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T. G. McLAUGHLIN

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COMPLETE INDEX SETS OF RECURSIVELY ENUMERABLE FAMILIES

by

T. G. McLaughlin

1. Introduction

This paper is an adjunct to [2]. In [2, § 2], we remarked that the index set $G(\mathscr{F})$ of a recursively enumerable family $\mathscr{F}$ of classes of r.e. sets can be $\sum_n^0$ complete for $1 \leqq n \leqq 5$ or $\prod_n^0$ complete for $1 \leqq n \leqq 4$. Here we shall give specific examples which verify that remark. All unexplained notation and background terminology in the present paper should be read according to the conventions laid down in [2, § 1]. We wish to take the opportunity to correct a minor technical error in [2]. The function ζ_0 referred to in the introductory section of [2] should not be alleged to be a *total* recursive function, but should rather be specified as a partial recursive function with $\delta\zeta_0 = \{e | W_e^F$ is a *nonempty* family consisting of *nonempty* classes$\}$. The only point at which ζ_0 enters into a proof in [2] is at the beginning of the proof of Lemma A, where, under the alias of ζ, it is mistakenly treated as being defined for all arguments e. However, it is easily seen that even with its domain limited as indicated above, ζ $(= \zeta_0)$ still permits the function ξ of [2, Lemma A] to be taken *total* recursive; none of the remaining discussion in [2] need then be modified or even reworded. (There are alternative ways of mending the error; but the way just indicated seems most direct.)

2. Complete index sets

Throughout this section, we let η denote a partial recursive function with the property that $(\forall x)[W_x \neq \emptyset \Rightarrow \eta(x) \in W_x]$; and we let μ denote a recursive function such that $(\forall x)[W_{\mu(x)} = \{x\}]$.

PROPOSITION 1. *If $\mathscr{F}$ is an r.e. family of classes then its index set, $G(\mathscr{F})$, is $\sum_5^0$.*

PROOF. Let $\mathscr{F} = \{W_e^C | e \in W_f\}$. Then

$$n \in G(\mathfrak{F}) \Leftrightarrow (\exists e)[e \in W_f \text{ and } W_e^C = W_n^C]$$
$$\Leftrightarrow (\exists e)[e \in W_f \text{ and } (\forall_j)[W_j \in W_e^C \Leftrightarrow W_j \in W_n^C]].$$

But $W_j \in W_k^C \Leftrightarrow (\exists r)[r \in W_k$ and $(\forall s)[s \in W_j \Leftrightarrow s \in W_r]]$. Hence, by means of the usual prenex transformation procedures, $W_j \in W_k^C$ is seen to be a $\sum_3^0$ predicate of j and k. It follows, again by the standard prenex operations, that $n \in G(\mathscr{F})$ is a $\sum_5^0$ predicate of n. Q.E.D.

PROPOSITION 2. *Let* $\mathscr{W} = \{W_e^C | e \in N\}$. *Then* $\mathscr{W}$ *is an r.e. family and* $G(\mathscr{W})$ *is recursive.*

Since $G(\mathscr{W}) = N$, Proposition 2 is obvious.

PROPOSITION 3. $\mathscr{W}$ *and* $\emptyset$ *are the only r.e. families* $\mathscr{F}$ *for which* $G(\mathscr{F})$ *is recursive.*

PROOF. The proposition is a precise analogue of a result of Rice's concerning *classes* ([3, Corollary B to Theorem 6]); and the proof follows the proof of Rice's result given in [1]. Thus, suppose $\mathscr{F}$ is a family of r.e. classes such that $G(\mathscr{F})$ is recursive. Suppose further that neither $\mathscr{F}$ nor $\mathscr{W} - \mathscr{F}$ is empty. Now, either $\emptyset \in \mathscr{F}$ or $\emptyset \in \mathscr{W} - \mathscr{F}$; let us first suppose that $\emptyset \in \mathscr{W} - \mathscr{F}$. Let Q be a fixed non-recursive r.e. subset of N; and let W_e^C be some fixed element of $\mathscr{F}$. Let g be a recursive function such that $n \in Q \Rightarrow W_{g(n)} = W_e$ and $n \notin Q \Rightarrow W_{g(n)} = \emptyset$. Then, since $W_e \neq \emptyset$, we have $n \in Q \Leftrightarrow g(n) \in G(\mathscr{F})$. But therefore Q is recursive: contradiction. A similar contradiction arises if we assume $\emptyset \in F$, since if $G(\mathscr{F})$ is recursive then so also is $G(\mathscr{W} - \mathscr{F})$. Hence either $\mathscr{F} = \mathscr{W}$ or $\mathscr{F} = \emptyset$. Q.E.D.

REMARK. The proof of Proposition 3 in fact shows that if $\mathscr{F} \neq \emptyset$ & $\emptyset \notin \mathscr{F}$ then every $\sum_1^0$ set is many-one reducible to $G(\mathscr{F})$. (Indeed, they are all one-one reducible to $G(\mathscr{F})$, since g can be taken one-one.)

PROPOSITION 4. *Let* $\mathscr{F}_0 = \{W_e^C | (\exists y)[y \in W_e$ *and* $W_y \neq \emptyset]\}$. *Then* $\mathscr{F}_0$ *is an r.e. family and* $G(\mathscr{F}_0)$ *is* $\sum_1^0$ *complete.*

PROOF. Clearly, we have

$$(\forall n)[n \in G(\mathscr{F}_0) \Leftrightarrow (\exists y)(\exists z)[y \in W_n \text{ and } z \in W_y]];$$

therefore $G(\mathscr{F}_0)$ is $\sum_1^0$. *A fortiori*, $\mathscr{F}_0$ is an r.e. family. Next, it is easy to see that there exists a recursive function $\psi_0(x, y)$ such that

$$(\forall w)(\forall z)(\forall x)(\forall y)[\varphi^3_{\psi_0(f,z)}(w, x, y)$$

is defined $\Leftrightarrow T_1(f, z, w)]$. Let $\omega(x)$ be a recursive function such that

$$(\forall x)[W_{\omega(x)} = \bigcup_{z \in x} W_z].$$

Then, since $(\forall f)[W_f = \{z | (\exists w) T_1(f, z, w)\}]$, we have that

$$\omega(\zeta_1(\psi_0(f, n))) \in G(\mathscr{F}_0) \Leftrightarrow n \in W_f;$$

here and subsequently, ζ_1 is as in § 1 of [2]. Thus $G(\mathscr{F}_0)$ is $\sum_1^0$ complete. (Alternatively, note that $\emptyset \notin \mathscr{F} \neq \emptyset$ and use the remark following the proof of Proposition 3.) Q.E.D.

PROPOSITION 5. *Let* $\mathscr{F}_1 = \{\emptyset, \{\emptyset\}\}$. *Then* $\mathscr{F}_1$ *is an r.e. family and* $G(\mathscr{F}_1)$ *is* $\prod_1^0$ *complete.*

PROOF. Since every *finite* family of r.e. classes is r.e., $\mathscr{F}_1$ is an r.e. family. We can easily construct a recursive function $\psi_1(x, y)$ such that

$$(\forall w)(\forall z)(\forall x)(\forall y)[\varphi^3_{\psi_1(f,z)}(w, x, y) \text{ is defined} \Leftrightarrow T_1(f, z, x)].$$

Therefore $\eta(\zeta_1(\psi_1(f, n))) \in G(\mathscr{F}_1) \Leftrightarrow (\forall x) \neg T_1(f, n, x)$, so that if $G(\mathscr{F}_1)$ is $\prod_1^0$ then it is $\prod_1^0$ *complete.* (Alternatively, note that $\emptyset \notin \mathscr{W} - \mathscr{F} \neq \emptyset$ and use the remark following the proof of Proposition 3.) But, since $n \in G(\mathscr{F}_1) \Leftrightarrow (\forall x)(\forall y)[x \in W_n \Rightarrow y \notin W_x]$, we have that $G(\mathscr{F}_1)$ is indeed $\prod_1^0$ and therefore $\prod_1^0$ complete. Q.E.D.

REMARK. The alternative proofs of completeness indicated for Propositions 4 and 5 show that we could have taken $\mathscr{F}_0 = \{W_e^{\mathbf{C}} \mid W_e \neq \emptyset\}$ for Proposition 4 and $\mathscr{F}_1 = \{\emptyset\}$ for Proposition 5. We prefer, however, the more involved choices of $\mathscr{F}_0$ and $\mathscr{F}_1$ since then the proofs can be given the common format shared by all the later proofs (with the single exception of our proof of Proposition 8).

PROPOSITION 6. *Let* $\mathscr{F}_2 = \{W_e^{\mathbf{C}} \mid \emptyset \in W_e^{\mathbf{C}}\}$. *Then* $\mathscr{F}_2$ *is an r.e. family and* $G(\mathscr{F}_2)$ *is* $\sum_2^0$ *complete.*

PROOF. $\mathscr{F}_2$ is r.e., since $\mathscr{F}_2 = \{W_e^{\mathbf{C}} \mid (\exists f)[W_e^{\mathbf{C}} = W_f^{\mathbf{C}} \cup \{\emptyset\}]\}$. Next, since $n \in G(\mathscr{F}_2) \Leftrightarrow (\exists j)[j \in W_n$ and $(\forall z)(z \notin W_j)]$, it is easily seen by routine prenex-form manipulation that $G(\mathscr{F}_2)$ is $\sum_2^0$. To show that $G(\mathscr{F}_2)$ is $\sum_2^0$ complete, we need only construct a recursive function $\psi_2(x, y)$ with the property that

$$(\forall f)(\forall n)[(\exists w)(\forall z) \neg T_2(f, n, w, z) \Leftrightarrow (\exists v)(\exists u)(\delta\varphi^3_{\psi_2(f,n)}(v, u, y) = \emptyset)];$$

for then we have that $n \in$ the f-th $\sum_0^2$ set $\Leftrightarrow \omega(\zeta_1(\psi_2(f, n))) \in G(\mathscr{F}_2)$. To obtain ψ_2, we first construct an auxiliary partial recursive function υ by stages, thus:

$$\upsilon^s_{f,n}(v, u, y) \simeq \begin{cases} 0, \text{ if } (\forall w \leqq v)(\exists z \leqq s)T_2(f, n, w, z), \\ \text{undefined, otherwise;} \end{cases}$$

and we set

$$\upsilon^s_{f,n} =_{df} \bigcup_{s=0}^{\infty} \upsilon^s_{f,n}.$$

Clearly, $\upsilon_{f,n}$ is partial recursive uniformly in the parameters f and n; so let $\psi_2(x, y)$ be a recursive function such that

$$(\forall f)(\forall n)[\varphi^3_{\psi_2(f,n)} \simeq v_{f,n}].$$

From the definition of $v_{f,n}$, it is easy to see that, for each pair $\langle f, n\rangle$,

$$(\exists w)(\forall z) \neg T_2(f, n, w, z) \Leftrightarrow (\exists v)(\exists u)(\delta\varphi^3_{\psi_2(f,n)}(v, u, y) = \emptyset).$$

Hence $G(\mathscr{F}_2)$ is $\sum^0_2$ complete. Q.E.D.

PROPOSITION 7. *Let $\mathscr{F}_3 = \{\{N\}\}$. Then $\mathscr{F}_3$ is an r.e. family and $G(\mathscr{F}_3)$ is $\prod^0_2$ complete.*

PROOF. $\mathscr{F}_3$ is r.e. since it is a finite family of r.e. classes. Since

$$n \in G(\mathscr{F}_3) \Leftrightarrow [W_n \neq \emptyset \text{ and } (\forall z)[z \in W_n \Rightarrow (\forall k)(k \in W_z)]],$$

and since $W_n \neq \emptyset$ is a $\sum^0_1$ predicate of n, we see by the usual prenex transformation procedures that $G(\mathscr{F}_3)$ is $\prod^0_2$. We shall construct a recursive function $\psi_3(x, y)$ with the property:

$$(\forall w)(\forall z)(\forall x)(\forall y)[\varphi^3_{\psi_3(f,z)}(w, x, y) \text{ is defined} \Leftrightarrow (\exists u)T_2(f, z, y, u)].$$

Since

$$\begin{aligned} z \in \text{the } f\text{-th } \textstyle\prod^0_2 \text{ set} &\Leftrightarrow (\forall y)(\exists u)T_2(f, z, y, u) \\ &\Leftrightarrow (\forall w)(\forall x)[\delta\varphi^3_{\psi_3(f,z)}(w, x, y) = N], \end{aligned}$$

the function $\eta\zeta_1\psi_3$ will then witness the $\prod^0_2$ completeness of $G(\mathscr{F}_3)$. The required function ψ_3 is very simply obtainable via a stage-by-stage construction of an auxiliary function τ: at stage s we set

$$\tau^s_{f,z} = \{\langle w, x, y, 0\rangle | (\exists u \leqq s)T_2(f, z, y, u)\},$$

for all pairs $\langle f, z\rangle$; then we take $\tau_{f,z} = \bigcup^\infty_{s=0} \tau^s_{f,z}$. Obviously, the construction of $\tau_{f,z}$ is effective uniformly in the parameters f and z; i.e., there is a recursive function ψ_3 such that $(\forall f)(\forall z)[\tau_{f,z} \simeq \varphi^3_{\psi_3(f,z)}]$. ψ_3 as so specified is plainly an indexing function of the kind that we require, and hence $G(\mathscr{F}_3)$ is $\prod^0_2$ complete. Q.E.D.

PROPOSITION 8. *Let $\mathscr{F}_4 = \{\{A\} | A$ is a cofinite subset of $N\}$. Then $\mathscr{F}_4$ is an r.e. family and $G(\mathscr{F}_4)$ is $\sum^0_3$ complete.*

PROOF. The class COF of cofinite subsets of N is r.e.; hence $\mathscr{F}_4$ is r.e. since $\mathscr{F}_4 = \{W^C_{\mu(e)} | W_e \in COF\}$. Now, it is shown in [4] that the set $C = \{e | W_e \in COF\}$ is $\sum^0_3$ complete. But hence $G(\mathscr{F}_4)$ is also $\sum^0_3$ complete, provided that it is $\sum^0_3$ at all; for if A is a $\sum^0_3$ subset of N and β is a recursive function such that $n \in A \Rightarrow \beta(n) \in C$ and $n \notin A \Rightarrow \beta(n) \notin C$, then $n \in A \Rightarrow \mu(\beta(n)) \in G(\mathscr{F}_4)$ and $n \notin A \Rightarrow \mu(\beta(n)) \notin G(\mathscr{F}_4)$. To see that $G(\mathscr{F}_4)$ is $\sum^0_3$, we first note that the predicate $(\forall z)[z \in W_n \Rightarrow W_z = W_e]$ is a $\prod^0_2$ predicate of n and e, and we then apply the standard prenex operations to the right-hand side of the equivalence $n \in G(\mathscr{F}_4) \Leftrightarrow [W_n \neq \emptyset$ and $(\exists e)[e \in C$ and $(\forall z)[z \in W_n \Rightarrow W_z = W_x]]]$. Thus, $G(\mathscr{F}_4)$ is $\sum_3^0$ complete. Q.E.D.

PROPOSITION 9. *Let* $\mathscr{F}_5 = \{W_e^C \mid \{W_j \mid W_r \textit{ is finite}\} \subseteq W_e^C\}$. *Then* $\mathscr{F}_5$ *is an r.e. family and* $G(\mathscr{F}_5)$ *is* $\prod_3^0$ *complete.*

PROOF. The class $\{W_j \mid W_j$ is finite$\}$ is r.e.; hence, since

$$\mathscr{F}_5 = \{W_e^C \mid (\exists k)[W_e^C = W_k^C \cup \{W_j \mid W_j \text{ is finite}\}]\},$$

it is easily deduced that $\mathscr{F}_5$ is an r.e. family. To show that $G(\mathscr{F}_5)$ is $\prod_3^0$, we make use of a 'canonical enumeration' of the class $\{W_j \mid W_j$ is finite$\}$. The particular enumeration that we shall apply is defined (as in [5, p. 70]) as follows:

$$D_0 \underset{df}{=} \emptyset;\ D_{n+1} \underset{df}{=} \{k_1, \cdots, k_l\}, \text{ where } n+1 = \sum_{m=1}^{l} 2^{k_m}$$

with $k_1 < k_2 < \cdots < k_m$. It is easily verified that the predicate $D_j \subseteq W_k$ is a $\sum_1^0$ predicate of j and k, and that the predicate $x \in D_j$ is a recursive predicate of x and j. Now, we have

$$\begin{aligned} n \in G(\mathscr{F}_5) &\Leftrightarrow (\forall j)(\exists m)[m \in W_n \text{ and } D_j = W_m] \\ &\Leftrightarrow (\forall j)(\exists m)[m \in W_n \text{ and } D_j \subseteq W_m \text{ and } W_m \subseteq D_j]; \end{aligned}$$

hence, by routine prenex manipulations we obtain a $\prod_3^0$ predicate form for $G(\mathscr{F}_5)$. We shall construct a recursive function $\psi_4(x, y)$ such that, for every pair of numbers $\langle f, n\rangle$, we have

$$(\forall z)(\exists w)(\forall y) \neg T_3(f, n, z, w, y) \Leftrightarrow (\forall j)(\exists l)(\exists k)[\delta\varphi^3_{\psi_4(f,n)}(l, k, u) = D_j].$$

It is then obviously the case that

$$n \in \text{the } f\text{-th } \textstyle\prod_3^0 \text{ set} \Leftrightarrow \omega(\zeta_1(\psi_4(f, n))) \in G(\mathscr{F}_5),$$

where ω is as in the proof of Proposition 4. Thus, the existence of such a function ψ_4 implies $\prod_3^0$ completeness of $G(\mathscr{F}_5)$. In order to specify ψ_4, we shall define an auxiliary partial recursive function $\bar{\tau}$ by stages, as follows:

$$\bar{\tau}^s_{f,n}(j, k, u) \simeq \begin{cases} 0, \text{ if } u \in D_j \text{ or if } (\exists y \leqq s)T_3(f, n, j, k, y), \\ \text{undefined, otherwise;} \end{cases}$$

$$\bar{\tau}_{f,n} =_{df} \bigcup_{s=0}^{\infty} \bar{\tau}^s_{f,n}.$$

It is obvious that the definition of $\bar{\tau}_{f,n}$ is effective uniformly in the parameters f and n; hence, there is a recursive function $\psi_4(x, y)$ such that

$$(\forall f)(\forall n)[\varphi^3_{\psi_4(f,n)} \simeq \bar{\tau}_{f,n}].$$

But then $\delta\varphi^3_{\psi_4(f,n)}(j, k, u)$, for a given pair $\langle j, k\rangle$, is either N or D_j, according as $(\exists y)T_3(f, n, j, k, y)$ or $(\forall y) \neg T_3(f, n, j, k, y)$. Thus, ψ_4 has the required property, and so $G(\mathscr{F}_5)$ is $\prod^0_3$ complete. Q.E.D.

Before stating Proposition 10 we remind the reader that p_n denotes the n-th prime number in order of magnitude, starting with $p_0 = 2$. We shall denote by P_n the set $\{p^m_n | m \in N - \{0\}\}$ of positive powers of p_n.

PROPOSITION 10. *Let* $\mathscr{F}_6 = \{W^C_e | (\exists n)[\{W_j | W_j$ *is a finite subset of* $P_n\} \subseteq W^C_e]\}$. *Then* $\mathscr{F}_6$ *is an r.e. family and* $G(\mathscr{F}_6)$ *is* $\sum^0_4$ *complete.*

PROOF. It is easily demonstrated that there is a recursive function $\chi(x, y)$, with $\chi(n, y)$ one-to-one for each n, such that $(\forall n)[\{D_{\chi(n,y)} | y \in N\} = \{W_j | W_j$ is a finite subset of $P_k\}]$. Hence $\mathscr{F}_6$ is r.e., since

$$\mathscr{F}_6 = \{W^C_e | (\exists f)(\exists n)[W^C_e = W^C_f \cup \{D_{\chi(n,y)} | y \in N\}]\}.$$

Next, we observe that

$$n \in G(\mathscr{F}_6) \Leftrightarrow (\exists k)(\forall j)(\exists m)[m \in W_n \text{ and } D_{\chi(k,j)} = W_m].$$

Therefore $G(\mathscr{F}_6)$ is $\sum^0_4$. To show that $G(\mathscr{F}_6)$ is $\sum^0_4$ complete, we need only make a slight modification of our above proof that $G(\mathscr{F}_5)$ is $\prod^0_3$ complete. Thus, we begin by defining

$$\zeta^s_{f,n}(k, j, y, z) \simeq \begin{cases} 0, \text{ if } z \in D_{x(k,j)} \text{ or if } (\exists w \leqq s)T_4(f, n, k, j, y, w), \\ \text{undefined, otherwise;} \end{cases}$$

and we then set

$$\zeta_{f,n} =_{df} \bigcup_{s=0}^{\infty} \zeta^s_{f,n},$$

for all pairs $\langle f, n\rangle$. Clearly, $\zeta_{f,n}$ is partial recursive uniformly in f and n; so there is a recursive function $\xi(x, y)$ such that

$$(\forall f)(\forall n)[\varphi^4_{\xi(f,n)} \simeq \zeta_{f,n}].$$

Let $\zeta_2(f, n)$ be a recursive function such that

$$(\forall f)(\forall n)[\varphi^3_{\zeta_2(f,n)}(\pi_2(k, j), y, z) \simeq \varphi^4_{\xi(f,n)}(k, j, y, z)];$$

π_2 is here a recursive pairing function as in [2, § 1]. Then, for all 5-tuples $\langle f, n, k, j, y\rangle$, we have $\delta\varphi^3_{\zeta_2(f,n)}(\pi_2(k, j), y, z) =$ either N or $D_{\chi(k,j)}$ according as $(\exists w)T_4(f, n, k, j, y, w)$ or $(\forall w) \neg T_4(f, n, k, j, y, w)$. It follows that, for every pair of numbers $\langle f, n\rangle$, we have

$$\begin{aligned}(\exists k)(\forall j)(\exists y)(\forall w) \neg T_4(f, n, k, j, y, w) \\ \Leftrightarrow [(\forall k)(\forall j)(\forall p)(\forall q)(\forall y)(p \neq k \\ \Rightarrow \delta\varphi^3_{\zeta_2(f,n)}(\pi_2(p, q), y, z) \neq D_{\chi(k,j)}) \text{ and} \\ (\exists k)(\forall j)(\exists y)[\delta\varphi^3_{\zeta_2(f,n)}(\pi_2(k, j), y, z) = D_{\chi(k,j)}]].\end{aligned}$$

Therefore if we define $\psi_5(x, y)$ by $\psi_5 = \omega\zeta_1\zeta_2$, we obtain the equivalence: $n \in$ the f-th $\sum^0_4$ set $\Leftrightarrow \psi_5(f, n) \in G(\mathscr{F}_6)$. Thus $G(\mathscr{F}_6)$ is $\sum^0_4$ complete. Q.E.D.

PROPOSITION 11. *Let* $\mathscr{F}_7 = \{W^{\boldsymbol{C}}_e \mid \{W_j \mid W_j \text{ is cofinite}\} \subseteq W^{\boldsymbol{C}}_e\}$. *Then* $\mathscr{F}_7$ *is an r.e. family and* $G(\mathscr{F}_7)$ *is* $\prod^0_4$ *complete.*

PROOF. The class $\{W_j \mid W_j$ is cofinite$\}$ is, of course, r.e.; say, $\{W_j \mid W_j$ is cofinite$\} = W^{\boldsymbol{C}}_{e_0}$. Hence $\mathscr{F}_7$ is an r.e. family, since

$$\mathfrak{F}_7 = \{W^{\boldsymbol{C}}_e \mid (\exists f)[W^{\boldsymbol{C}}_e = W^{\boldsymbol{C}}_f \cup W^{\boldsymbol{C}}_{e_0}]\}.$$

Next, observe that

$$n \in G(\mathscr{F}_7) \Leftrightarrow (\forall h)[W_h \text{ is cofinite} \Rightarrow (\exists l)[l \in W_n \text{ and } (W_l = W_h)]].$$

But $W_l = W_h$ is a $\prod^0_2$ predicate of l and h; and the assertion that W_h is cofinite is ([4]) a $\sum^0_3$ predicate of h. Hence, by the usual prenex operations (carried out with several alternations of priority between the antecedent and the consequent inside the main quantifier), we see that $n \in G(\mathscr{F}_7)$ is a $\prod^0_4$ predicate of n. To show $\prod^0_4$ completeness of $G(\mathscr{F}_7)$, we construct a recursive function $\psi_6(x, y)$ with the property that

$$\begin{aligned}(\forall f)(\forall n)[(\forall u)(\exists v)(\forall w)(\exists z)T_4(f, n, u, v, w, z) \\ \Leftrightarrow (\forall j)(\exists l)(\exists k)[\delta\varphi^3_{\psi_6(f,n)}(l, k, y) = N - D_j]].\end{aligned}$$

To this end we define a partial recursive function ζ as follows:

$$\zeta^s_{f,n}(l, k, y) \simeq \begin{cases} 0, \text{ if } (\forall w \leqq y)(\exists z \leqq s)T_4(f, n, l, k, w, z) \text{ and} \\ y \in N - D_l, \\ \text{undefined, otherwise;} \end{cases}$$

$$\zeta_{f,n} =_{df} \bigcup_{s=0}^{\infty} \zeta^s_{f,n}.$$

$\zeta_{f,n}$ is partial recursive uniformly in f and n; so let $\psi_6(x, y)$ be a recursive function such that

$$(\forall f)(\forall n)[\varphi^3_{\psi_6(f,n)} \simeq \zeta_{f,n}].$$

Now, it is easily seen from the definition of ζ that for each pair $\langle l, k\rangle$ we have either

$$\delta\varphi^3_{\psi_6(f,n)}(l,k,y) = N - D_l \text{ or } \delta\varphi^3_{\psi_6(f,n)}(l,k,y) = \text{a}$$

finite set, according as $(\forall w)(\exists z)T_4(f,n,l,k,w,z)$ or $\neg(\forall w)(\exists z)T_4(f,n,l,k,w,z)$. It readily follows that for every pair $\langle f, n\rangle$ we have

$$(\forall u)(\exists v)(\forall w)(\exists z)T_4(f,n,u,v,w,z) \Leftrightarrow (\forall j)(\exists l)(\exists k)[\delta\varphi^3_{\psi_6(f,n)}(l,k,y) = N - D_j].$$

Hence $n \in$ the f-th $\prod^0_4$ set $\Leftrightarrow \omega(\zeta_1(\psi_6(f,n))) \in G(\mathscr{F}_7)$, and $G(\mathscr{F}_7)$ is $\prod^0_4$ complete. Q.E.D.

PROPOSITION 12. *Let* $\mathscr{F}_8 = \{W^C_e \mid (\exists n)[\{W_j \mid W_j$ *is a relatively cofinite subset of* $P_n\} \subseteq W^C_e]\}$. *Then* $\mathscr{F}_8$ *is an r.e. family and* $G(\mathscr{F}_8)$ *is* $\sum^0_5$ *complete.*

PROOF. Let $\bar{\chi}(x,y)$ be a recursive function with $\bar{\chi}(n,y)$ one-to-one for each number n, such that $(\forall n)[\{P_n - D_{\bar{\chi}(n,y)} \mid y \in N\} = \{W_j \mid W_j$ is a relatively cofinite subset of $P_n\}]$. It follows that $\mathscr{F}_8$ is r.e., since

$$\mathscr{F}_8 = \{W^C_e \mid (\exists f)(\exists n)[W^C_e = W^C_f \cup \{P_n - D_{\bar{\chi}(n,y)} \mid y \in N\}]\}.$$

Next, since $n \in G(\mathscr{F}_8) \Leftrightarrow (\exists k)(\forall j)(\exists m)[m \in W_n$ and $P_k - D_{\bar{\chi}(k,j)} = W_m]$, and since the predicate $P_k - D_{\bar{\chi}(k,j)} = W_m$ is a $\prod^0_2$ predicate of k, j and m, we have that $G(\mathscr{F}_8)$ is $\sum^0_5$. In the same way in which we showed $G(\mathscr{F}_6)$ to be $\sum^0_4$ complete by modifying our proof of the $\prod^0_3$ completeness of $G(\mathscr{F}_5)$, we shall now show $G(\mathscr{F}_8)$ to be $\sum^0_5$ complete by making appropriate alterations in our proof of the $\prod^0_4$ completeness of $G(\mathscr{F}_7)$. First, we define a partial recursive function γ by stipulating that

$$\gamma^s_{f,n}(l,k,j,y) \simeq \begin{cases} 0, \text{ if } y \in P_l - D_{\bar{\chi}(l,k)} \text{ and} \\ (\forall w \leqq y)(\exists z \leqq s)T_5(f,n,l,k,j,w,z), \\ \text{undefined, otherwise;} \end{cases}$$

$$\gamma_{f,n} =_{df} \bigcup_{s=0}^{\infty} \gamma^s_{f,n}.$$

$\gamma_{f,n}$, as so defined, is partial recursive uniformly in f and n; hence there is a recursive function $\beta(x,y)$ such that $(\forall f)(\forall n)[\gamma_{f,n} \simeq \varphi^4_{\beta(f,n)}]$. Let ζ_3 be a recursive function such that $(\forall f)(\forall n)[\varphi^3_{\zeta_3(f,n)}(\pi_2(l,k),j,y) \simeq \varphi^4_{\beta(f,n)}(l,k,j,y)]$. Then, for all 5-tuples $\langle f,n,l,k,j\rangle$, we have that $\delta\varphi^3_{\zeta_3(f,n)}(\pi_2(l,k),j,y) =$ either $P_l - D_{\bar{\chi}(l,k)}$ or a finite set, according as $(\forall w)(\exists z)T_5(f,n,l,k,j,w,z)$ or not. It follows that, for every pair $\langle f,n\rangle$ of numbers, we have

$$(\exists l)(\forall k)(\exists j)(\forall w)(\exists z)T_5(f,n,l,k,j,w,z) \Leftrightarrow (\exists l)(\forall k)(\exists j)[\delta\varphi^3_{\zeta_3(f,n)}(\pi_2(l,k),j,y) = P_l - D_{\bar{\chi}(l,k)}].$$

So, if we set $\psi_7 \underset{df}{=} \omega\zeta_1\zeta_3$ then

$$n \in \text{the } f\text{-th } \textstyle\sum_5^0 \text{ set} \Leftrightarrow \psi_7(f, n) \in G(\mathscr{F}_8).$$

Thus $G(\mathscr{F}_8)$ is $\sum_5^0$ complete. Q.E.D.

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Dept. of Mathematics
The University of Illinois at Urbana
URBANA, Ill. 61801
U.S.A.