionen. I. Proc. Kon. Ned. Akad. v. Wetensch. 38 (1935), 813–821.

P. Erdös

[4] Some unsolved problems. Publ. Math. Inst. Hung. Acad. Sci. 6 (1961), 221–254.

P. Erdös

[5] Problems and results on diophantine approximations. Compositio Math. 16 (1964), 52–56. (Nijenrode lecture 1962).

E. Hecke

[6] Über analytische Funktionen und die Verteilung von Zahlen mod Eins. Abh. Math. Sem. Hamburg 1 (1922), 54–76.

H. Kesten
[7] On a conjecture of Erdös and Szüsz related to uniform distribution mod 1. Acta Arith. 12 (1966/67), 193–212.

A. Ostrowski
[8] Math. Miszellen. IX. Notiz zur Theorie der Diophantischen Approximationen. Jber. Deutsch. Math. Ver. 36 (1927), 178–180.

K. F. Roth
[9] On Irregularities of Distribution. Mathematika 7 (1954), 73–79.

W. M. Schmidt
[10] Irregularities of Distribution. Quarterly J. of Math. (Oxford) 19 (1968), 181–191.

(Oblatum 30–XI–1970)

Department of Mathematics
University of Colorado
Boulder, Colorado 80302, U.S.A.