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AN ANALYTIC CONSTRUCTION OF DEGENERATING CURVES OVER COMPLETE LOCAL RINGS

by

David Mumford

This is the first half of a 2 part paper, the first of which deals with the construction of curves and the second with abelian varieties. The idea of investigating the p-adic analogs of classical uniformizations of curves and abelian varieties is due to John Tate. In a very beautiful and influential piece of unpublished work, he showed that if K is a complete non-Archimedean valued field, and E is an elliptic curve over K *whose j-invariant is not an integer*, then E can be analytically uniformized. This uniformization is *not* a holomorphic map:

$$\pi : \boldsymbol{A}_K^1 \to E$$

generalizing the universal covering space

$$\pi : \boldsymbol{C} \to E \ (= \text{closed points of an elliptic curve over } \boldsymbol{C}),$$

but instead is a holomorphic map:

$$\pi_2 : \boldsymbol{A}_K^1 - \{0\} \to E$$

generalizing an infinite cyclic covering π_2 over $\boldsymbol{C}$:

$$\begin{tikzcd} & C^* \arrow[dr, "\pi_2"] & \\ C \arrow[ur, "\pi_1"] \arrow[rr, "\pi"'] & & E \end{tikzcd}$$

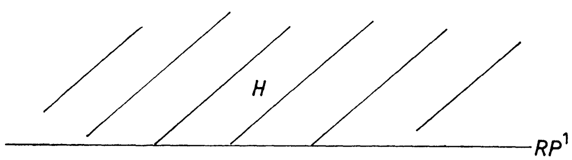
$$\pi_1(z) = e^{2\pi i(z/\omega_1)}$$

ω_1 one of the 2 periods of E. Here you can take holomorphic map to mean holomorphic in the sense of the non-Archimedean function theory of Grauert and Remmert [G-R]. But the uniformization π_2 is more simply expressed by embedding E in $\boldsymbol{P}_K^2$ and defining the three homogeneous coordinates of $\pi(z)$ by three everywhere convergent Laurent series.

The purpose of my work is 2-fold: The first is to generalize Tate's results both to curves of higher genus and to abelian varieties. This gives a very useful tool for investigating the structures at infinity of the moduli spaces. It gives for instance an abstract analog of the Fourier series development of modular forms. Our work here overlaps to some extent with the work of Morikawa [Mo] and McCabe [Mc] generalizing Tate's uniformization to higher-dimensional abelian varieties. The second purpose is to understand the *algebraic* meaning of these uniformizations. For instance, in Tate's example, π defines not only a holomorphic map but also a formal morphism from the Néron model of $\boldsymbol{G}_m$ to the Néron model of E over the ring of integers $A \subset K$. And from an algebraic point of view, it is very unnatural to uniformize only curves over the quotient fields K of complete one-dimensional rings A: one wants to allow A to be a higher-dimensional local ring as well, (this is essential in the applications to moduli for instance). But when $\dim A > 1$, there is no longer any satisfactory theory of holomorphic functions and spaces over K.

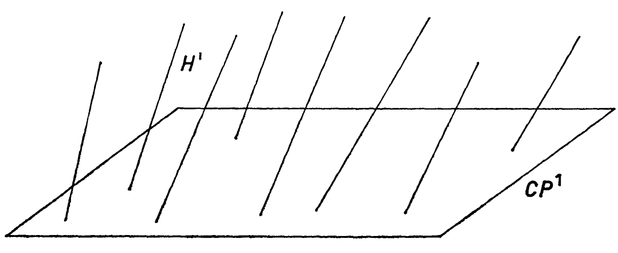
In this introduction, I would first like to explain (in the case K is a discretely-valued complete local field) what to expect for curves of higher genus. We can do this by carrying a bit further the interesting analogies between the real, complex and p-adic structures of $PGL(2)$ as developed recently by Bruhat, Tits and Serre:

(A) *real case*: $PSL(2, \boldsymbol{R})$ acts *isometrically* and transitively on the upper $\frac{1}{2}$-plane and the boundary can be identified with $\boldsymbol{RP}^1$ (the real line, plus ∞):



$$z \mapsto \frac{az+b}{cz+d} \qquad ds^2 = \frac{1}{y^2}(dx^2+dy^2)$$

(B) *complex case*: $PGL(2, \boldsymbol{C})$ acts isometrically and transitively on the upper $\frac{1}{2}$-space [1] H' and the boundary can be identified with $\boldsymbol{CP}^1$:



$$\text{coordinates } z \in \boldsymbol{C},\ x \in \boldsymbol{R},\ x \geqq 0$$

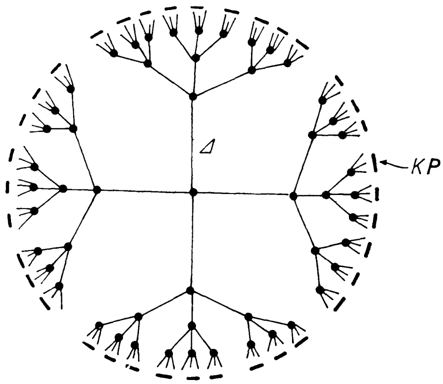
$$\text{metric } ds^2 = \frac{1}{x^2}(|dz|^2+dx^2)$$

[1] The action of $SL(2, \boldsymbol{C})$ on H' is given by:

$$(z, x) \mapsto \left(\frac{\overline{(cz+d)}(az+b)+a\bar{c}x^2}{|cz+d|^2+|c|^2x^2}, \frac{x}{|cz+d|^2+|c|^2x^2}\right)$$

$(z \in \boldsymbol{C}, x \in \boldsymbol{R}^+)$.

(C) *p-adic case*: $PGL(2, K)$ acts isometrically and transitively on the tree Δ of Bruhat-Tits, (whose vertices correspond to the subgroups $gPGL(2, A)g^{-1}$, and whose edges have length 1 and correspond to the subgroups gBg^{-1}, $B = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in A, c \in m, ad \notin m\right\}$ modulo A^*) and the set of whose ends can be identified with KP^1 (for details, cf. § 1, below and Serre [S]):



[the case card $(k) = 3$].

In Δ, for any vertex v the set of edges meeting v is naturally isomorphic to $k\boldsymbol{P}^1$, the isomorphism being canonical up to an element of $PGL(2, k)$ (where $k = A/m$).

In the first case, if $\Gamma \subset PSL(2, \boldsymbol{R})$ is a discrete subgroup with no elements of finite order such that $PSL(2, \boldsymbol{R})/\Gamma$ is compact, we obtain Koebe's uniformization

$$H \to H/\Gamma = X$$

of an arbitrary compact Riemann surface X of genus $g \geqq 2$.

In the second, if $\Gamma \subset PGL(2, \boldsymbol{C})$ is a discrete subgroup which acts discontinuously at at least one point of $\boldsymbol{CP}^1$ (a *Kleinian group*) and which moreover is free with n generators and has no unipotent elements in it, then according to a Theorem of Maskit [Ma], Γ is a so-called Schottky group, i.e. if Ω = set of points of $\boldsymbol{CP}^1$ where Γ acts discontinuously, then Ω is connected and up to homeomorphism we get a uniformization:

$$\begin{array}{ccccc} (H' \cup \Omega) & \to & (H' \cup \Omega)/\Gamma & \underset{\text{homeo}}{\cong} & \text{solid torus with } n \text{ handles} \\ \cup & & \cup & & \cup \\ \Omega & \xrightarrow{\pi} & \Omega/\Gamma & \underset{\text{homeo}}{\cong} & \{\text{boundary, a surface of genus } n\} \end{array}$$

In particular Ω/Γ is a compact Riemann surface of genus n and for a suitable standard basis $a_1, \cdots, a_n, b_1, \cdots, b_n$ of $\pi_1(\Omega/\Gamma)$, π is the partial covering corresponding to the subgroup

$$N \subset \pi_1(\Omega/\Gamma)$$
$$N = \text{least normal subgroup containing } a_1, \cdots, a_n.$$

The uniformization π is what is called in the classical literature the Schottky uniformization. It is the one which has a p-adic analog.

In the third case, let $\Gamma \subset PGL(2, K)$ be any discrete subgroup consisting entirely of hyperbolic elements [2]. Then Ihara [I] proved that Γ is free: let Γ have n generators. Again, let Ω = set of closed points of $\boldsymbol{P}_K^1$ where Γ acts discontinuously (equivalently, Ω is the set of points which are not limits of fixed points of elements of Γ).

Then I claim that there is a curve C of genus n and a holomorphic isomorphism:

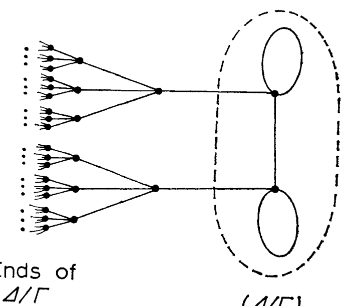
$$\pi : \Omega/\Gamma \overset{\approx}{\to} C.$$

Moreover Δ/Γ has a very nice interpretation as a graph of the specialization of C over the ring A. In fact

a) there will be a smallest subgraph $(\Delta/\Gamma)_0 \subset \Delta/\Gamma$ such that

$$\pi_1((\Delta/\Gamma)_0) \overset{\sim}{\to} \pi_1(\Delta/\Gamma) \text{ and } (\Delta/\Gamma)_0$$

will be finite:



b) C will have a canonical specialization $\bar{C}$ over A, where $\bar{C}$ is a singular curve of arithmetic genus n made up from copies of $\boldsymbol{P}_K^1$ with a finite number of distinct pairs of k-rational points identified to form ordinary double points. Such a curve $\bar{C}$ will be called a *k-split degenerate curve of genus n.*

c) $C(K)$, the set of K-rational points of C, will be naturally isomorphic to the set of ends of Δ/Γ; $\bar{C}(k)$, the set of k-rational points of $\bar{C}$, will be naturally isomorphic to the set of edges of Δ/Γ that meet vertices of

[2] If K is locally compact, this is equivalent to asking simply that Γ has no elements of finite order, since suitable powers of a non-hyperbolic element not of finite order must converge to the identity.

$(\Delta/\Gamma)_0$ (so that the components of $\bar{C}$ correspond to the edges of Δ/Γ meeting a fixed vertex of $(\Delta/\Gamma)_0$ and the double points of $\bar{C}$ correspond to the edges of $(\Delta/\Gamma)_0$); and finally the specialization map

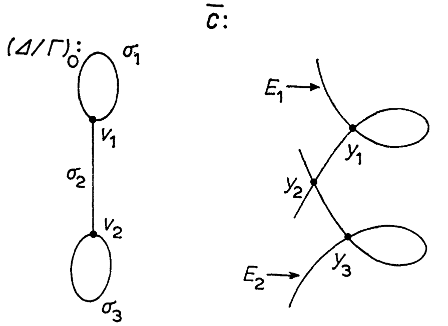
$$C(K) \to \bar{C}(k)$$

is equal, under the above identifications, to the map:

$$(\text{Ends of } \Delta/\Gamma) \to \begin{pmatrix} \text{edges of } \Delta/\Gamma \\ \text{meeting}(\Delta/\Gamma)_0 \end{pmatrix}$$

which takes an end to the last edge in the shortest path from that end to $(\Delta/\Gamma)_0$.

Examples. We have illustrated a case where the genus is 2, $\bar{C}$ has 2 components, each with one double point and meeting each other once:



$E_i \leftrightarrow v_i$
$y_i \leftrightarrow \sigma_i$
$\bar{C} = E_1 \cup E_2$

Because all the curves C which we construct have property (b), we refer to them as *degenerating curves*. Our main theorem implies that every such degenerating curve C has a unique analytic uniformization $\pi : \Omega/\Gamma \xrightarrow{\sim} C$.

Next I would like to give an idea of how I intend to construct *algebraic* objects which imply the existence of the analytic uniformization π, which express the way the analytic map specializes over A, and which will generalize to the case $\dim A > 1$. Given a discretely-valued complete local field K, one has:

a) the category of holomorphic spaces X over K, in the sense e.g. of Grauert and Remmert [G-R],

b) the category of formal schemes $\mathscr{X}$ over A, locally of topological finite type over A, with $m\mathcal{O}_{\mathscr{X}}$ as a defining sheaf of ideals.

There is a functor:

$$\mathscr{X} \to \mathscr{X}_{an}$$

from the category of formal schemes to the category of holomorphic spaces given as follows:

i) as a point set $\mathscr{X}_{an} \cong$ set of reduced irreducible formal subschemes $Z \subset \mathscr{X}$ such that Z is finite over A, but $Z \not\subset$ the closed fibre $\mathscr{X}_0$;

ii) if $\emptyset : \mathscr{X}_{an} \to \text{Max}\,(\mathscr{X}) =$ (closed points of $\mathscr{X}$) is the specialization map $Z \mapsto Z \cap \mathscr{X}_0$, then for all $U \subset \mathscr{X}$ affine open, $\emptyset^{-1}(\text{Max}\,U) = V$ is an affinoid subdomain of $\mathscr{X}_{an}$ with affinoid ring $\Gamma(U, \mathcal{O}_{\mathscr{X}})$.

According to results of Raynaud, the category of holomorphic spaces looks like a kind of localization of the category of formal schemes with respect to blowings-up of subschemes concentrated in the closed fibre. In fact, he has proven that the category of holomorphic spaces admitting a finite covering by affinoids is equivalent to this localization of the category of formal schemes of finite type. What happens in our concrete situation is that the holomorphic spaces Ω and C both have *canonical* liftings into the category of formal schemes and the analytic map $\pi : \Omega \to C$ is induced by a formal morphism. For the uniformization of abelian varieties, discussed in the 2nd paper of this series, the lifting turns out not to be canonical; however, a whole class of such liftings can be singled out, which is non-empty and for which π lifts too. Thus the whole situation is lifted into the category of formal schemes where it can be generalized to higher-dimensional base rings A. Let me illustrate this lifting in Tate's original case of an elliptic curve. First of all, what formal scheme over A gives rise to the holomorphic space $\boldsymbol{A}^1_K - \{0\} = \boldsymbol{G}_{m,K}$? If we take the formal completion of the algebraic group $\boldsymbol{G}_m$ over Spec (A), the holomorphic space that we get is only the unit circle:

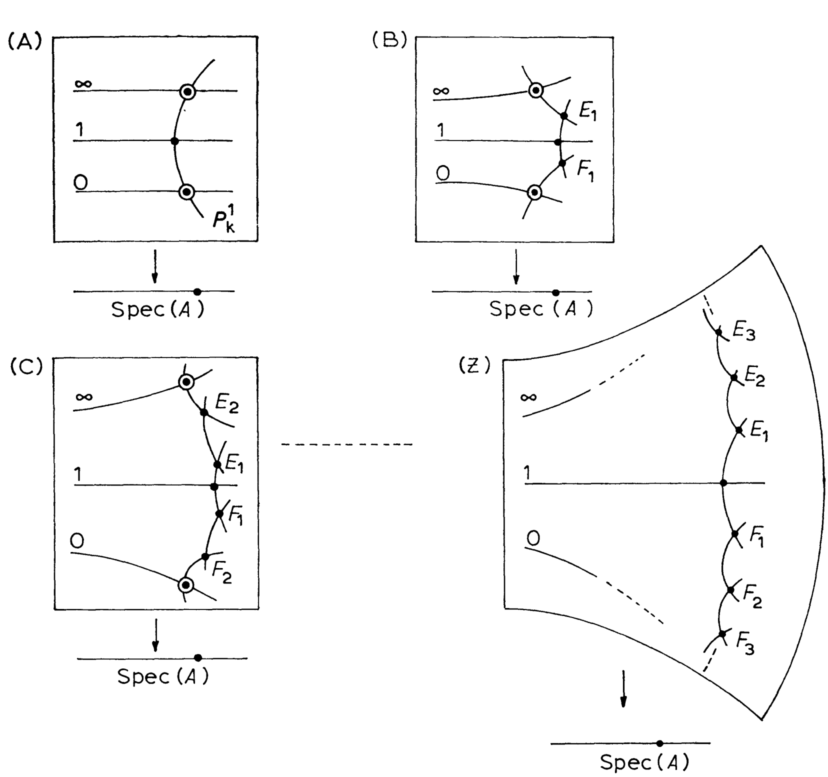
$$T \subset \boldsymbol{G}_{m,K}$$
$$T = \{z \mid |z| = 1\}.$$

If we take formal completion of Raynaud's 'Néron model' of $\boldsymbol{G}_m$ over Spec (A) (cf. [R]), we get the subgroup:

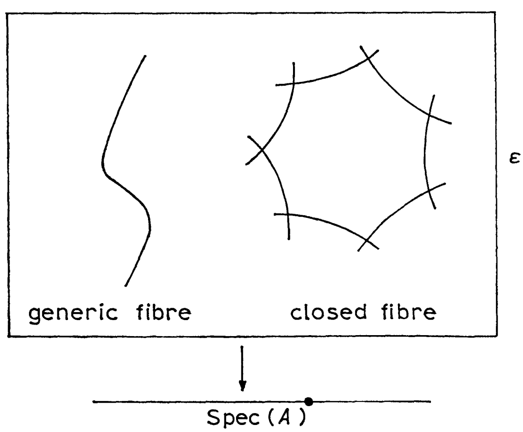
$$\bigcup_{n=-\infty}^{+\infty} \pi^n T \subset \boldsymbol{G}_{m,K}$$
$$(\pi) = \text{max. ideal of } A.$$

To get the full $\boldsymbol{G}_{m,K}$ start with $\boldsymbol{P}^1 \times \text{Spec}\,(A)$. Blow up (0), (∞) in the closed fibre $\boldsymbol{P}^1_K$; then blow up again the points where the 0-section and ∞-section meet the closed fibre; repeat infinitely often. (See figure on next page.)

The result is a scheme P_∞, only locally of finite type over Spec (A). If we omit the double points of the closed fibre, we get Raynaud's 'Néron model'. However, if we take the whole affair, the holomorphic space associated to its formal completion is $\boldsymbol{G}_{m,K}$. On the other hand, the Néron model of E over Spec (A) will have a canonical 'compactifica-



tion': it can be embedded in a unique normal scheme $\mathscr{E}$ proper over Spec (A) by adding a finite set of points. It will look like this (possibly, after replacing K by a suitable quadratic extension):



$\mathscr{E}$ will in fact be regular. Finally, the analytic uniformization which we denoted π_2 will come from a formal étale morphism from P_∞ to $\mathscr{E}$

which simply wraps the infinite chain in the closed fibre of P_∞ around and around the polygon which is the closed fibre of $\mathscr{E}$.

We can now state fully our main result:

For every Schottky group $\Gamma \subset PGL(2, K)$, there is a canonical formal scheme $\mathscr{P}$ over A on which Γ acts freely and whose associated holomorphic space is the open set $\Omega \subset \boldsymbol{P}^1_K$. There is a one-one correspondence between a) conjugacy classes of Schottky groups Γ, and b) isomorphism classes of curves C over K which are the generic fibres of normal schemes $\mathscr{C}$ over A whose closed fibre $\bar{C}$ is a k-split degenerate curve, set up by requiring that $\mathscr{P}/\Gamma$ is formally isomorphic to $\mathscr{C}$.

Some notation

$\boldsymbol{P}^1_K$ = projective line over K

$K\boldsymbol{P}^1$ = K-rational points of $\boldsymbol{P}^1_K = K+K-(0, 0)/K^*$

$\langle u, v\rangle$ = module generated by u, v

$R(X)$ = field of rational functions on an integral scheme X.

$\hat{X}$ = formal completion of a scheme X over a complete local ring A, along its closed fibre.

1. Trees

Let A be a complete integrally closed noetherian local ring, with quotient field K, maximal ideal m and residue field $k = A/m$. Let $S = \text{Spec}(A)$, $S_\eta = \text{Spec}(K)$ and $S_0 = \text{Spec}(k)$:

$$S_\eta \hookrightarrow S \hookleftarrow S_0.$$

We are interested in certain finitely generated subgroups of $PGL(2, K)$ that we will call Schottky groups. First of all define a morphism:

$$t : PGL(2) \to \boldsymbol{A}^1$$

by

$$t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{(a+d)^2}{ad-bc}$$

Here and below we will describe elements of $PGL(2)$ by 2×2 matrices, considered modulo multiplication by a scalar, without further comment.

LEMMA (1.1). *Let $\gamma \in PGL(2, K)$. Then $t^{-1}(\gamma) \in m$ if and only if γ can be represented*:

$$A\cdot\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}\cdot A^{-1}, \quad \mu \in m.$$

PROOF. On the one hand:

$$t^{-1}\left(A\cdot\begin{pmatrix}\mu & 0\\ 0 & 1\end{pmatrix}\cdot A^{-1}\right) = \frac{\mu}{(\mu+1)^2}\in m.$$

Conversely, suppose $t^{-1}(\gamma) = v \in m$, and let the matrix C represent γ. Then $\det C = v\cdot(\mathrm{Tr}\, C)^2$ and the characteristic polynomial of C is

$$X^2-(\mathrm{Tr}\, C)\cdot X+v\cdot(\mathrm{Tr}\, C)^2 = 0.$$

By Hensel's lemma, this has 2 distinct roots in K of the form

$$X_1 = u\cdot \mathrm{Tr}\, C$$
$$X_2 = v\cdot u^{-1}\cdot \mathrm{Tr}\, C$$
where u = a unit in A.

Then γ is also represented by the matrix $C' = C/u\cdot \mathrm{Tr}\, C$ with eigenvalues 1 and $v/u^2 \in m$. Therefore C' has the required form.

Q.E.D.

DEFINITION (1.2). The elements $\gamma\in PGL(2, K)$ such that $t^{-1}(\gamma)\in m$ will be called *hyperbolic*.

From the lemma it follows immediately that if γ is hyperbolic, then γ as an automorphism of $\boldsymbol{P}^1_K$ has 2 distinct fixed points P and Q, both rational over K, and such that the differential $d\gamma|_P$ = mult. by μ in T_P (the tangent space to $\boldsymbol{P}^1_K$ at P), $\mu\in m$, while $d\gamma|_Q$ = mult. by μ^{-1}; P is called the *attractive fixed point* of γ and Q the *repulsive fixed point*.

DEFINITION (1.3). A *Schottky group* $\Gamma\subset PGL(2, K)$ is a finitely generated subgroup such that every $\gamma\in\Gamma$, $\gamma\neq e$, is hyperbolic.

These are probably the most natural class of groups to look at. However, there is a particular type which is easier to prove theorems about and which include *all* Schottky groups in the case $\dim A = 1$:

DEFINITION (1.4). *A flat Schottky group* $\Gamma\subset PGL(2, K)$ has the extra property that if $\Sigma\subset K\boldsymbol{P}^1$ is the set of fixed points of the elements $\gamma\in\Gamma$, then for any $P_1, P_2, P_3, P_4\in\Sigma$, $R(P_1, P_2; P_3, P_4)$ or its inverse is in A, i.e. the cross-ratio R defines a morphism from S to $\boldsymbol{P}^1$.

The construction of flat Schottky groups is not so easy and we postpone this until § 4. For the time being, we simply assume that one is given.

The structure of $PGL(2, K)$ and of Γ is best displayed, following the method of Bruhat and Tits [B-T] by introducing:

$\Delta^{(0)} \cong$ {set of sub A-modules $M\subset K+K$, M free of rank 2, modulo the identification $M\sim\lambda\cdot M$, $\lambda\in K^*$, (the image $\{M\}$ in $\Delta^{(0)}$ of a module M will be called the *class* of M)}

$\cong$ {set of schemes P/S with generic fibre $\boldsymbol{P}^1_K$, such that $P\cong\boldsymbol{P}^1_S$, modulo isomorphism}.

These sets will be identified by the map

$$M \mapsto P = \text{Proj (Symmetric } A\text{-algebra on Hom } (M, A)).$$

This is easily seen to be a bijection under which the set of A-valued points of P equals the set of elements $x \in M - mM$, modulo A^*. Intuitively, P is the scheme of one-dimensional subspaces of the rank 2 vector bundle M.

DEFINITION (1.5). If $\{M\} \in \Delta^{(0)}$, we denote the corresponding scheme P/S by $\mathbf{P}(M)$.

Note that $PGL(2, K)$ acts on $\Delta^{(0)}$:

$\forall X \in GL(2, K), \forall M \subset K+K$,
let $X(M) = \{X \cdot x | x \in M\}$.
Then the class $\{X(M)\}$ depends only on the image $\{X\}$ of X in $PGL(2, K)$ and on the class $\{M\}$.

The stabilizer of the module $A+A$ is:

$PGL(2, A)$ = elements of $PGL(2, K)$ represented by matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in A$, $ad - bc \in A^*$,

and the stabilizers of the other modules M are conjugates of $x \cdot PGL(2, A) \cdot x^{-1}$ in $PGL(2, K)$.

Moreover $PGL(2, K)$ acts transitively on $\Delta^{(0)}$, so $\Delta^{(0)}$ can be naturally identified with the coset space $PGL(2, K)/PGL(2, A)$.

Less obvious is the fact that any 3 distinct points $x_1, x_2, x_3 \in K\mathbf{P}^1$ determined canonically an element of $\Delta^{(0)}$: let $w_1, w_2, w_3 \in K+K$ be homogeneous coordinates for x_1, x_2, x_3. Then there is a linear equation: $a_1 w_1 + a_2 w_2 + a_3 w_3 = 0$, unique up to scalar. Let $M = \sum_{i=1}^{3} A \cdot a_i w_i$. The class of multiples $\{M\}$ of M is determined by the x_i alone. We will write this class as $\{M(x_1, x_2, x_3)\}$.

Unlike the case where $\dim A = 1$, the full set $\Delta^{(0)}$ is rather unmanageable. We need to introduce the concept:

DEFINITION (1.6). $\{M_1\}, \{M_2\} \in \Delta^{(0)}$ are *compatible* if there exists a basis u, v of M_1, and elements $\lambda \in K^*$, $\alpha \in A$ such that λu, $\lambda \alpha v$ is a basis of M_2, (M_i representatives of $\{M_i\}$).

It is easy to check that this definition is symmetric and that the principal ideal (α) is uniquely determined by $\{M_1\}$ and $\{M_2\}$. Since (α) measures the 'distance' of $\{M_1\}$ from $\{M_2\}$, we write:

$$(\alpha) = \rho(\{M_1\}, \{M_2\}).$$

Moreover, when $\dim A = 1$, every pair $\{M_1\}, \{M_2\}$ is compatible. If M_i' are representatives of the classes $\{M_i\}$ such that

$$M_1' \supset M_2' \supset \alpha \cdot M_1',$$

we call M_1', M_2' *representatives in standard position.*

Now then, let

$$\Delta_\Gamma^{(0)} = \text{set of classes } \{M(x_1, x_2, x_3)\}, \text{ where } x_1, x_2, x_3 \in \Sigma,$$
$$\Sigma = \text{set of fixed points of elements of } \Gamma.$$

The flatness of the Schottky group Γ is obviously equivalent to the property:

$$* \begin{cases} \forall x_1, x_2, x_3, x_4 \in \Sigma, \text{ these points have homogeneous coordinates} \\ w_1, w_2, w_3, w_4 \in K+K \text{ such that} \\ \qquad w_3 = w_1 + w_2 \\ \qquad w_4 = a_1 w_1 + a_2 w_2, \qquad a_i \in A \\ \text{and either } a_1 \text{ or } a_2 \text{ is not in } m. \end{cases}$$

This now gives us:

PROPOSITION (1.7). *Any 2 classes* $\{M_1\}, \{M_2\} \in \Delta_\Gamma^{(0)}$ *are compatible.*

PROOF. First note the

LEMMA (1.8). *If* $x_1, x_2, x_3, x_4 \in \boldsymbol{P}_K^1$ *have property* $(*)$, *then for some* $i, j \in \{1, 2, 3\}$, *with* $i \neq j$, $\{M(x_1, x_2, x_3)\} = \{M(x_i, x_j, x_4)\}$.

PROOF. Let w_i be coordinates for x_i as in $(*)$. Then if a_1 and $a_2 \notin m$, one checks that $M(x_1, x_2, x_3) = M(x_1, x_2, x_4)$; if $a_1 \notin m$, $a_2 \in m$, then $M(x_1, x_2, x_3) = M(x_2, x_3, x_4)$; and if $a_1 \in m$, $a_2 \notin m$, then

$$M(x_1, x_2, x_3) = M(x_1, x_3, x_4).$$

Now let $\{M_1\} = M(x_1, x_2, x_3)$, $\{M_2\} = M(y_1, y_2, y_3)$. Choose coordinates w_i for x_i and u_i for y_i such that

a) $M_1 = A \cdot w_1 + A \cdot w_2, w_3 = w_1 + w_2$,

b) $u_i = a_i w_1 + b_i w_2$, where $a_i, b_i \in A$ but if $a_i \in m$, then $b_i \notin m$.

Next, if the ratios $a_i : b_i \bmod m$ in $k\boldsymbol{P}^1$ are all distinct, one checks immediately that the u_i are related by $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$ where $\lambda_i \in A$, $\lambda_i \notin m$. This implies that

$$M_2 = (\text{module generated by the } u_i) = M_1,$$

hence M_1 and M_2 are obviously compatible. Ncw if the ratios $a_i : b_i \bmod m$ are not all distinct, then at least one of the triples $(0:1)$, $(1:0)$, $(1:1)$ is different from all three ratios $a_i : b_i \bmod m$. Permuting the three w_i's, we may as well assume that $(1:0)$ does not cocur, i.e. $b_i \notin m$ for all i. Multiplying u_i by a unit, we can normalize it so that now:

b′) $u_i = a_i w_1 + w_2$.

Now by the lemma, $\{M_2\} = \{M(y_i, y_j, x_1)\}$ for some i and j. The linear equation relating u_i, u_j and w_1 is:

$$u_i - a_i w_1 = u_j - a_j w_1.$$

Therefore $M(y_i, y_j, x_1)$ is the module generated by u_i, u_j and $(a_i - a_j)w_1$. Thus u_i and w_1 are a basis of M_1 and u_i and $(a_i - a_j)w_1$ are a basis of M_2. Hence $\{M_1\}$ and $\{M_2\}$ are compatible.

Q.E.D.

I claim that for any 3 compatible classes of modules, there is a multiplicative triangle 'inequality' relating their 'distances' from each other:

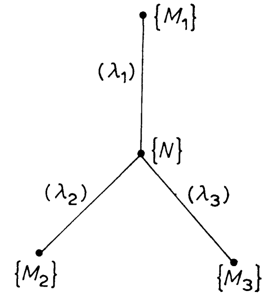
PROPOSITION (1.9). *Let $\{M_1\}, \{M_2\}, \{M_3\} \in \Delta^{(0)}$ be distinct but compatible. Let $(\alpha_{ij}) = \rho(\{M_i\}, \{M_j\})$ and let*

$$M_1 \supset M_2 \supset \alpha_{12} M_1$$
$$M_1 \supset M_3 \supset \alpha_{13} M_1$$

be representatives in standard position. Then if $N = M_2 + M_3$, there exist $u, v \in M_1$ and $\lambda_1, \lambda_2, \lambda_3 \in A$ such that:

$$\begin{aligned} M_1 &= \langle u, v\rangle \\ N &= \langle u, \lambda_1 v\rangle \\ M_2 &= \langle u, \lambda_1 \lambda_2 v\rangle \\ M_3 &= \langle u + \lambda_1 v, \lambda_1 \lambda_3 v\rangle \\ (\alpha_{ij}) &= (\lambda_i \lambda_j). \end{aligned}$$

In particular, for all permutations i, j, k of 1, 2, 3, $\alpha_{ij} | \alpha_{ik}\alpha_{jk}$.



CLUMSY PROOF. First choose $u \in M_2$ such that $u \notin m \cdot M_1$. Secondly choose $\bar{v} \in M_1/mM_1$ such that $\bar{v}$ is not in either of the 1-dimensional subspaces $M_2/M_2 \cap mM_1$ or $M_3/M_3 \cap mM_1$ of M_1/mM_1. Lift $\bar{v}$ to

$v \in M_1$. Then first of all $\bar{u}$ and $\bar{v}$ generate M_1/mM_1, hence u and v generate M_1. Secondly u and $\alpha_{12}v$ lie in M_2 and since M_1 and M_2 are in standard position, it is easy to see that they must generate M_2. Thirdly, for some $\lambda_1 \in A$, $u+\lambda_1 v \in M_3$. Then since M_1 and M_3 are in standard position, $u+\lambda_1 v$ and $\alpha_{13}v$ must generate M_3. Now use the fact that M_2 and M_3 are compatible: for some $\xi \in K^*$, $M_2 \supset \xi \cdot M_3$ and this pair is in standard position. Then

$$\xi \cdot (u+\lambda_1 v) \in M_2$$
$$\xi \cdot \alpha_{13} v \in M_2$$

and one of these is not in $m \cdot M_2$. This implies that $\xi \in A$, $\xi\lambda_1 = \zeta \cdot \alpha_{12}$, $\xi\alpha_{13} = \eta \cdot \alpha_{12}$, (where ζ and $\eta \in A$), and furthermore that either ξ, ζ or η is a unit. Firstly, suppose ξ is a unit but ζ is not. Note that we may replace u by $u' = u+\alpha_{12}v$ then u' and $\alpha_{12}v$ still generate M_2 and $u'+\lambda_1' v$ and $\alpha_{13}v$ generate M_3 where $\lambda_1' = \lambda_1-\alpha_{12}$. But then $\xi\lambda_1' = \zeta'\alpha_{12}$, where $\zeta' = \zeta-1$ is a unit. Therefore by suitable choice of u, we can assume that ζ is a unit. Secondly, suppose η is a unit but ζ is not. In this case, note that $u+\lambda_1' v$ and $\alpha_{13}v$ still generate M_3 where $\lambda_1' = \lambda_1+\alpha_{13}$. And then $\xi\lambda_1' = \xi\lambda_1+\xi\alpha_{13} = (\zeta+\eta)\alpha_{12} = \zeta'\alpha_{12}$ where ζ' is a unit. Thus we can always assume that ζ is a unit. Then if $\lambda_2 = \xi \cdot \zeta^{-1}$ and $\lambda_3 = \eta \cdot \zeta^{-1}$ it follows that $\alpha_{12} = \lambda_1\lambda_2$ and $\alpha_{13} = \lambda_1\lambda_3$ hence M_2 and M_3 are generated as required. It follows immediately that M_2+M_3 is generated by u and $\lambda_1 v$. To evaluate α_{23}, note that the 2 modules $M_2 \supset \lambda_2 M_3$ are in standard position (since $\lambda_2 u+\lambda_1\lambda_2 v$ is in $\lambda_2 M_3$ but not in mM_2) and that $\lambda_2 M_3$ is generated by $\lambda_2 u+\lambda_1\lambda_2 v$ and by $\lambda_2\lambda_3 u$ hence $(\lambda_2\lambda_3) = \rho(\{M_2\}, \{M_3\})$.

Q.E.D.

COROLLARY (1.10). *If* $M_1 \supset M_2 \supset \alpha_{12}M_1$, $M_1 \supset M_3 \supset \alpha_{13}M_1$ *are representatives of* 3 *compatible classes in standard position, then*

a) $M_2 \supset M_3 \Leftrightarrow \rho(\{M_1\}, \{M_3\}) = \rho(\{M_1\}, \{M_2\}) \cdot \rho(\{M_2\}, \{M_3\})$,

b) $M_1 = M_2+M_3 \Leftrightarrow \rho(\{M_2\}, \{M_3\})$
$= \rho(\{M_2\}, \{M_1\}) \cdot \rho(\{M_1\}, \{M_3\})$.

PROOF. In the notation of the proposition, both parts of (a) are equivalent to λ_2 being a unit; both parts of (b) are equivalent to λ_1 being a unit.

This Proposition motivates:

DEFINITION (1.11). A subset $\Delta_*^{(0)} \subset \Delta^{(0)}$ is *linked* if a) every pair of elements $\{M_1\}, \{M_2\} \in \Delta_*^{(0)}$ is compatible, b) for every triple $\{M_1\}$, $\{M_2\}, \{M_3\} \in \Delta_*^{(0)}$, if we pick representatives $M_1 \supset M_2$, $M_1 \supset M_3$ in standard position, then M_2+M_3 (which is a free A-module by the proposition) defines a class $\{M_2+M_3\}$ in $\Delta_*^{(0)}$.

We must check that $\Delta_\Gamma^{(0)}$ has both these fine properties:

THEOREM (1.12). *$\Delta_\Gamma^{(0)}$ is linked.*

PROOF. Suppose $\{M_i\} = \{M(x_i, y_i, z_i)\}$, $i = 1, 2, 3$, where all these points come from Σ. We saw above that all these classes are compatible. Choose representatives $M_1 \supset M_2, M_1 \supset M_3$ in standard position, and choose homogeneous coordinates $u_i, v_i, w_i \in M_i$ for x_i, y_i and z_i such that the linear relation $\alpha_i u_i + \beta_i v_i + \gamma_i w_i = 0$ has the property $\alpha_i, \beta_i, \gamma_i \in A^*$. Since $M_2 \not\subset mM_1$, one of u_2, v_2, or w_2 is in the set $M_1 - mM_1$. Renaming, we can assume $u_2 \notin mM_1$. Similarly, we can assume $u_3 \notin mM_1$. Next, the images $\bar{u}_1, \bar{v}_1, \bar{w}_1$ of u_1, v_1, w_1 in M_1/mM_1 are all distinct, so one of them is different from both $\bar{u}_2$ and $\bar{u}_3$. Renaming, we can assume $\bar{u}_1 \neq \bar{u}_2$ or $\bar{u}_3$. Let us construct a module in the class $\{M(x_1, x_2, x_3)\} \in \Delta_\Gamma^{(0)}$. We must find the linear equation relating u_1, u_2, u_3: since $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in M_1/mM_1$ are related by an equation $\bar{\alpha}\bar{u}_1 + \bar{\beta}\bar{u}_2 + \bar{u}_3 = 0$, where $\bar{\alpha}, \bar{\beta} \in A/m$, and $\bar{\beta} \neq 0$, it follows that u_1, u_2, u_3 are related by an equation $\alpha u_1 + \beta u_2 + u_3 = 0$, where $\alpha, \beta \in A$, $\beta \notin m$. Therefore

$$\langle u_2, u_3 \rangle \in \{M(x_1, x_2, x_3)\}.$$

On the other hand, if we choose generators $u, v \in M_1$ as in the previous Proposition, it follows that

$$\begin{aligned} u_2 &= \sigma_2 u + \tau_2(\lambda_1 \lambda_2 v), \qquad && \sigma_2, \tau_2 \in A, \sigma \notin m \\ u_3 &= \sigma_3(u + \lambda_1 v) + \tau_3(\lambda_1 \lambda_3 v), \qquad && \sigma_3, \tau_3 \in A, \sigma_3 \notin m. \end{aligned}$$

If $\lambda_2, \lambda_3 \in m$, then since $M_2 + M_3 = \langle u, \lambda_1 v \rangle$, it follows that u_2 and u_3 have distinct images $\bar{u}_2, \bar{u}_3 \in M_2 + M_3/m \cdot (M_2 + M_3)$. Therefore $M_2 + M_3 = \langle u_2, u_3 \rangle$ whose class is in $\Delta_\Gamma^{(0)}$. If either λ_2 or λ_3 is in A^*, then $M_2 \supset M_3$ or $M_3 \supset M_2$ and $M_2 + M_3$ equals either M_2 or M_3, whose class is in $\Delta_\Gamma^{(0)}$. *Q.E.D.*

Linked subsets $\Delta_*^{(0)} \subset \Delta^{(0)}$ are very nice objects. They can be fitted together in a natural way into a *tree*.

TREE THEOREM (1.13). *If $\Delta_*^{(0)}$ is a linked subset of $\Delta^{(0)}$, then $\Delta_*^{(0)}$ is the set of a vertices of a connected tree Δ_* in which a principal ideal (α_σ) is associated to each edge σ and such that for every pair of classes $\{P\}, \{Q\} \in \Delta_*^{(0)}$, if they are linked in the tree as follows:*

$$\begin{tikzcd}[arrows=dash]
\bullet \arrow[r, "\sigma_1"] & \bullet \arrow[r, "\sigma_2"] & \bullet \arrow[r, "\sigma_3"] & \cdots \arrow[r] & \bullet \arrow[r, "\sigma_{n-1}"] & \bullet \\
\{M_1\} \arrow[d, phantom, "\|"] & \{M_2\} & \{M_3\} & & \{M_{n-1}\} & \{M_n\} \arrow[d, phantom, "\|"] \\
\{P\} & & & & & \{Q\}
\end{tikzcd}$$

then

$$(*) \qquad \rho(\{P\}, \{Q\}) = \prod_{i=1}^{n-1} (\alpha_{\sigma_i}).$$

PROOF. First, let us call $\{P\}, \{Q\} \in \Delta_*^{(0)}$ *adjacent* if there is no $\{R\} \in \Delta_*^{(0)}$ such that:

$$\rho(\{P\}, \{Q\}) = \rho(\{P\}, \{R\}) \cdot \rho(\{R\}, \{Q\}),$$
$$\{R\} \neq \{P\} \text{ or } \{Q\}.$$

Join 2 adjacent classes by an edge σ and set $(\alpha_\sigma) = \rho(\{P\}, \{Q\})$. This gives us a graph in any case. Starting now with any $\{P\}, \{Q\}$, consider all sequences

$$\{P\} = \{M_1\}, \{M_2\}, \cdots, \{M_n\} = \{Q\} \quad \text{in } \Delta_*^{(0)}$$

such that

$$\rho(\{P\}, \{Q\}) = \prod_{i=1}^{n-1} \rho(\{M_i\}, \{M_{i+1}\}).$$

By the noetherian assumption on A, there is a maximal sequence of this type. Then each pair $\{M_i\}, \{M_{i+1}\}$ must be adjacent and this proves that $\{P\}$ and $\{Q\}$ are joined in our graph by a sequence of edges. Therefore the graph is connected.

To prove that our graph is a tree and to prove $(*)$, it suffices, by an obvious induction, to prove:

LEMMA (1.14). *Let* $\{M_1\}, \{M_2\}, \cdots, \{M_n\} \in \Delta_*^{(0)}$ *such that* $\{M_i\}$, *and* $\{M_{i+1}\}$ *are adjacent* ($1 \leqq i \leqq n-1$). *Assume*

$$\rho(\{M_1\}, \{M_{n-1}\}) = \prod_{i=1}^{n-2} \rho(\{M_i\}, \{M_{i+1}\}).$$

Then either

$$\rho(\{M_1\}, \{M_n\}) = \prod_{i=1}^{n-1} \rho(\{M_i\}, \{M_{i+1}\})$$

or

$$\{M_n\} = \{M_{n-2}\}.$$

PROOF OF LEMMA. Let $M_{n-1} \supset M_n$, $M_{n-1} \supset M_{n-2}$, $M_{n-1} \supset M_1$ be representatives in standard position. By the Corollary (1.10) $M_{n-1} \supset M_{n-2} \supset M_1$. Consider $M_{n-2}+M_n$. Since $\Delta_*^{(0)}$ is linked, $\{M_{n-2}+M_n\} \in T$. Since $\{M_{n-1}\}$ is adjacent to $\{M_{n-2}\}$ and $M_{n-2} \subset M_{n-2}+M_n \subset M_{n-1}$, $M_{n-2}+M_n$ equals M_{n-1} or M_{n-2}; similarly since $\{M_{n-1}\}$ is adjacent to $\{M_n\}$, $M_{n-2}+M_n$ equals M_n or M_{n-1}. Thus either $M_{n-2}+M_n = M_{n-1}$ or, if not, then $M_n = M_{n-2}+M_2 = M_{n-2}$. In the first case, $M_n/M_n \cap mM_{n-1}$ and $M_{n-2}/M_{n-2} \cap mM_{n-1}$ are distinct one-dimensional subspaces of M_{n-1}/mM_{n-1}. But $(0) \subsetneqq M_1/M_1 \cap mM_{n-1} \subset M_{n-2}/M_{n-2} \cap$

mM_{n-1}, so $M_1/M_1 \cap mM_{n-1}$ must be the same subspace as $M_{n-2}/M_{n-2} \cap m \cdot M_{n-1}$. Therefore M_1 and M_n together generate M_{n-1}/mM_{n-1}, hence they generate M_{n-1}. By Cor. (1.10) applied to the triple M_1, M_{n-1}, M_n this means that

$$\begin{aligned}\rho(\{M_1\}, \{M_n\}) &= \rho(\{M_1\}, \{M_{n-1}\}) \cdot \rho(\{M_{n-1}\}, \{M_n\}) \\ &= \prod_{i=1}^{n-1} \rho(\{M_i\}, \{M_{i+1}\}). \qquad \textit{Q.E.D.}\end{aligned}$$

Corollary (1.15). *In a canonical way,* $\Delta_\Gamma^{(0)}$ *is the set of vertices of a tree* Δ_Γ *on which* Γ *acts.*

Corollary (1.16) (Ihara). Γ *is a free group.*

Proof. I claim that Γ acts freely on Δ_Γ. In fact, if $\gamma \in \Gamma$, $\gamma \neq e$, has a fixed point P, then P is either a vertex or the midpoint of an edge. In the latter case, γ^2 fixes the 2 endpoints of this edge. But the stabilizers in $PGL(2, K)$ of the elements of $\Delta^{(0)}$ are the various subgroups

$$xPGL(2, A)x^{-1} \subset PGL(2, K).$$

Since every $\gamma \in \Gamma$ is hyperbolic, neither γ nor γ^2 can belong to any such subgroup. Thus Γ acts freely on a tree, hence Γ itself must be free. *Q.E.D.*

Corollary (1.17) (Bruhat-Tits). *If* $\dim A = 1$, *the whole of* $\Delta^{(0)}$ *is, in a canonical way, the set of vertices of a tree* Δ *on which the whole group* $PGL(2, K)$ *acts.*

It can be shown further when $\dim A = 1$ that for all $\gamma \in PGL(2, K)$, either γ is hyperbolic and has no fixed point on Δ; or γ is not hyperbolic and γ has a fixed point on Δ in which case then γ^2 is in some subgroup $g \cdot PGL(2, A) \cdot g^{-1}$.

For any linked subset $\Delta_*^{(0)} \subset \Delta^{(0)}$, let Δ_* be the associated tree. We can add a boundary to Δ_* that has an interesting interpretation: let

$$\partial\Delta_* = \text{the set of ends of } \Delta_*$$

[Where an *end* is an equivalence class of subtrees of Δ_* isomorphic to:

$$\bullet\text{———}\bullet\text{———}\bullet\text{———}\bullet\text{———} \cdots$$

two such being 'equivalent' if they differ only in a finite set of vertices]. Let $\bar{\Delta}_* = \Delta_* \cup \partial\Delta_*$: this is a topological space if an open set is a subset $U \cup V$, where $U \subset \Delta_*$ is open and $V \subset \partial\Delta_*$ is the set of ends represented by subtrees in U.

Proposition (1.18). a) *There is a natural injection*

$$i : \partial\Delta_* \hookrightarrow KP^1.$$

b) *If* $\Delta_* = \Delta_\Gamma$, $\Sigma \subset i(\partial\Delta_\Gamma)$.

c) *If* $\dim A = 1$, $\Delta_* = \Delta$, *then* i *is a bijection of* $\partial\Delta$ *and* $\boldsymbol{KP}^1$.

PROOF. Let $\{M_1\}, \{M_2\}, \cdots$ be an infinite sequence of adjacent vertices of Δ_* which defines an end $e \in \partial\Delta_*$. Represent these by modules in standard position:

$$M_1 \supset M_2 \supset M_3 \supset \cdots.$$

Then it is easy to check that $\bigcap_{n=1}^{\infty} M_n$ is a free A-module of rank 1 in $K+K$. If $u \in \cap M_n$ is a generator, u defines the point $i(e) \in \boldsymbol{KP}^1$. Note that $u \notin mM_1$ and if $v \in M_1$ is such that $\bar{u} \neq \bar{v}$ in M_1/mM_1, then

$$M_1 = \langle u, v\rangle, M_2 = \langle u, \alpha_2 v\rangle, M_3 = \langle u, \alpha_3 v\rangle, \cdots,$$

where $(\alpha_n) = \rho(\{M_1\}, \{M_n\})$. Next, let e' be a 2nd end and assume $e \neq e'$. From general properties of trees, it follows that there is a unique subtree of Δ_* isomorphic to:

$$\cdots \;\text{——}\bullet\text{——}\bullet\text{——}\bullet\text{——}\bullet\text{——}\bullet\text{——}\bullet\text{——}\bullet\text{——}\; \cdots$$

(we call such a subtree a *line*) defining e at one end, e' at the other. Pick a base point $\{P\}$ on this, and let its vertices in the 2 directions be $\{M_1\}, \{M_2\}, \cdots$ and $\{N_1\}, \{N_2\}, \cdots$. Represent these by modules in standard positions:

$$P \supset M_1 \supset M_2 \supset \cdots$$
$$P \supset N_1 \supset N_2 \supset \cdots$$

By Cor. (1.10), since $\rho(\{M_k\}, \{N_k\}) = \rho(\{M_k\}, \{P\}) \cdot \rho(\{P\}, \{N_k\})$, it follows that $P = M_k + N_k$. Thus if u is a generator of $\cap M_n$ and v is a generator of $\cap N_n$, then $P = \langle u, v\rangle$. In particular $K \cdot u$ and $K \cdot v$ are distinct subspaces of $K+K$. But $K \cdot u$ represents $i(e)$, $K \cdot v$ represents $i(e')$. Therefore $i(e) \neq i(e')$. To prove (b), note that when $\Delta_* = \Delta_\Gamma$, Γ acts on Δ_Γ, on $\partial\Delta_\Gamma$ and on $\boldsymbol{KP}^1$ and i is Γ-linear. On the other hand, any fixed point free automorphism γ of a tree leaves invariant a unique line (its *axis*) and it acts on its axis by a translation. Thus γ fixes 2 distinct ends of the tree. In our case, every $\gamma \in \Gamma$, $\gamma \neq e$, has therefore 2 fixed points in $\partial\Delta_\Gamma$ hence in $i(\partial\Delta_\Gamma)$, and hence $i(\partial\Delta_\Gamma)$ contains the 2 fixed points of γ in $\boldsymbol{KP}^1$. To prove (c), let $x \in \boldsymbol{KP}^1$ and let $u \in K+K$ be homogeneous coordinates for x. Let $v \in K+K$ be any vector that is not a multiple of u and let $M_n = \langle u, \pi^n v\rangle$, where $(\pi) = m$, the maximal ideal in A. Then $\{M_n\}_{n \geq 0}$ is a half-line in the full tree Δ whose end is mapped by i to x. *Q.E.D.*

DEFINITION (1.19). $i(\partial\Delta_\Gamma) \subset \boldsymbol{KP}^1$ will be denoted $\bar{\Sigma}$ and called the *limit point set* of Γ.

In fact, when $\dim A = 1$, it is easy to check that $\bar{\Sigma}$ is precisely the closure of Σ in the natural topology on $K\boldsymbol{P}^1$.

We can link together via the map i several of the ideas we have been working with:

PROPOSITION (1.20). *Let $\Delta_*^{(0)}$ be any linked subset of $\Delta^{(0)}$, and let Δ_* be the corresponding tree. For any $x, y, z \in \partial\Delta_*$, the class $\{M(ix, iy, iz)\}$ is in $\Delta_*^{(0)}$ and equals to unique vertex of Δ_* such that the paths from v to the 3 ends x, y, z all start off on different edges.*

PROOF. In fact, let the module M represent v, and let $M \supset M_x$, $M \supset M_y$, $M \supset M_z$ be representatives in standard positions of M and the module classes next on the path from v to x, y and z respectively. Then ix, iy, iz are represented by homogeneous coordinates $u, v, w \in K+K$ such that

$$\begin{aligned} u &\in M_x - mM, \\ v &\in M_y - mM, \\ w &\in M_z - mM. \end{aligned}$$

Since $\{M\}$ is between $\{M_x\}$ and $\{M_y\}$ in Δ_*, it follows from (1.10) that $M_x + M_y = M$ hence $\bar{u}, \bar{v} \in M/mM$ are distinct. Similarly $\bar{w}$ is distinct from $\bar{u}$ and $\bar{v}$. Therefore, if $\alpha u + \beta v + \gamma w = 0$ is the linear relation on u, v, w, it follows that $\alpha, \beta, \gamma \in A - m$. Therefore

$$\{M(x, y, z)\} = \{A \cdot u + A \cdot v + A \cdot w\} = \{M\} = v.$$

COROLLARY (1.21). *Every vertex of Δ_Γ meets at least 3 distinct edges.*

PROPOSITION (1.22). *Let $\Delta_*^{(0)}$ be any linked subset of $\Delta^{(0)}$ and let Δ_* be the corresponding tree. Then for any $x_1, x_2, x_3, x_4 \in \partial\Delta_*$, $R(ix_1, ix_2, ix_3, ix_4) \in A$ or $R(ix_1, ix_2, ix_3, ix_4)^{-1} \in A$.*

PROOF. Let v be the vertex of Δ_* joined to x_1, x_2, x_3 by paths starting on different edges. Represent v by M and ix_1, ix_2, ix_3 by coordinates $u_1, u_2, u_3 \in M - mM$ as in the proof of the previous Proposition.

Moreover we can represent x_4 by coordinates $u_4 \in M - mM$ too. Then

$$\begin{aligned} u_3 &= \alpha u_1 + \beta u_2, \qquad && \alpha, \beta \in A - m \\ u_4 &= \gamma u_1 + \delta u_2, \qquad && \gamma, \delta \in A \text{ and } \gamma \text{ or } \delta \notin m. \end{aligned}$$

In this case,

$$R(ix_1, ix_2, ix_3, ix_4) = \frac{\delta\alpha}{\beta\gamma} \begin{cases} \in A & \text{if } \gamma \notin m, \\ \in A^{-1} & \text{if } \delta \notin m. \end{cases} \qquad \textit{Q.E.D.}$$

We can use Proposition (1.20) to prove the important:

THEOREM (1.23). *For all flat Schottky groups Γ, Δ_Γ/Γ is a finite graph.*

PROOF. Let $\gamma_1, \cdots, \gamma_n$ be free generators of Γ and let v be any vertex of Δ_Γ. Let σ_i be the path in Δ_Γ from v to $\gamma_i(v)$ and let $S = \sigma_1 \cup \cdots \cup \sigma_n$. S is a finite tree and I claim that S maps *onto* Δ_Γ/Γ. This is equivalent to saying that

$$\tilde{S} = \bigcup_{\gamma \in \Gamma} \gamma(S) = \bigcup_{\gamma \in \Gamma} \bigcup_{i=1}^{n} (\text{path from } \gamma(v) \text{ to } \gamma\gamma_i(v))$$

is equal to Δ_Γ. Note first that $\tilde{S}$ is connected. In fact, if $\gamma \in \Gamma$, write γ as a word:

$$\gamma = \gamma_{i_1}^{\varepsilon_1}\gamma_{i_2}^{\varepsilon_2} \cdots \gamma_{i_N}^{\epsilon_N}, \quad \varepsilon_i = \pm 1, \ 1 \leqq i_l \leqq n, \text{ all } l.$$

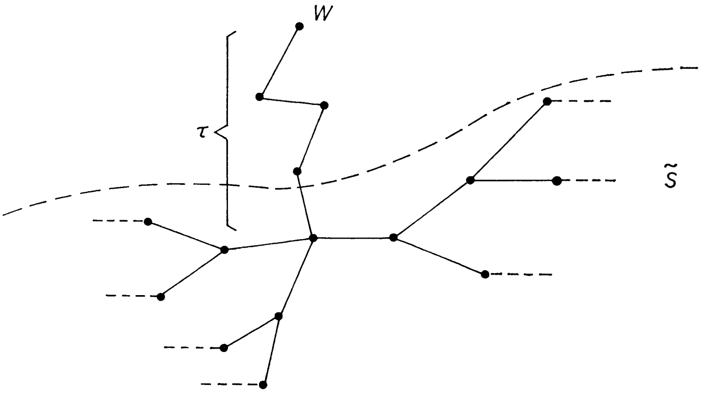
Then a typical point of $\tilde{S}$ is on the path from $\gamma(v)$ to $\gamma\gamma_i(v)$. It is joined to v by the sequence of paths:

$$\begin{array}{rl}
\gamma\gamma_i(v) \ \bullet & \\
 & \gamma(\sigma_i) \\
\gamma_{i_1}^{\varepsilon_1} \cdots \gamma_{i_N}^{\varepsilon_N}(v) = \gamma(v) \ \bullet & \\
 & \gamma_{i_1}^{\varepsilon_1} \cdots \gamma_{i_{N-1}}^{\varepsilon_{N-1}}(\sigma_{i_N}^{\varepsilon_N}) \\
\gamma_{i_1}^{\varepsilon_1} \cdots \gamma_{i_{N-1}}^{\varepsilon_{N-1}}(v) \ \bullet & \\
\vdots & \\
\gamma_{i_1}^{\varepsilon_1}\gamma_{i_2}^{\varepsilon_2}(v) \ \bullet & \\
 & \gamma_{i_1}^{\varepsilon_1}(\sigma_{i_2}^{\varepsilon_2}) \\
\gamma_{i_1}^{\varepsilon_{i_1}}(v) \ \bullet & \\
 & \sigma_{i_1}^{\varepsilon_1} \\
v \ \bullet &
\end{array}$$

where $\sigma_i^1 = \sigma_i$ and $\sigma_i^{-1} = \gamma_i^{-1}(\sigma_i) = (\text{path from } v \text{ to } \gamma_i^{-1}(v))$. Note secondly that for every $x \in \Sigma$, the end $i^{-1}x$ of Δ_Γ is actually an end of the subtree $\tilde{S}$. In fact, if x is a fixed point of $\gamma \in \Gamma$, it suffices to join the points

$$\cdots, \gamma^{-2}v, \gamma^{-1}v, v, \gamma v, \gamma^2 v, \cdots$$

by paths in $\tilde{S}$. The result is a line in $\tilde{S}$, invariant under γ, plus an infinite set of spurs one leading to each of the points $\gamma^n v$. The 2 ends of this line are the 2 ends of Δ_Γ fixed by γ and one of these is $i^{-1}x$. Thus $i^{-1}x$ is, in fact, an end of $\tilde{S}$. Finally, suppose w is a vertex of $\Delta_\Gamma - \tilde{S}$. Then $w = \{M(x, y, z)\}$ for some $x, y, z \in \Sigma$. Since $w \notin \tilde{S}$, w is connected to $\tilde{S}$ by a unique path τ:



Thus all the paths from w to all ends of $\tilde{S}$ start with the same edge. Since $i^{-1}x$, $i^{-1}y$, $i^{-1}z$ are all ends of $\tilde{S}$, this contradicts Prop. (1.20). Hence $\Delta_\Gamma = \tilde{S}$. *Q.E.D.*

COROLLARY (1.24). *Δ_Γ is a locally finite tree.*

DEFINITION (1.25). Let $\{M\}$, $\{N\} \in \Delta^{(0)}$ be compatible, and let $z \in K\boldsymbol{P}^1$. Then $\{M\}$ *separates* $\{N\}$ from z if there are representatives $N \supset M$ in standard positions and homogeneous coordinates $z^* \in K+K$ of z such that:

$$z^* \in M, z^* \notin m \cdot N.$$

The following is almost immediate:

PROPOSITION (1.26). *Let $\Delta_*^{(0)} \subset \Delta^{(0)}$ be a linked subset and let e be an end of $\partial\Delta_*$. Let $\{M\}$, $\{N\} \in \Delta_*^{(0)}$. Then $\{M\}$ separates $\{N\}$ from $i(e)$ if and only if the line in Δ_* from $\{N\}$ to the end e passes through $\{M\}$.*

DEFINITION (1.27). Let $\Delta_*^{(0)} \subset \Delta^{(0)}$ be any linked subset and let $z \in K\boldsymbol{P}^1$. The *base* of z of $\Delta_*^{(0)}$ is the set of all $\{M\} \in \Delta_*^{(0)}$ which are not separated from z by any $\{N\} \in \Delta_*^{(0)}$.

PROPOSITION (1.28). *The base of z on $\Delta_*^{(0)}$ is empty if and only if $z \in i(\partial\Delta_*)$.*

PROOF. If $z \in i(\partial\Delta_*)$, the base of z is empty by (1.26). Conversely, suppose the base is empty. Choose homogeneous coordinates $z^* \in K+K$ for z. Then for all $\{M\} \in \Delta_*^{(0)}$, there is a representative $M \in \{M\}$ such that $z^* \notin m \cdot M$ and an $\{N\} \in \Delta_*^{(0)}$ such that

$$(*) \qquad M \not\supseteq N \ni z^*.$$

Start with any $\{M_1\} \in \Delta_*^{(0)}$. Call the N satisfying $(*)$ M_2. With M as M_2, call the N satisfying $(*)$ M_3. Continuing in this way, we get an infinite sequence:

$$M_1 \not\supseteq M_2 \not\supseteq M_3 \not\supseteq \cdots \ni z^*.$$

Then the sequence $\{M_i\}$ defines an end of Δ_* which is mapped by i to z. *Q.E.D.*

PROPOSITION (1.29). *If* $\dim A = 1$, *then for any locally finite tree* Δ_* *and for any* $z \in KP^1$, *the base of* z *on* Δ_* *consists of zero, one or two points.*

PROOF. Consider Δ_* inside the big tree Δ. It is easy to see that if the edges of Δ_* are suitably subdivided, Δ_* becomes a subtree Δ'_* of Δ; i.e. for all adjacent $M_1 \supset M_2$ in Δ_* let

$$\begin{aligned} M_1 &= \langle u, v\rangle \\ M_2 &= \langle u, \pi^r v\rangle, \\ (\pi) &= \text{max. ideal of } A. \end{aligned}$$

Then adding the intermediate classes

$$\{\langle u, \pi^i v\rangle\}, \qquad 1 \leqq i \leqq r-1.$$

has the effect of subdividing the edge in Δ_* between $\{M_1\}$ and $\{M_2\}$ so that it becomes a path in Δ. Now every $z \in KP^1$ is an end of Δ, and every end of Δ which is not an end of Δ_* can be joined to the subtree Δ'_* by a unique shortest path. If $z \in KP^1 - i(\partial\Delta_*)$, let $v(z)$ be the vertex of Δ'_* where this path meets Δ'_*. Then it follows from (1.26) that the base of z on Δ_* is $\{v(z)\}$ if $v(z)$ is a vertex of Δ_*; or it equals the 2 endpoints of the edge of Δ_* containing $v(z)$ if $v(z)$ is not a vertex of Δ_*. *Q.E.D.*

In case $\dim A > 1$, the points $z \in KP^1 - i(\partial\Delta_*)$ can have wildly diverse kinds of bases on Δ_*. Take the case $\Delta_* = \Delta_\Gamma$. Then heuristically, Γ does not act equally discontinuously at all the points of $KP^1 - \overline{\Sigma}$. An important definition is this:

DEFINITION (1.30). If Γ is a flat Schottky group, then $\Omega_\Gamma = \{z \in KP^1 \mid$ the base of z on Δ_Γ is finite and non-empty$\}$; Ω_Γ is called the set of *strict discontinuity*, or the set of points where Γ acts *strictly discontinuously*. (Note that if $\dim A = 1$, Ω_Γ is simply $KP^1 - \overline{\Sigma}$, the usual set of discontinuity.)

2. From trees to schemes

We turn now from the construction of trees to the construction of actual or formal schemes over S[3]. Consider the set of all reduced and

[3] It is interesting that the trees which Bruhat and Tits associated to $PGL(2, K)$ are, in fact, a highly developed special case of the graphs that have been used for a long time in the theory of algebraic surfaces to plot the configurations of intersecting curves. To be precise, if $\dim A = 1$, we consider the inverse system of surfaces obtained by blowing up closed points on the 2-dimensional scheme $P^1 \times S$. The graph which plots the components of their closed fibres over S and their intersection relations is canonically isomorphic to Δ.

irreducible schemes Z over S whose generic fibre is $\boldsymbol{P}^1_K$. These form a partially ordered set if $Z_1 > Z_2$ means that there is an S-morphism from Z_1 to Z_2 which restricts to the identity on the generic fibre. In this partially ordered set, any 2 elements have a least upper bound, called their *join*. We want to study

1. the joins of finite sets of schemes $\boldsymbol{P}(M)$, $\{M\} \in \Delta^{(0)}$,
2. certain special infinite joins that exist as formal schemes over S but not as actual schemes.

PROPOSITION (2.1). *Let $\{M_1\}, \{M_2\} \in \Delta^{(0)}$. Let P_{12} be the join of $\boldsymbol{P}(M_1)$ and $\boldsymbol{P}(M_2)$. Then*

(a) *if $\{M_1\}, \{M_2\}$ are not compatible, the closed fibre of P_{12} is isomorphic to $\boldsymbol{P}(M_1)_0 \times \boldsymbol{P}(M_2)_0$; in particular, the fibres of P_{12} over S do not all have the same dimension hence P_{12} is not flat over S.*

(b) *if $\{M_1\}, \{M_2\}$ are compatible and $(\alpha) = \rho(\{M_1\}, \{M_2\})$, then P_{12} is a normal scheme, flat over S, and its fibre over $s \in S$ is:*

(b$_1$) $\boldsymbol{P}(M_i)_s$ *if* $\alpha(s) \neq 0$ $\qquad$ $(i = 1$ *or* $2)$,

(b$_2$) $\left\{\begin{array}{l} \boldsymbol{P}(M_1)_s \cup \boldsymbol{P}(M_2)_s \\ \textit{meeting transversely in one} \\ k(s)\textit{-rational point} \end{array}\right\}$ $\qquad$ *if* $\alpha(s) = 0$.

PROOF. Let u_1, v_1 be a basis of M_1, and let $u_2 = au_1 + bv_1$, $v_2 = cu_1 + dv_1$ be a basis of M_2. Define map s:

$$\left.\begin{array}{r} X_i, Y_i : M_i \to A \\ X_i(u_i) = Y_i(v_i) = 1 \\ X_i(v_i) = Y_i(u_i) = 0 \end{array}\right\} \qquad i = 1, 2.$$

Then $\boldsymbol{P}(M_i) = \operatorname{Proj} A[X_i, Y_i]$. But

$$\begin{aligned} X_1 &= aX_2 + cY_2 \\ Y_1 &= bX_2 + dY_2 \end{aligned}$$

and these equations define the generic isomorphism of $\boldsymbol{P}(M_1)$ and $\boldsymbol{P}(M_2)$. Now P_{12} is just the closure in $\boldsymbol{P}(M_1) \times_S \boldsymbol{P}(M_2)$ of the graph of this generic isomorphism, i.e. P_{12} is the closure in

$$\operatorname{Proj} A[X_1X_2, X_1Y_2, Y_1X_2, Y_1Y_2]$$

of the curve in the generic fibre defined by

$$(*) \qquad aX_2Y_1 - bX_1X_2 + cY_1Y_2 - dY_2X_1 = 0.$$

According to the lemma of Ramanujam-Samuel (EGA IV.21) this closure either contains the whole closed fibre of $\boldsymbol{P}(M_1) \times_S \boldsymbol{P}(M_2)$ or else is a relative Cartier divisor over S. If the latter is true, then the closure must be defined as a subscheme of $\boldsymbol{P}(M_1)\times_S\boldsymbol{P}(M_2)$ by a suitable multiple of equation $(*)$ with all coefficients in A, and not all coefficients in m. In particular P_{12} is then flat over S and its fibres are curves in $\boldsymbol{P}(M_1)_s\times\boldsymbol{P}(M_2)_s$ defined by equations of type $(*)$. But over any field L, an equation of type $(*)$ which is not identically zero defines a curve in $\boldsymbol{P}^1_L\times\boldsymbol{P}^1_L$ which is (i) a graph of an isomorphism of the 2 factors if $ad-bc\neq 0$ or (ii) equal to $\boldsymbol{P}^1_L\times(\alpha)\cup(\beta)\times\boldsymbol{P}^1_L$ for some $\alpha,\beta\in L\boldsymbol{P}^1$ if $ad-bc=0$.

To tie these possibilities up with compatibility of the $\{M_i\}$, note that on the one hand if $\lambda a,\lambda b,\lambda c,\lambda d\in A$ and not all are in m, then $M_1\supset\lambda M_2$ and $mM_1\not\supset\lambda\cdot M_2$: thus $\{M_1\}$ and $\{M_2\}$ are compatible and $M_1,\lambda M_2$ are representatives in standard position. It is easy to check that

$$(\lambda a\cdot\lambda d-\lambda\beta\cdot\lambda c)=\rho(\{M_1\},\{M_2\}).$$

On the other hand, if $\{M_1\}$ and $\{M_2\}$ are compatible, choose $M_1\supset M_2$ to be representatives in standard position. Then $a,b,c,d\in A$, and not all are in m.

Finally, the normality of P_{12} in the 2nd clase is a formal consequence of the rest: since S is normal and P_{12} is flat and generically smooth over S, it is certainly non-singular in codimension one. And if $f\in m$, then the ideal $f\cdot A\subset A$ has no embedded components, and since none of the fibres of P_{12}/S have embedded components, $f\cdot\mathcal{O}_{P_{12}}$ has no embedded components either. Thus P_{12} is normal. *Q.E.D.*

PROPOSITION (2.2). *Let $\{M_1\},\{M_2\}\in\Delta^{(0)}$ be compatible and let $z\in K\boldsymbol{P}^1$. Then $\{M_1\}$ separates $\{M_2\}$ from z if and only if*

$$\mathrm{cl}_{P_{12}}(z)\cap\boldsymbol{P}(M_2)_0=\emptyset,$$

where $\mathrm{cl}_{P_{12}}(z)$ is the closure of $\{z\}$ in P_{12} and $\boldsymbol{P}(M_2)_0$ is the component of the closed fibre of P_{12} isomorphic to the closed fibre of $\boldsymbol{P}(M_2)$.

PROOF. Since the closed fibre of P_{12} minus $\boldsymbol{P}(M_2)_0$ is isomorphic to $\boldsymbol{P}(M_1)_0$ minus a point, $\mathrm{cl}_{P_{12}}(z)\cap\boldsymbol{P}(M_2)_0=\emptyset$ implies that $\mathrm{cl}_{P_{12}}(z)$ meets the closed fibre in P_{12} in a finite set of points where P_{12} is smooth over S. Therefore by the lemma of Ramanujam-Samuel, $\mathrm{cl}_{P_{12}}(z)$ will be a relative Cartier divisor in this case; hence $\mathrm{cl}_{P_{12}}(z)$ will be the image of a section of P_{12} over S. Thus $\mathrm{cl}_{P_{12}}(z)\cap\boldsymbol{P}(M_2)_0=\emptyset$ is equivalent to z extending to a section of P_{12} not meeting the component $\boldsymbol{P}(M_2)_0$ of the closed fibre. But if

$$M_2=\langle u,v\rangle,$$
$$M_1=\langle u,\alpha v\rangle$$

are representatives in standard position, then for z to define a section of $\boldsymbol{P}(M_1)$ means that z has homogeneous coordinates:

$$z^* = \beta u + \gamma(\alpha v), \qquad \beta, \gamma \in A, \qquad \beta \text{ or } \gamma \notin m.$$

In this case $\mathrm{cl}_{P_{12}}(z)$ meets $\boldsymbol{P}(M_1)_0$ in a point other than the one where $\boldsymbol{P}(M_2)_0$ meets $\boldsymbol{P}(M_1)_0$ if and only if $\beta \notin m$. Thus $\{M_1\}$ separates $\{M_2\}$ and z if and only if z defines a section of $\boldsymbol{P}(M_1)$ passing through a closed point other than the point where $\boldsymbol{P}(M_2)_0$ meets it in P_{12}; which is the same as saying that z defines a section of P_{12} not meeting $\boldsymbol{P}(M_2)_0$. *Q.E.D.*

PROPOSITION (2.3): *Let $\{M_1\}, \cdots, \{M_k\} \in \Delta^{(0)}$ be pairwise compatible. Let Z be the join of $\boldsymbol{P}(M_1), \cdots, \boldsymbol{P}(M_k)$. Then Z is a normal scheme, proper and flat over S and its closed fibre Z_0 satisfies:*

i) *it is reduced, connected and 1-dimensional,*

ii) *its components are naturally isomorphic to the schemes $\boldsymbol{P}(M_1)_0, \cdots \boldsymbol{P}(M_k)_0$ respectively,*

iii) 2 *components meet in at most one point and no set of components meets to form a loop,*

(iv) *every singular point z is locally isomorphic over k to the union W_l of the coordinate axes in A^l.*

PROOF. By definition, Z is the closure in $\boldsymbol{P}(M_1) \times_S \cdots \times_S \boldsymbol{P}(M_k)$ of the graph of the generic isomorphism of all the factors. Therefore Z_0 is connected by Zariski's connectedness theorem (EGA, III. 4.3.1). For every i and $j (1 \leqq i, j \leqq k)$, let p_{ij} be the projection onto $\boldsymbol{P}(M_i) \times_S \boldsymbol{P}(M_j)$. Since p_{ij} is proper, it follows from the proof of the previous proposition that:

$$(*) \qquad p_{ij}(Z_0) = \boldsymbol{P}(M_i) \times (a) + (b) \times \boldsymbol{P}(M_j) \qquad \text{some } a, b \in k\boldsymbol{P}^1.$$

Therefore each component of Z_0 is 'parallel to one of the coordinate axes', i.e. has the form:

$$(a_1) \times (a_2) \times \cdots \times \boldsymbol{P}(M_i)_0 \times \cdots \times (a_k)$$

for some i, and is naturally isomorphic to $\boldsymbol{P}(M_i)_0$ for this i. Moreover it follows immediately from $(*)$ that for each i, exactly one of the components of Z_0 is parallel to the i^{th} coordinate axis. This proves (ii). For any union Z_0 of coordinate axes, $(Z_0)_{\text{red}}$ is locally isomorphic to the scheme W_l in (iv). Moreover, if Z_0 had a loop, then for some i there would have to be 2 or more components parallel to the i^{th} axis. The only point which is not very clear is that Z_0 is reduced at its singular points. But note that the ideal of W_l in A^l is generated by the monomials $(X_i X_j)$ which are the defining equations of $p_{ij}(W_l)$; therefore the scheme-theoretic intersection $\bigcap_{i,j} p_{ij}^{-1}[\boldsymbol{P}(M_i)_0 \times (a) + (b) \times \boldsymbol{P}(M_n)_0]$ is already reduced, so *a fortiori* Z_0 is reduced. Finally, Z is flat over S by EGA 4.15.2, and normal by the same argument used in Prop. (2.1). *Q.E.D.*

Finally, we have:

PROPOSITION (2.4): *Let $\Delta_*^{(0)} \subset \Delta^{(0)}$ be a finite linked subset. Let $\boldsymbol{P}(\Delta_*)$ be the join of all the schemes $\boldsymbol{P}(M_i)$, $\{M_i\} \in \Delta_*^{(0)}$. Then in addition to the above properties, we have also:*

i) *Z_0 has only double points,*

ii) *in the one-one correspondence between the components of the closed fibre Z_0 and the elements of $\Delta_*^{(0)}$ 2 components meet if and only if the corresponding elements of $\Delta_*^{(0)}$ are adjacent in the tree Δ_*.*

In other words, if we make a tree out of Z_0 by taking a vertex for each component and an edge for each point of intersection, we obtain geometrically the tree Δ_*.

PROOF. In fact, if Z_0 has a point of multiplicity $\geqq 3$, this would mean that $\geqq 3$ components of Z_0 all met each other. Since we know Δ_* is a tree, this would contradict (ii). Thus it suffices to prove (ii). Suppose $\{M_1\}$, $\{M_2\} \in \Delta_*^{(0)}$ are *not* adjacent. This means there is an $\{M_3\} \in \Delta_*^{(0)}$ different from $\{M_1\}$ and $\{M_2\}$ such that

$$\rho(\{M_1\}, \{M_2\}) = \rho(\{M_1\}, \{M_3\}) \cdot \rho(\{M_3\}, \{M_2\}).$$

Therefore we can find representatives of these classes in standard position:

$$\begin{aligned} M_1 &= \langle u, v\rangle, \\ M_3 &= \langle u, \alpha v\rangle, (\alpha) = \rho((M_1\}, (M_3\}) \subset m, \\ M_2 &= \langle u, \alpha \cdot \beta v\rangle, (\beta) = \rho((M_3\}, (M_2\}) \subset m. \end{aligned}$$

Let X, Y be defined by

$$\begin{aligned} X(u) &= Y(v) = 1, \\ X(v) &= Y(u) = 0. \end{aligned}$$

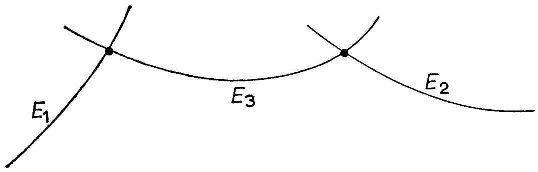
Then:

$$\begin{aligned} \boldsymbol{P}(M_1) &= \operatorname{Proj} A[X, Y], \\ \boldsymbol{P}(M_3) &= \operatorname{Proj} A[X, Y/\alpha], \\ \boldsymbol{P}(M_2) &= \operatorname{Proj} A[X, Y/\alpha\beta]. \end{aligned}$$

Now form the join Z_{123} of $\boldsymbol{P}(M_1), \boldsymbol{P}(M_2), \boldsymbol{P}(M_3)$:

$$\begin{aligned} Z_{123} &= \operatorname{Proj} A[X^3, X^2Y/\alpha\beta, XY^2/\alpha^2\beta, Y^3/\alpha^2\beta] \\ &\cong \operatorname{Proj} A[u_1, u_2, u_3, u_4]/(\beta u_2^2 - u_1u_3, \alpha u_3^2 - u_2u_4, \alpha\beta u_2u_3 - u_1u_4). \end{aligned}$$

From this it follows easily that the closed fibre of Z_{123} has 3 components, connected like this:



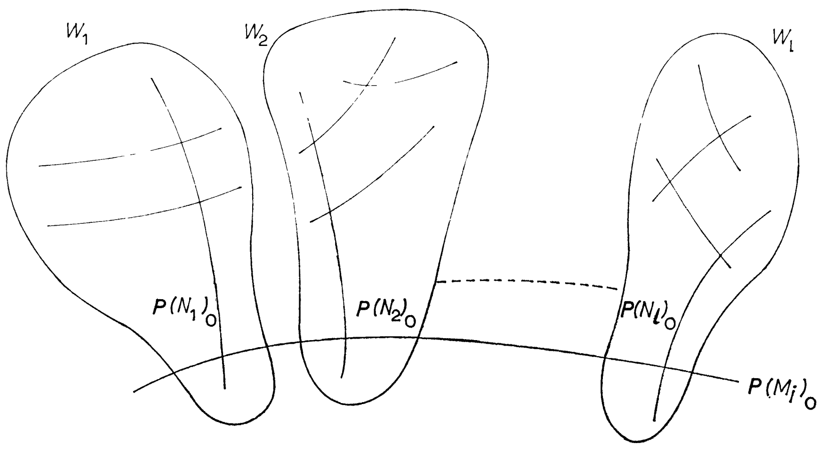
where E_i maps onto $\boldsymbol{P}(M_i)_0$. But now there is a natural map of Z onto Z_{123}, and the component of Z_0 corresponding to $\{M_i\}$ must map onto $E_i \subset (Z_{123})_0$. Since $E_1 \cap E_2 = \emptyset$, it follows that in Z_0 the components corresponding to $\{M_1\}$ and $\{M_2\}$ do not meet! This proves that the only components of Z_0 that can meet are those corresponding to adjacent vertices of Δ_*. But Δ_* is a tree so it is disconnected by leaving out any edge. Thus if any pair of components of Z_0 corresponding to adjacent vertices of Δ_0 did not actually meet, Z_0 would be disconnected. Thus (ii) is completely proven. *Q.E.D.*

PROPOSITION (2.5): *Let $\Delta_*^{(0)} \subset \Delta^{(0)}$ be a finite linked subset, and let $\boldsymbol{P}(\Delta_*)$ be the join of the $\boldsymbol{P}(M_i)$, $\{M_i\} \in \Delta_*^{(0)}$. Let $z \in K\boldsymbol{P}^1$, and let* cl(z) *denote the closure of $\{z\}$ in $\boldsymbol{P}(\Delta_*)$. Then for all $\{M_i\} \in \Delta_*^{(0)}$*

$$\mathrm{cl}(z) \cap \boldsymbol{P}(M_i)_0 \neq \emptyset$$

if and only if $\{M_i\}$ is in the base of z on $\Delta_*^{(0)}$.

PROOF. The statement is equivalent to: $[\mathrm{cl}(z) \cap \boldsymbol{P}(M_i)_0 = \emptyset] \Leftrightarrow [\exists \{M_j\} \in \Delta_*^{(0)}$ separating $\{M_i\}$ and $z]$. The implication $\Leftarrow$ is an immediate consequence of (2.2). Conversely, suppose $\mathrm{cl}(z) \cap \boldsymbol{P}(M_i)_0 = \emptyset$. Let $x_1, \cdots, x_l$ be the double points on $\boldsymbol{P}(M_i)_0$, let $\boldsymbol{P}(N_1)_0, \cdots, \boldsymbol{P}(N_l)_0$ be the other components of $\boldsymbol{P}(\Delta_*)_0$ through $x_1, \cdots, x_l$ respectively, and let $W_t \subset \boldsymbol{P}(\Delta_*)_0$ be the union of the components of $\boldsymbol{P}(\Delta_*)_0$ which are connected to $\boldsymbol{P}(N_t)_0$ without passing through $\boldsymbol{P}(M_i)_0$:



By Zariski's connectedness theorem, $\mathrm{cl}(z)_0$ is connected, hence there is some j $(1 \leqq j \leqq 1)$ such that

$$\mathrm{cl}(z)_0 \subset W_j - \boldsymbol{P}(M_i)_0.$$

Now project everything into the join P_{ij} of $\boldsymbol{P}(M_i)$ and $\boldsymbol{P}(N_j)$. Since the projection is proper, and all components of $\boldsymbol{P}(\Delta_*)_0$ except $\boldsymbol{P}(M_i)_0$, $P(N_j)_0$ are mapped to one point, it follows that $p_{ij}(\mathrm{cl}\,(z)_0) =$ one point, and it is still disjoint from $\boldsymbol{P}(M_i)_0$. Therefore by (2.2), $\{N_j\}$ separates z from $\{M_i\}$.

In the case dim $A = 1$, an important class of trees Δ_* are the subtrees of the full Δ, i.e. linked sets $\Delta_*^{(0)}$ such that if $\{M_1\}$, $\{M_2\} \in \Delta_*^{(0)}$ are adjacent, then equivalently $\rho(\{M_1\}, \{M_2\}) =$ max. ideal of A, or $\{M_1\}$, $\{M_2\}$ are adjacent in Δ. These have the following easy characterization:

PROPOSITION (2.6). *If* dim $A = 1$, *and* $\Delta_*^{(0)} \subset \Delta^{(0)}$ *is a finite linked set then* $\boldsymbol{P}(\Delta_*)$ *is regular if and only if* Δ_* *is a subtree of* Δ.

We omit the proof, which is easy. When dim $A = 1$, $\mathrm{cl}(z)$ is necessarily isomorphic to S, i.e. it is the image of a section of $\boldsymbol{P}(\Delta_*)$ over S. Therefore $\mathrm{cl}(z) \cap \boldsymbol{P}(\Delta_*)_0$ is a single k-rational point of $\boldsymbol{P}(\Delta_*)_0$. If, moreover, $\boldsymbol{P}(\Delta_*)$ is regular, it must be a non-singular point of $\boldsymbol{P}(\Delta_*)_0$ and we have the following nice interpretation of the map $z \mapsto \mathrm{cl}\,(z) \cap \boldsymbol{P}(\Delta_*)_0$.

PROPOSITION (2.7): *Let* dim $A = 1$ *and let* $\Delta_* \subset \Delta$ *be a finite subtree Consider the maps*:

$$\begin{array}{ccc} z & \mapsto & \mathrm{cl}(z) \cap \boldsymbol{P}(\Delta_*)_0 \\ \rotatebox[origin=c]{-90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\ K\boldsymbol{P}^1 & \mapsto & [\text{non-singular } k\text{-rational points of } \boldsymbol{P}(\Delta_*)_0] \\ & & \wr\| \\ \text{ends of } \Delta & \rightarrow & [\text{edges of } \Delta - \Delta_* \text{ that meet } \Delta_*] \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ e & \mapsto & \text{last edge in path from } e \text{ to } \Delta_* \end{array}$$

The horizontal arrows are surjective and there is a unique isomorphism of the set of non-singular k-rational points of $\boldsymbol{P}(\Delta_*)_0$ *and the set of edges of* $\Delta - \Delta_*$ *meeting* Δ_* *making the diagram commute.*

(Proof left to reader).

The next step is to generalize Prop. (2.4) to infinite but locally finite trees Δ_*. We cannot do this in the category of schemes, but only in the category of formal schemes. Here is the construction:

given $\Delta_*^{(0)} \subset \Delta^{(0)}$ a linked subset with Δ_* locally finite

I) for all finite subtrees $S \subset \Delta_*$ let $\boldsymbol{P}(S)$ be the finite join as above,

II) when $S_1 \subset S_2$, there is a natural morphism

$$p : \boldsymbol{P}(S_2) \to \boldsymbol{P}(S_1)$$

giving us an inverse system,

III) let $\mathscr{P}(S) =$ formal completion of $\boldsymbol{P}(S)$ along its closed fibre $\boldsymbol{P}(S)_0$. We get again an inverse system:

$$p : \mathscr{P}(S_2) \to \mathscr{P}(S_1).$$

IV) For all S, let $\mathscr{P}(S)'$ be the maximal open subset $U \subset \mathscr{P}(S)$ such that for all finite subtrees $S \subset T \subset \Delta_*$, the morphism $\mathrm{res}_U\, p$ is an isomorphism:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\mathrm{res}_U\, p]{} & U \\ \cap & & \cap \\ \mathscr{P}(T) & \xrightarrow[p]{} & \mathscr{P}(S). \end{array}$$

V) Then the inverse system of $\mathscr{P}(S)$'s become a direct system of $\mathscr{P}(S)'$'s in which all morphisms:

$$\mathscr{P}(S_2)' \to \mathscr{P}(S_1)'$$

are open immersions. Let

$$\mathscr{P}(\Delta_*) = \varinjlim_{S} \mathscr{P}(S)'.$$

PROPOSITION (2.8): *$\mathscr{P}(\Delta_*)$ is a normal formal scheme flat over S, such that $m \cdot \mathcal{O}_{\mathscr{P}(\Delta)}$ is a defining sheaf of ideals. The closed fibre has the properties*:

i) it is reduced, connected, 1*-dimensional and locally of finite type over k,*

ii) it has at most ordinary double points and these are k-rational,

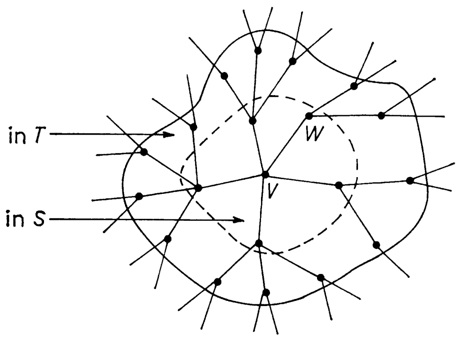
iii) its components are all isomorphic to $\boldsymbol{P}^1_k$ and are in one-one correspondence with the elements of $\Delta_*^{(0)}$,

iv) 2 *components meet if and only if the corresponding vertices of $\Delta_*^{(0)}$ are adjacent and then in exactly one point.*

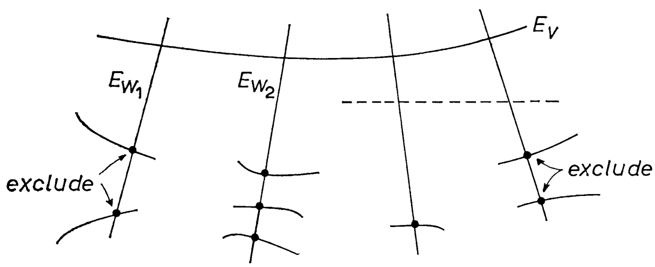
PROOF. This follows immediately from the previous Propositions once we have proven the following lemma.

LEMMA (2.9). Let $S \subset \Delta_*$ *be a finite subtree. If a vertex v of S is such that all edges of Δ_* which meet v lie in S, then the component E_v of $\boldsymbol{P}(S)_0$ corresponding to v lies entirely in the open set $\mathscr{P}(S)'$.*

PROOF OF LEMMA. Let $S \subset T \subset \Delta_*$, where T is another finite subtree containing all edges of Δ_* meeting edges meeting v, i.e.



Let $U \subset \boldsymbol{P}(S)_0$ be the open set consisting of E_v, plus those points x of the components E_w which meet E_v such that $p^{-1}(x)$ does not meet any other component of $\boldsymbol{P}(T)_0$:



Then even if $T' \supset T$ is a bigger finite subtree, it follows from Prop. (2.3) that in the diagram:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\mathrm{res}_U\, p_0} & U \\ \cap & & \cap \\ \boldsymbol{P}(T')_0 & \xrightarrow[p_0]{} & \boldsymbol{P}(S)_0 \end{array}$$

$\mathrm{res}_U\, p_0$ is an isomorphism. Since $p : \boldsymbol{P}(T') \to \boldsymbol{P}(S)$ is proper and surjective and $\boldsymbol{P}(S)$ is reduced, there is an open set $V \subset \boldsymbol{P}(S)$ such that $V \supset U$ and $p^{-1}(V) \to V$ is an isomorphism. Therefore the inverse image of U in the formal scheme $\mathscr{P}(T')$ is mapped isomorphically to the open subscheme U of $\mathscr{P}(S)$. Therefore $U \subset \mathscr{P}(S)'$. *Q.E.D.*

Speaking heuristically, $\mathscr{P}(\Delta_*)$ is the *infinite formal join* of the schemes $\boldsymbol{P}(M)$, for all $\{M\} \in \Delta_*^{(0)}$.

For every $z \in K\boldsymbol{P}^1$, we can talk about the closure of z, $\mathrm{cl}(z)^\wedge$, in $\mathscr{P}(\Delta_*)$. In fact, we can form $\mathrm{cl}_S(z)$ in $\boldsymbol{P}(S)$, take its formal completion $\mathrm{cl}_S(z)^\wedge$ in $\mathscr{P}(S)$, and restrict it to $\mathscr{P}(S)'$. Then if $S_1 \subset S_2$, $\mathrm{cl}_{S_1}(z)^\wedge$ is just the restriction of $\mathrm{cl}_{S_2}(z)^\wedge$ to $\mathscr{P}(S_1)$, hence there is a unique formal subscheme:

$$\mathrm{cl}(z)^\wedge \subset \mathscr{P}(\Delta_*)$$

such that

$$\mathrm{cl}(z)^\wedge \cap \mathscr{P}(S)' = \mathrm{cl}_S(z)^\wedge \cap \mathscr{P}(S)',$$

all finite subtrees S.

PROPOSITION (2.10). $\mathrm{cl}(z)^\wedge$ *is proper over S if and only if the base of S on Δ_* is finite. If* $\mathrm{cl}(z)^\wedge$ *is non-empty as well then it is the formal completion of the proper scheme* $\mathrm{cl}_S(z)$ *for all sufficiently large finite subtrees $S \subset \Delta_*$.*

This is an immediate consequence of (2.5).

COROLLARY (2.11). *If Γ is a flat Schottky group and $z \in K\mathbf{P}^1$, then Γ acts strictly discontinuously at z if and only if* $\mathrm{cl}(z)^\wedge \subset \mathscr{P}(\Delta_\Gamma)$ *is non-empty and proper over S.*

In case $\dim A = 1$, we have the infinite generalization of (2.7):

PROPOSITION (2.12). *Let* $\dim A = 1$ *and let Δ_* be a locally finite subtree of Δ. Consider the maps*

$$\begin{array}{rcl}
z & \mapsto & \mathrm{cl}(z) \cap (\Delta_*)_0 \\
\rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\
K\mathbf{P}^1 - i(\partial\Delta_*) & \to & [\textit{non-singular k-rational points of } (\Delta_*)_0] \\
\wr\| & & \\
\partial\Delta - \partial\Delta_* & \to & [\textit{edges of } \Delta - \Delta_* \textit{ meeting } \Delta_*] \\
\rotatebox[origin=c]{-90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\
e & \mapsto & \textit{last edge in path from } e \textit{ to } \Delta_*
\end{array}$$

The horizontal maps are surjective and there is a unique isomorphism of the set of non-singular k-rational points of $\mathscr{P}(\Delta_*)_0$ and the set of edges of $\Delta - \Delta_*$ meeting Δ_* making the diagram commute.

(Proof left to reader).

3. The construction of the quotient

We now restrict ourselves to the case $\Delta_* = \Delta_\Gamma$. Then the group Γ acts on $\mathscr{P}(\Delta_\Gamma)$. The final step in our construction is to form a quotient $\mathscr{P}(\Delta_\Gamma)/\Gamma$ and to algebrize it.

THEOREM (3.1). *There exists a unique pair $(\mathscr{X}, \pi)$ consisting of a formal scheme $\mathscr{X}$ proper over S and a surjective étale S-morphism*

$$\pi : \mathscr{P}(\Delta_\Gamma) \to \mathscr{X}$$

such that

a) $\forall \gamma \in \Gamma$, *if* $[\gamma]$ *represents the induced automorphism of* $\mathscr{P}(\Delta_\Gamma)$, *then* $\pi \circ [\gamma] = \pi$,

b) $\forall x, y \in \mathscr{P}(\Delta_\Gamma)$, $\pi(x) = \pi(y) \Leftrightarrow x = [\gamma]y$, some $\gamma \in \Gamma$.

Moreover $\mathscr{X}$ is normal, is flat and projective over S, and is algebraizable. *$\mathscr{X}$ will be written $\mathscr{P}(\Delta_\Gamma)/\Gamma$.*

PROOF. As a topological space, $\mathscr{X}$ must equal the quotient of the underlying topological space to $\mathscr{P}(\Delta_\Gamma)$ by Γ, and its structure sheaf must be the subsheaf of $\pi_*(\mathcal{O}_{\mathscr{P}(\Delta_\Gamma)})$ of Γ-invariants. Therefore $\mathscr{X}$ is unique. To construct $\mathscr{X}$, we proceed in two stages:

(i) prove the results for a suitable $\Gamma_0 \subset \Gamma$ of finite index,

(ii) prove them for Γ.

The point is that since Γ acts freely on the tree Δ_Γ, no $\gamma \in \Gamma$ $(\gamma \neq e)$ takes any component of $\mathscr{P}(\Delta_\Gamma)_0$ into itself. Even better, there is a normal subgroup $\Gamma_0 \subset \Gamma$ of finite index such that no $\gamma \in \Gamma_0(\gamma \neq e)$ takes a vertex of Δ_Γ into itself or to an adjacent vertex. Therefore no $\gamma \in \Gamma_0(\gamma \neq e)$ takes a component of $\mathscr{P}(\Delta_\Gamma)_0$ into itself or into a second component meeting the fiirst one. From this it follows that Γ_0 acts on $\mathscr{P}(\Delta_\Gamma)_0$ discontinuously in the Zariski topology: i.e. every $x \in \mathscr{P}(\Delta_\Gamma)_0$ has an open neighbourhood U such that $U \cap \gamma U = \emptyset$, all $\gamma \in \Gamma_0(\gamma \neq e)$. But in this case a quotient $\mathscr{P}(\Delta_\Gamma)/\Gamma_0$ can be constructed simply by re-glueing! To be precise cover $\mathscr{P}(\Delta_\Gamma)$ by affine open subschemes $\mathrm{Spf}(A_i)$ whose underlying open subsets have the above property. Then for every i, j, there is at most one element $\gamma_{ij} \in \Gamma_0$ such that

$$\gamma_{ij}(\mathrm{Spf}\,(A_i)) \cap \mathrm{Spf}\,(A_j) \neq \emptyset.$$

Glue $\mathrm{Spf}(A_i)$ to $\mathrm{Spf}(A_j)$ on this overlap via the map $[\gamma_{ij}]$. This gives a formal scheme $\mathscr{Y}$ and a morphism

$$\pi_0 : \mathscr{P}(\Delta_\Gamma) \to \mathscr{Y}$$

which is surjective and locally an isomorphism such that

(a) $\pi \circ [\gamma] = \pi$, all $\gamma \in \Gamma_0$, and

(b) $\pi(x) = \pi(y)$ implies $x = [\gamma]y$, some $\gamma \in \Gamma_0$.

Note that since Δ_Γ/Γ is a finite graph, so is Δ_Γ/Γ_0. Therefore $\mathscr{Y}_0$ has only a finite number of components and is proper over k. Therefore $\mathscr{Y}$ is proper over S. Obviously $\mathscr{Y}$ is normal and flat over S too since it is locally isomorphic to $\mathscr{P}(\Delta_\Gamma)$.

Now for each component E of $\mathscr{Y}_0$, choose a point $x \in E$ which is not in any other component of $\mathscr{Y}_0$. Let $\bar{d}_E \in \mathcal{O}_{x,\mathscr{Y}_0} = \mathcal{O}_{x,\mathscr{Y}}/m \cdot \mathcal{O}_{x,\mathscr{Y}}$ be a generator of the maximal ideal and let $d_E \in \mathcal{O}_{x,\mathscr{Y}}$ lift $\bar{d}_E$. Let $d_E = 0$ define the relative Cartier divisor $D_E \subset \mathscr{Y}$, and let $D = \Sigma D_E$. Then D is relatively ample on $\mathscr{Y}$ over S, hence $\mathscr{Y}$ is projective. Now we can apply Grothendieck's algebraizability theorem (EGA III.5) to conclude that

$\mathscr{Y}$ is the formal completion of a unique scheme Y, projective over S, along its closed fibre Y_0.

Finally, Y is a projective scheme, hence any finite subset of Y is contained in an affine. Therefore its Γ/Γ_0-orbits are contained in affines and there exists a quotient $Y/(\Gamma/\Gamma_0)$(cf. [M1], § 7). Let $\mathscr{X}$ be the formal completion of $Y/(\Gamma/\Gamma_0)$ along its closed fibre. This $\mathscr{X}$ has all the required properties.

Q.E.D.

DEFINITION (3.2). P_Γ is the scheme, projective over S, whose formal completion is $\mathscr{P}(\Delta_\Gamma)/\Gamma$.

We recall the concept of a *stable curve* over S in the sense of Deligne and Mumford: this is a scheme C, proper and flat over S, whose geometric fibres are reduced connected and 1-dimensional; have at most ordinary double points; and such that their non-singular rational components, if any, meet the remaining components in at least 3 points. Moreover a stable curve C over a field k will be call *degenerate* if Pic^0_C is a torus, or equivalently if the normalizations of all the components of $C \times_k \bar{k}$ ($\bar{k}$ an algebraic closure of k) are rational curves. C is called *k-split degenerate* if the normalizations of all the components of C are isomorphic to $\boldsymbol{P}^1_k$, and if all the double points are k-rational with 2 k-rational branches. This means that C is gotten by identifying in pairs a finite set of distinct k-rational points of a finite union of copies of $\boldsymbol{P}^1_k$.

THEOREM (3.3). *If n is the number of generators of Γ, then P_Γ is a stable curve over S of genus n, whose generic fibre is smooth over K and whose special fibre is k-split degenerate. Moreover its special fibre $(P_\Gamma)_0$ has the property*:

(*) *there is a $1-1$ correspondence between components of $(P_\Gamma)_0$ and vertices of Δ_Γ/Γ, and between double points of $(P_\Gamma)_0$ and edges of Δ_Γ/Γ, such that a component contains a double point if and only if the corresponding vertex is an endpoint of the corresponding edge.*

PROOF. Since $(P_\Gamma)_0$ is the quotient of $\mathscr{P}(\Delta_\Gamma)_0$ by Γ, the asserted properties of a stable curve are clear except for the requirement that every non-singular rational component meets the other components in $\geqq 3$ points. But by Prop. (1.20), every vertex of Δ_Γ/Γ is met by at least 3 edges so this is *OK*. Now a deformation of a stable curve is stable, so P_Γ is stable. Finally, the formal completion of P_Γ is normal, so P_Γ is normal. Therefore, its generic fibre is regular. Since it is also a stable curve, it is smooth over K too. *Q.E.D.*

Let $P_\Gamma(K)$ denote the set of K-rational points of P_Γ. We can now construct a map

$$\pi : \Omega_\Gamma \to P_\Gamma(K)$$

from the set of strict discontinuity to the set of rational points of the smooth curve $(P_\Gamma)_\eta$. In fact if $z \in \Omega_\Gamma$, we have seen that we can form

$$\mathrm{cl}(z)^\wedge \subset \mathscr{P}(\Delta_\Gamma)$$

and that $\mathrm{cl}(z)^\wedge$ is the formal completion of scheme $\mathrm{cl}(z)$ proper over S, and birational to it. This gives us a formal morphism.

$$\hat{p}_z : \mathrm{cl}\,(z)^\wedge \to \mathscr{P}(\Delta_\Gamma)/\Gamma$$

which by Grothendieck's theorem (EGA III.5) comes from a morphism:

$$p_z : \mathrm{cl}(z) \to P_\Gamma$$

Let $\pi(z)$ be the image of the generic point under this map.

PROPOSITION (3.4). *If $z_1, z_2 \in \Omega_\Gamma$, then*

$$[\pi(z_1) = \pi(z_2)] \Leftrightarrow [\exists \gamma \in \Gamma, \text{ such that } \gamma(z_1) = z_2].$$

PROOF. '$\Leftarrow$' is obvious. Conversely, say $\pi(z_1) = \pi(z_2)$. Let Z be the join of $\mathrm{cl}(z_1)$, $\mathrm{cl}(z_2)$: Z is proper over S and birational to it. In particular, Z_0 is connected. By assumption the 2 morphisms:

$$\begin{tikzcd} & \mathrm{cl}(z_1) \arrow[dr, "p_{z_1}"] & \\ Z \arrow[ur] \arrow[dr] & & P_\Gamma \\ & \mathrm{cl}(z_2) \arrow[ur, "p_{z_2}"'] & \end{tikzcd}$$

are equal. Therefore the 2 formal morphisms:

$$\begin{tikzcd} & \mathrm{cl}(z_1)^\wedge \subset \mathscr{P}(\Delta_\Gamma) \\ \hat{Z} \arrow[ur] \arrow[dr] & \\ & \mathrm{cl}(z_2)^\wedge \subset \mathscr{P}(\Delta_\Gamma) \end{tikzcd}$$

both lift the same formal morphism to $\mathscr{P}(\Delta_\Gamma)/\Gamma$. Since Z_0 is connected and $\mathscr{P}(\Delta_\Gamma)$ is étale over $\mathscr{P}(\Delta_\Gamma)/\Gamma$, these 2 differ by the action of some $\gamma \in \Gamma$. Therefore for a large finite subtree $S \subset \Delta_\Gamma$, we get a commutative diagram:

$$\begin{tikzcd} & \mathrm{cl}(z_1) \subset & \mathbf{P}(S) \arrow[d, "{[\gamma]}"] \\ Z \arrow[ur] \arrow[dr] & & \\ & \mathrm{cl}(x_2) \subset & \mathbf{P}(\gamma(S)). \end{tikzcd}$$

Evaluating these on the generic point, it follows that $\gamma(z_1) = z_2$. *Q.E.D.*

THEOREM (3.5). ***If A is a regular local ring, then π is surjective.***

PROOF. Note that if $\Gamma_1 \subset \Gamma$ is a subgroup of finite index, then

i) the set of fixed points Σ_1 of Γ_1 and Σ of Γ are the same (since $\forall \gamma \in \Gamma$, γ^n is in Γ_1 for some n) hence

ii) $\Delta_{\Gamma_1} = \Delta_\Gamma$ and

iii) $\mathscr{P}(\Delta_{\Gamma_1}) = \mathscr{P}(\Delta_\Gamma)$, hence

iv) P_{Γ_1} is a finite étale covering of P_Γ.

The first step in our proof is that the map

$$P_{\Gamma_1}(K) \rightarrow P_\Gamma(K)$$

is surjective. To see this, take some $z \in P_\Gamma(K)$ and let $\mathrm{cl}(z)$ be the closure of z in P_Γ, and let $\mathrm{cl}(z)'$ be the normalization of $\mathrm{cl}(z)$. Taking the fibre product:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & P_{\Gamma_1} \arrow[d] \\ \mathrm{cl}(z)' \arrow[r, "f"] & P_\Gamma \end{tikzcd}$$

we get an induced finite étale covering W of $\mathrm{cl}(z)'$. Since $\mathrm{cl}(z)'$ is normal, so is W and hence the components W_i of W are disjoint and are each finite and étale over $\mathrm{cl}(z)'$. Let K_i be the function field of W_i and let A_i be the normalization of A in K_i. Since the projection

$$\mathrm{cl}(z)' \rightarrow \operatorname{Spec} A$$

is birational, $\mathrm{cl}(z)'$ and Spec A are isomorphic outside a closed set Z of codimension two in $\operatorname{Spec}(A)$. Therefore over $\operatorname{Spec}(A)-Z$, W_i is isomorphic to $\operatorname{Spec}(A_i)$. In particular, $\operatorname{Spec}(A_i)$ is unramified over $\operatorname{Spec}(A)$ in codimension one. But by the theorem of the purity of the branch locus, which applies since A is regular, this proves that A_i is unramified everywhere over A; hence by Hensel's lemma, A_i is isomorphic to A. Therefore $K_i = K$ and $W_i \cong \mathrm{cl}(z)'$. This moves that not only is

$$P_{\Gamma_1}(K) \rightarrow P_\Gamma(K)$$

surjective, but also that there is a lifting f_1:

$$\begin{tikzcd} & P_{\Gamma_1} \arrow[d] \\ \mathrm{cl}(z)' \arrow[ur, dashed, "f_1"] \arrow[r, "f"] & P_\Gamma \end{tikzcd}$$

for every $\Gamma_1 \subset \Gamma$ if finite index.

The second step is that by passing to the formal completion of $\mathrm{cl}(z)'$ along the closed fibre, there exists a lifting $\hat{g}$:

$$\begin{tikzcd}
& P(\Delta_\Gamma) \arrow[d, dotted] \\
& P(\Delta_{\Gamma_1})/\Gamma_1 \arrow[d] \\
\widehat{cl(z)}' \arrow[uur, dashed, "\hat{g}"] \arrow[ur, "\hat{f}_1"] \arrow[r, "\hat{f}"] & P(\Delta_\Gamma)/\Gamma
\end{tikzcd}$$

In fact let k be the maximum number of edges meeting any vertex of Δ_Γ and let n be the number of components of the closed fibre of $\text{cl}(z)'$. Then if Γ_1 is sufficiently small, every $\gamma \in \Gamma_1$ ($\gamma \neq e$) maps every vertex v of Δ_Γ to a vertex $\gamma(v)$ joined to v by a line with more than $(k+1)n$ edges. Let $U \subset (P_{\Gamma_1})_0$ be the open subset consisting of all points of all the components that meet $f_1(\text{cl}(z)'_0)$ except those that lie also on components disjoint from $f_1(\text{cl}(z)'_0)$. Then U has at most $(k+1)n$ components. Let $\hat{U}$ be the corresponding open sub-formal scheme of $\mathscr{P}(\Delta_{\Gamma_1})/\Gamma_1 = P^{\wedge}_{\Gamma_1}$. Let $\hat{V}$ be the inverse image of $\hat{U}$ in $\mathscr{P}(\Delta_\Gamma)$. By our assumption on the way Γ_1 operates in Δ_Γ, no 2 components E and $\gamma(E)$ of $\hat{V}(\gamma \in \Gamma_1, \gamma \neq e)$ can be joined by a line of components of $\hat{V}$: hence $\hat{V}$ is the disjoint union of copies of $\hat{U}$. Choosing one of these, we can lift $\hat{f}_1$ uniquely to a morphism $\hat{g}$ of $\widehat{\text{cl}(z)}'$ into this component.

Thirdly, $\hat{g}(\text{cl}(z)'_0)$ is proper over k, hence it lies in one of the approximating pieces:

$$\mathscr{P}(S)' \subset \mathscr{P}(\Delta_\Gamma)$$

$$S \subset \Delta_\Gamma \text{ finite subtree.}$$

Thus $\hat{g}$ can be algebraized to a true morphism:

$$g : \text{cl}(z)' \to \boldsymbol{P}(S).$$

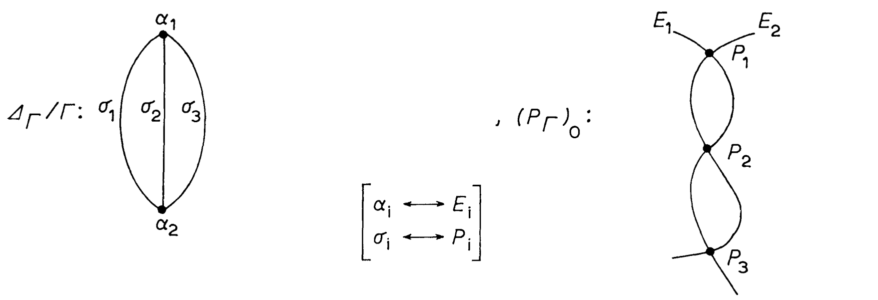
The image of the generic point here is a point of $K\boldsymbol{P}^1$ which clearly lies in Ω_Γ and is mapped by π to z. *Q.E.D.*

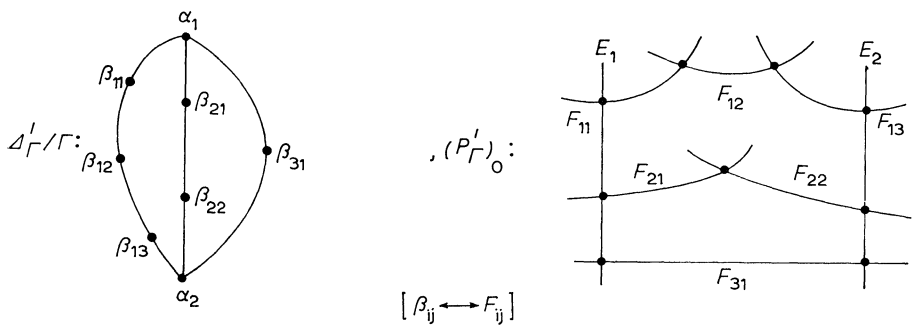
Before ending this section, I would like to discuss briefly the special features of the dim $A = 1$ case and indicate how the somewhat more precise and elegant formulation given in the Introduction can be worked out. When dim $A = 1$, one has the big tree Δ and it usually is more convenient to replace the tree Δ_Γ by the tree Δ'_Γ where vertices are:

a) the vertices of Δ_Γ
b) the vertices of Δ intermediate between 2 vertices of Δ_Γ.

Then Δ'_Γ is a subtree of Δ hence $\mathscr{P}(\Delta'_\Gamma)$ is regular by (2.6): in fact, $\mathscr{P}(\Delta'_\Gamma)$ is just the minimal resolution of the normal surface $\mathscr{P}(\Delta_\Gamma)$. Let $\mathscr{P}(\Delta'_\Gamma)/\Gamma$ be the formal completion of P'_Γ. Then P'_Γ is regular and is the minimal resolution of the normal surface P_Γ. Generically, $P'_\Gamma = P_\Gamma$, but the closed fibre is now only a semi-stable curve – i.e. a reduced connected 1-dimensional scheme with at most ordinary double points and such that every non-singular rational component meets the other components in at least *2* points. P'_Γ is the so-called *minimal model* of the curve $(P_\Gamma)_\eta$ over A (cf. [$D-M$], p. 87, [L] and [$\check{S}$]). Moreover, the closed fibre $(P'_\Gamma)_0$ has only rational components one for each vertex of the graph Δ'_Γ/Γ; and one double point for each edge of the graph Δ'_Γ/Γ. Δ'_Γ/Γ is the graph referred to as $(\Delta/\Gamma)_0$ in the Introduction.

EXAMPLE.





All this is an immediate generalization of (3.3) and is proven in exactly the same way. Finally applying Prop. (2.12) to Δ'_Γ, we get a commutative diagram on the upper-left.

(*See figure on next page.*)

Then the commutative diagram on the lower-right is deduced by taking the quotient of each set by Γ.

$$\begin{tikzcd}
\Omega_\Gamma \arrow[rr, two heads, "z \mapsto \mathrm{cl}(z)_0"] \arrow[dr, "\pi"] \arrow[dd, equal] & & {[\text{n.s. } k\text{-rat. pts. of } \mathscr{P}(\Delta'_\Gamma)_0]} \arrow[dr] \arrow[dd, equal] & \\
& P_\Gamma(K) \arrow[rr, two heads, "z \mapsto \mathrm{cl}(z)_0"] \arrow[dd, equal] & & {[\text{n.s. } k\text{-rat. pts. of } (P'_\Gamma)_0]} \arrow[dd, equal] \\
\partial\Delta - \partial\Delta'_\Gamma \arrow[rr, two heads] \arrow[dr] & & {[\text{edges of } \Delta - \Delta'_\Gamma \text{ meeting } \Delta'_\Gamma]} \arrow[dr] & \\
& \partial(\Delta/\Gamma) \arrow[rr, two heads] & & {[\text{edges of } \Delta_\Gamma/\Gamma - \Delta'_\Gamma/\Gamma \text{ meeting } \Delta'_\Gamma/\Gamma]}
\end{tikzcd}$$

4. Existence and uniqueness

Summarising the discussion up to this point, we have started with a flat Schottky group $\Gamma \subset PGL(2, K)$, and have then constructed:

a) a tree Δ_Γ,

b) a formal scheme $\mathscr{P}(\Delta_\Gamma)$,

c) a stable curve P_Γ/S such that $\hat{P}_\Gamma \cong \mathscr{P}(\Delta_\Gamma)/\Gamma$ with generic fibre non-singular and with closed fibre k-split degenerate.

Let $(P_\Gamma)_{\bar{\eta}}$ be the generic *geometric* fibre of P_Γ over an algebraic closure $\bar{K}$ of K. The next question to study is whether the non-singular curve $(P_\Gamma)_{\bar{\eta}}$ determines Γ uniquely. We shall invert our construction step by step. The first is this:

PROPOSITION (4.1): *If C_1, C_2 are 2 stable curves over S with k-split degenerate fibres, then the scheme $\underline{\mathrm{Isom}}_S(C_1, C_2)$ exists and is isomorphic to a disjoint union $\coprod_{i=1}^N S_i$ of closed subschemes of S.*

PROOF. According to Theorem (1.11) of [$D-M$], $\underline{\mathrm{Isom}}_S(C_1, C_2)$ exists and is finite and unramified over S. Since $S = \mathrm{Spec}\,(A)$, A complete, by Hensel's lemma $\underline{\mathrm{Isom}}$ is a disjoint union of schemes S_i with only a single point over the closed point of S. But because $C_{1,0}$ and $C_{2,0}$ are k-split degenerate every isomorphism of $C_{1,0}$ and $C_{2,0}$ is uniquely determined by specifying which components of $C_{1,0}$ go to which components of $C_{2,0}$ and which double points go to which double points; once this combinatorial data is specified, there is at most one such isomorphism of $C_{1,0}$ and $C_{2,0}$ and if it exists at all, it is rational over k. Therefore the closed points $s_i \in S_i$ are k-rational. Since S_i is finite and unramified over S, $S_i \to S$ must in fact be a closed immersion. *Q.E.D.*

COROLLARY (4.2): *Every isomorphism of the generic geometric fibres $(P_{\Gamma_1})_{\bar{\eta}}$, $(P_{\Gamma_2})_{\bar{\eta}}$ extends uniquely to an isomorphism of P_{Γ_1} and P_{Γ_2}.*

PROOF. The hypothesis means that one of the components S_i of $\underline{\mathrm{Isom}}_S(P_{\Gamma_1}, P_{\Gamma_2})$ has a point over the generic point of S, hence S_i is isomorphic to S and defines an S-isomorphism of P_{Γ_1}, P_{Γ_2}.

PROPOSITION (4.3). *The morphism* $\mathscr{P}(\Delta_\Gamma) \to \hat{P}_\Gamma$ *makes* $\mathscr{P}(\Delta_\Gamma)$ *into the universal covering space of* $\hat{P}_\Gamma$.

PROOF. In fact, the category of formal étale coverings of $\hat{P}_\Gamma$ is isomorphic to the category of étale coverings of $(P_\Gamma)_0$. Since the closed fibre $\mathscr{P}(\Delta_\Gamma)_0$ is connected and is a tree-like union of copies of $\boldsymbol{P}^1_k$, it is simply connected and must be the universal covering space of $(P_\Gamma)_0$. *Q.E.D.*

DEFINITION (4.4). An *exterior isomorphism* of 2 groups G_1, G_2 is the set of isomorphisms $\alpha\varphi\alpha^{-1}$ conjugate to an ordinary isomorphism φ.

When we talk of $\pi_1(X)$, for a connected scheme X, in order to have a well-defined group depending functorially on X we have to fix a geometric base point $x : \operatorname{Spec}(\Omega) \to X$ (Ω an algebraically closed field), and then π_1 should be written $\pi_1(X, x)$. But up to exterior isomorphism, π_1 is independent of x, hence so long as we only talk of exterior isomorphisms, we can write $\pi_1(X)$.

COROLLARY (4.5). *Γ, as an abstract group, is canonically exterior-isomorphic to* $\pi_1(\hat{P}_\Gamma)$*(or* $\pi_1((P_\Gamma)_0)$*).*

COROLLARY (4.6). *Starting with an isomorphism*

$$\bar{\varphi} : (P_{\Gamma_1})_{\bar{\eta}} \simeq (P_{\Gamma_2})_{\bar{\eta}},$$

$\bar{\varphi}$ first extends uniquely to an isomorphism

$$\varphi : P_{\Gamma_1} \to P_{\Gamma_2}$$

hence to an isomorphism

$$\hat{\varphi} : \hat{P}_{\Gamma_1} \to P_{\Gamma_2},$$

hence to a pair consisting of an isomorphism

$$\alpha : \Gamma_1 \to \Gamma_2$$

and an α-equivariant isomorphism

$$\tilde{\varphi} : \mathscr{P}(\Delta_{\Gamma_1}) \to \mathscr{P}(\Delta_{\Gamma_2}).$$

Then $(\alpha, \tilde{\varphi})$ is unique up to a change $(\alpha', \tilde{\varphi}') = (\gamma\alpha\gamma^{-1}, \gamma \circ \tilde{\varphi})(\gamma \in \Gamma_2)$.

The last step is to show how the function field $R(\boldsymbol{P}^1_K)$ can be identified inside the field of meromorphic functions on $\mathscr{P}(\Delta_\Gamma)$, so that Γ as *a subgroup of* $\mathrm{PGL}(2, \mathrm{K})$ $(= \operatorname{Aut}_K R(\boldsymbol{P}^1_K))$ can be recovered from Γ as a group of automorphisms of $\mathscr{P}(\Delta_\Gamma)$.

PROPOSITION (4.7): *Let* $\mathscr{D} \subset \mathscr{P}(\Delta_\Gamma)$ *be a positive relative Cartier divisor such that* $\mathscr{D}_0$ *meets only one component of the closed fibre* $\mathscr{P}(\Delta_\Gamma)_0$. *Then* $R(\boldsymbol{P}^1_K)$*, as a field of meromorphic functions on* $\mathscr{P}(\Delta_\Gamma)$ *is the quotient field of the A-algebra:*

$$\bigcup_{n=1}^{\infty} \Gamma(\mathscr{P}(\Delta_\Gamma), \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)}(n\mathscr{D})).$$

PROOF. Let $\mathscr{D}_0$ meet the component of $\mathscr{P}(\Delta_\Gamma)_0$ corresponding to the vertex $\{M\} \in \Delta_\Gamma^{(0)}$. Consider the projection:

$$p : \mathscr{P}(\Delta_\Gamma) \to \boldsymbol{P}(M)^\wedge.$$

Then $\mathscr{D}$ is the inverse image of a relative Cartier divisor in $\boldsymbol{P}(M)^\wedge$. By Grothendieck's existence Theorem (EGA III.5), this divisor is the formal completion of a relative Cartier divisor $D \subset \boldsymbol{P}(M)$. We have homomorphisms:

$$\Gamma(\boldsymbol{P}(M), \mathcal{O}_{\boldsymbol{P}(M)}(nD)) \overset{\approx}{\to} \Gamma(\boldsymbol{P}(M)^\wedge, \mathcal{O}_{\boldsymbol{P}(M)^\wedge}(n\hat{D})) \underset{p^*}{\hookrightarrow} \Gamma(\mathscr{P}(\Delta_\Gamma), \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)}(n\mathscr{D})),$$

the first being an isomorphism by EGA III.4.1. Since $R(\boldsymbol{P}_K^1)$ is the quotient field of

$$\bigcup_{n=1}^{\infty} \Gamma(\boldsymbol{P}(M), \mathcal{O}_{\boldsymbol{P}(M)}(nD)),$$

the proposition will follow if we show that the second of these maps is an isomorphism too. This follows from:

LEMMA (4.8). *Let $\Delta_* \subset \Delta_\Gamma$ be any finite subtree. Let $p : \mathscr{P}(\Delta_\Gamma) \to \boldsymbol{P}(\Delta_*)^\wedge$ be the projection. Then*

$$p_*(\mathcal{O}_{\mathscr{P}(\Delta_\Gamma)}) = \mathcal{O}_{\boldsymbol{P}(\Delta_*)^\wedge}.$$

PROOF. It suffices to prove that for affine open affine $U \subset \boldsymbol{P}(\Delta_*)_0$ and every ideal $I \subset A$ of finite codimension, that

$$\Gamma(U, \mathcal{O}_{\boldsymbol{P}(\Delta_*)}/I \cdot \mathcal{O}_{\boldsymbol{P}(\Delta_*)}) \to \Gamma(p^{-1}(U), \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)}/I \cdot \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)})$$

is an isomorphism. We check this by induction on $\dim_k(A/I)$. If $I = m$, note that the open subscheme $p^{-1}(U)$ of the closed fibre $\mathscr{P}(\Delta_\Gamma)_0$ is obtained from U by adding infinite trees of $\boldsymbol{P}_k^1$'s at a finite set of points of U. Since global sections of $\mathcal{O}_{\boldsymbol{P}_k^1}$ are constants in k, a section on $p^{-1}(U)$ of $\mathcal{O}_{\mathscr{P}(\Delta_\Gamma)_0}$ is just a section on U of $\mathcal{O}_{\boldsymbol{P}(\Delta_*)_0}$ extended as a constant to each of these trees. The result is true when $I = m$. In general, if $I_0 = I + A \cdot \eta$, where $m \cdot \eta \subset I$, then by flatness of $\mathscr{P}(\Delta_\Gamma)$ and $\boldsymbol{P}(M)$ over S, we get a diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathcal{O}_{\boldsymbol{P}(\Delta_*)}/m \cdot \mathcal{O}_{\boldsymbol{P}(\Delta_*)} & \xrightarrow{\eta} & \mathcal{O}_{\boldsymbol{P}(\Delta_*)}/I \cdot \mathcal{O}_{\boldsymbol{P}(\Delta_*)} & \to & \mathcal{O}_{\boldsymbol{P}(\Delta_*)}/I_0 \mathcal{O}_{\boldsymbol{P}(\Delta_*)} & \to 0 \\
 & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\
0 \to & p_* \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)}/m \cdot \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)} & \xrightarrow{\eta} & p_* \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)}/I \cdot \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)} & \to & p_* \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)}/I_0 \mathcal{O}_{\mathscr{P}(\Delta_\Gamma)}. &
\end{array}$$

Then α and γ are isomorphisms by induction, hence so is β.

Note that plenty of such $\mathscr{D}$'s exist: just take a closed point x of the closed fibre $\mathscr{P}(\Delta_\Gamma)_0$ where $\mathscr{P}(\Delta_\Gamma)$ is smooth over S: let $f \in \mathcal{O}_{x, \mathscr{P}(\Delta_\Gamma)}$ be an element such that

$$m_{x, \mathscr{P}(\Delta_\Gamma)} = m \cdot \mathcal{O}_{x, \mathscr{P}(\Delta_\Gamma)} + f \cdot \mathcal{O}_{x, \mathscr{P}(\Delta_\Gamma)}.$$

Then $f = 0$ defines such a $\mathscr{D}$. Therefore we have:

COROLLARY (4.9). *In the situation of Corollary* (4.6), *the map induced by* $\tilde{\varphi}$ *on the meromorphic functions restricts to an isomorphism*:

$$\varphi^* : R(\boldsymbol{P}^1_K) \to R(\boldsymbol{P}^1_K).$$

If φ^* *is given by an element* $g \in \mathrm{PGL}(2, K)$. *then* $\alpha : \Gamma_1 \to \Gamma_2$ *is given by*

$$\alpha(\gamma) = g\gamma g^{-1}$$

PROOF. The first part follows immediately from (4.8) since such $\mathscr{D}$'s exist and $\tilde{\varphi}$ takes such a $\mathscr{D}$ to another such $\mathscr{D}$. Since $\tilde{\varphi}$ is α-equivariant, so is φ^*, i.e. acting on $R(\boldsymbol{P}^1_K)$, we find

$$\varphi^* \circ \gamma = \alpha(\gamma) \circ \varphi^*, \text{ all } \gamma \in \Gamma_1.$$

Hence if φ^* is given by the action of $g \in PGL(2, K)$ on $R(\boldsymbol{P}^1_K)$, $g\gamma = \alpha(\gamma)g$, all $\gamma \in \Gamma_1$. *Q.E.D.*

COROLLARY (4.10).

$$\begin{aligned}\mathrm{Isom}_{\bar{K}}((P_\Gamma)_{\bar{\eta}}, (P_{\Gamma_2})_{\bar{\eta}}) &= \mathrm{Isom}_S(P_{\Gamma_1}, P_{\Gamma_2}) \\ &= \{g \in PGL(2, K) | g\Gamma_1 g^{-1} = \Gamma_2\}/\text{modulo } g \sim g',\end{aligned}$$

where $g \sim g'$ *if* $\exists \gamma_2 \in \Gamma_2$ *such that* $g' = \gamma_2 g$, *or equivalently* $\exists \gamma_1 \in \Gamma_1$ *such that* $g' = g\gamma_1$.

PROOF. It suffices to note that everything can be reversed: given $g \in PGL(2, K)$ such that $g\Gamma_1 g^{-1} = \Gamma_2$, then g defines a $\tilde{\varphi}$ and an α, hence a $\hat{\varphi}$, hence a φ. *Q.E.D.*

COROLLARY (4.11). $(P_{\Gamma_1})_{\bar{\eta}}$ *is isomorphic to* $(P_{\Gamma_2})_{\bar{\eta}}$ *if and only if* Γ_1 *is conjugate to* Γ_2 *in* $PGL(2, K)$. *All isomorphisms are, moreover, rational over K.*

COROLLARY (4.12). Aut $((P_\Gamma)_{\bar{\eta}})$ *is isomorphic to* $N(\Gamma)/\Gamma$, *where* $N(\Gamma)$ *is the normalizer of* Γ *in* $PGL(2, K)$. *All automorphisms are, moreover, rational K.*

We turn finally to the question of the existence of these uniformizations. We wish to start only with a stable curve C/S with non-singular generic fibre and k-split degenerate closed fibre and reverse the construction step-by-step.

Step I: Let $\mathscr{C}$ be the formal completion of C along its closed fibre. Note that C must be normal, hence so is $\mathscr{C}$.

Step II: Let $p_0 : P_0 \to C_0$ be the universal covering scheme of C_0 and let Γ be the group of cover transformations. It is clear that P_0 is an infinite union of copies of $\boldsymbol{P}^1_k$, each one joined to a finite number of others (at least 3 others) at k-rational double points, but the whole being connected as a tree. More precisely, if we make a graph Δ (resp. G) to illustrate P_0 (resp. C_0) in the usual way (one vertex for each component, one edge for each double point), then Δ is a tree, Γ acts freely on Δ and $\Delta/\Gamma \cong G$.

Step III: Since the category of étale coverings of $\mathscr{C}$ and of C_0 are equivalent, there is a unique formal scheme $\mathscr{P}$ with closed fibre P_0, and formal étale morphism

$$p : \mathscr{P} \to \mathscr{C}$$

extending p_0. Moreover, Γ acts on $\mathscr{P}$ so that $\mathscr{C} \cong \mathscr{P}/\Gamma$.

Step IV: For each component M of $\mathscr{P}_0$, let $\mathscr{D} \subset \mathscr{P}$ be a positive relative Cartier divisor such that $\mathscr{D}_0$ meets only the component M of $\mathscr{P}_0$. Let:

$$\boldsymbol{P}(M) = \operatorname{Proj} \sum_{n=0}^{\infty} \Gamma(\mathscr{P}, \mathcal{O}_{\mathscr{P}}(n\mathscr{D})).$$

PROPOSITION (4.13). *$\boldsymbol{P}(M) \cong \boldsymbol{P}^1 \times S$, and there is a canonical formal morphism*:

$$p_M : \mathscr{P} \to \boldsymbol{P}(M)^\wedge$$

which on the closed fibre $\mathscr{P}_0$ maps every component $M' \neq M$ to a point and maps M isomorphically onto $\boldsymbol{P}(M)_0$.

PROOF. First we need:

LEMMA (4.14). $H^1(\mathscr{P}_0, \mathcal{O}_{\mathscr{P}_0}(n\mathscr{D})) = (0)$, all $n \geqq 0$.

PROOF. Let $\mathscr{P}'_0$ be the disjoint union of the components of $\mathscr{P}_0$, and let $q : \mathscr{P}'_0 \to \mathscr{P}_0$ be the obvious morphism. We have an exact sequence:

$$0 \to \mathcal{O}_{\mathscr{P}_0} \to q_*(\mathcal{O}_{\mathscr{P}'_0}) \to \bigoplus_{\substack{\text{double}\\ \text{points } x}} k(x) \to 0.$$

It is obvious that

$$H^1(q_*(\mathcal{O}_{\mathscr{P}'_0}) \cong H^1(\mathcal{O}_{\mathscr{P}'_0}) \cong \prod_{\substack{\text{components}\\ M}} [H^1(\mathcal{O}_M)] = (0),$$

and that

$$\Big(\prod_{\substack{\text{components}\\ M}} [H^0(\mathcal{O}_M)]\Big) = H^0(\mathcal{O}_{\mathscr{P}'_0}) \to \prod_{\substack{\text{double}\\ \text{points } x}} k(x)$$

is surjective (since $\mathscr{P}_0$ is connected together like a tree!). Therefore $H^1(\mathcal{O}_{\mathscr{P}_0}) = (0)$. Moreover, if $n > 0$, use

$$0 \to \mathcal{O}_{\mathscr{P}_0} \to \mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0) \to \mathscr{F} \to 0$$

where dim (Supp $\mathscr{F}$) = 0. From this it follows that $H^1(\mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0)) = (0)$ too.

LEMMA (4.15). *For all $n \geqq 0$, $\Gamma(\mathscr{P}, \mathcal{O}_{\mathscr{P}}(n\mathscr{D}))$ is a finitely generated free A-module such that*

$$\Gamma(\mathscr{P}, \mathcal{O}_{\mathscr{P}}(n\mathscr{D})) \underset{A}{\otimes} k \cong \Gamma(\mathscr{P}_0, \mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0)) \cong \Gamma(M, \mathcal{O}_M(n\mathscr{D}_0)).$$

PROOF. It suffices to prove that for all $I \subset A$ of finite codimension, $\Gamma(\mathscr{P}, \mathcal{O}_{\mathscr{P}}(n)/I \cdot \mathcal{O}_{\mathscr{P}}(n\mathscr{D}))$ is a finitely generated free A/I-module such that

$$\Gamma(\mathscr{P}, \mathcal{O}_{\mathscr{P}}(n\mathscr{D})/I \cdot \mathcal{O}_{\mathscr{P}}(n\mathscr{D})) \underset{A}{\otimes} A/m = \Gamma(\mathscr{P}_0, \mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0)).$$

If $I = m$, note that every section of $\mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0)$ is just a section of $\mathcal{O}_M(n\mathscr{D}_0)$ extended as a constant to all other components of $\mathscr{P}_0$, hence

$$\text{res}: \Gamma(\mathscr{P}_0, \mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0)) \xrightarrow{\approx} \Gamma(M, \mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0))$$

is an isomorphism. In general, use induction of dim A/I. If $I_0 = I + A \cdot \eta$, where $m \cdot \eta \subset I$, we get a diagram (because of *flatness* of $\mathscr{P}$ over S):

$$\begin{aligned} 0 \to H^0(\mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0)) &\xrightarrow{\eta} H^0(\mathcal{O}_{\mathscr{P}}(n\mathscr{D})/I \cdot \mathcal{O}_{\mathscr{P}}(n\mathscr{D})) \\ &\to H^0(\mathcal{O}_{\mathscr{P}}(n\mathscr{D})/I_0 \cdot \mathcal{O}_{\mathscr{P}}(n\mathscr{D})) \to H^1(\mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0)) = (0). \end{aligned}$$

Using this, the assertion for I_0 implies immediately the assertion for I.

Q.E.D.

It follows from (4.15) that $\sum_{n=0}^{\infty} \Gamma(\mathscr{P}, \mathcal{O}_{\mathscr{P}}(n\mathscr{D}))$ is a free finitely generated graded A-algebra, hence its Proj is a flat and proper scheme over S; moreover, its closed fibre is just

$$\text{Proj}\,\{\sum_{n=0}^{\infty} \Gamma(M, \mathcal{O}_M(n\mathscr{D}_0))\},$$

which is M itself. Since $M \cong \boldsymbol{P}^1_k$ and all deformations of $\boldsymbol{P}^1$ are trivial, this proves that $\boldsymbol{P}(M) \cong \boldsymbol{P}^1 \times S$. Finally, $\mathcal{O}_{\mathscr{P}_0}(\mathscr{D}_0)$ is generated by its global sections, hence by (4.15), $\mathcal{O}_{\mathscr{P}}(\mathscr{D})$ is generated by its sections. Therefore there is a formal morphism from $\mathscr{P}$ to $\boldsymbol{P}(M)^\wedge$. Since $\mathscr{D}_0$ is very ample on M and since all sections of $\mathcal{O}_{\mathscr{P}_0}(n\mathscr{D}_0)$ (any $n \geqq 0$) are constant on all other components of $\mathscr{P}_0$, the last assertions of the Proposition are obvious. *Q.E.D.*

PROPOSITION (4.16). *For any 2 components M_1, M_2 of $\mathscr{P}_0$, consider the morphism $p_{M_1,M_2} : \mathscr{P} \to (\boldsymbol{P}(M_1) \times_S \boldsymbol{P}(M_2))^\wedge$. There is a unique* *relative Carter divisor* $Z \subset \boldsymbol{P}(M_1) \times_S \boldsymbol{P}(M_2)$ *defined by an equation*:

$$ax_1x_2 + bx_1y_2 + cy_1x_2 + dy_1y_2 = 0, \; a, b, c, d \in A$$

(*where* x_i, y_i *are homogeneous coordinates on* $\boldsymbol{P}(M_i)$) *such that* p_{M_1, M_2} *factors through* Z. *Moreover* $ad - bc \neq 0$ *but* $ad - bc \in m$.

PROOF. Via the isomorphism $\boldsymbol{P}(M_i) \cong \boldsymbol{P}^1 \times S$, let the sheaf $\mathcal{O}(1) \otimes \mathcal{O}_S$ go over to the sheaf L_i on $\boldsymbol{P}(M_i)$. Then $\Gamma(\boldsymbol{P}(M_i), L_i) \cong A \cdot x_i \oplus A \cdot y_i$. Let $K = p_1^* L_1 \otimes p_2^* L_2$ on $\boldsymbol{P}(M_1) \times_S \boldsymbol{P}(M_2)$. Then:

$$\Gamma(\boldsymbol{P}(M_1) \times_S \boldsymbol{P}(M_2), K) \cong Ax_1x_2 \oplus Ay_1x_2 \oplus Ax_1y_2 \oplus Ay_1y_2.$$

Let $\hat{L}_i = p_{M_i}^*(L_i)$, $\hat{K} = p_{M_1, M_2}^*(K)$ be the induced sheaves on $\mathscr{P}$, and let $L_{i,0}$, K_0 be the induced sheaves on $\mathscr{P}_0$. The first step is to check that:

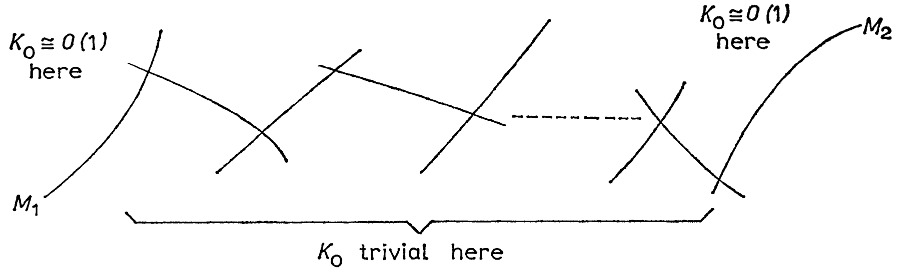
a) $H^0(\mathscr{P}_0, K_0) \cong H^0(\boldsymbol{P}(M_1)_0 \times \boldsymbol{P}(M_2)_0, K_0)$/modulo 1-dimensional subspace of form $\lambda(\alpha x_1 + \beta y_1) \cdot (\gamma x_2 + \delta y_2)$,

b) $H^1(\mathscr{P}_0, K_0) = (0)$.

In fact, $L_{i,0}$ is a trivial invertible sheaf on all components of $\mathscr{P}_0$ except M_i; it follows easily that if we pick arbitrary sections in the 2-dimensional spaces:

$$H^0(M_1, K_0 \otimes \mathcal{O}_{M_1}), \quad H^0(M_2, K \otimes \mathcal{O}_{M_2}),$$

they extend to at most one section of $\mathscr{P}_0$, and that there is one condition for them to do so, namely that they induce the same constant section in the link between M_1 and M_2:



Thus dim $H^0(\mathscr{P}_0, K_0) = 3$. Now $H^1(\mathscr{P}_0, K_0) = (0)$ follows as in lemma (4.14). Finally, from what we know about the images $p_{M_1}(\mathscr{P}_0)$, it follows that $p_{M_1, M_2}(\mathscr{P}_0)$ must be a union $\boldsymbol{P}(M_1)_0 \times (a) \cup (b) \times \boldsymbol{P}(M_2)_0$, hence the kernel of

$$H^0(\boldsymbol{P}(M_1)_0 \times \boldsymbol{P}(M_2)_0, K_0) \xrightarrow{p_{M_1, M_2}} H^0(\mathscr{P}_0, K_0)$$

is 1-dimensional and generated by an element $(\alpha x_1 + \beta y_1)(\gamma x_2 + \delta y_2)$.

The second step is that $H^0(\mathscr{P}, K)$ is a free A-module of rank 3 such that

$$H^0(\mathscr{P}, \hat{K}) \underset{A}{\otimes} k \cong H^0(\mathscr{P}_0, K_0).$$

This follows from (a) and (b) just as in lemma (4.15). Therefore

$$H^0(\boldsymbol{P}(M_1)\times_S \boldsymbol{P}(M_2), K) \to H^0(\mathscr{P}, \hat{K})$$

is a homomorphism of a free rank 4 module to a free rank 3 module; after $\otimes_A k$, it becomes surjective, so it is already surjective. Therefore $H^0(\boldsymbol{P}(M_1)\times_S \boldsymbol{P}(M_2), K) \cong H^0(\mathscr{P}, \hat{K}) \oplus A\cdot f$, where $f = ax_1x_2 + \cdots + dy_1y_2$ and $f \equiv (\alpha x_1 + \beta y_1)(\gamma x_2 + \delta y_2) \bmod m$. Thus $f = 0$ defines a relative Cartier divisor Z through which p_{M_1,M_2} factors. Finally, if $ad-bc = 0$, then f splits into a product over A as well as over k; then Z is reducible: say $Z = Z_1 \cup Z_2$. If $\mathscr{P}_i = p^{-1}_{M_1,M_2}(Z_i)$, then $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$. But $\mathscr{P}$ is normal and connected so this is absurd. *Q.E.D.*

COROLLARY (4.17). *Let $R(\boldsymbol{P}(M))$ be the field of rational functions on $\boldsymbol{P}(M)$ and let $p_M^*R(\boldsymbol{P}(M))$ be the induced field of meromorphic functions on $\mathscr{P}$. Then*

$$p^*_{M_1}R(\boldsymbol{P}(M_1)) = p^*_{M_2}R(\boldsymbol{P}(M_2))$$

for any 2 components M_1, M_2 of $\mathscr{P}_0$.

PROOF. In fact, p_{M_i} factors:

$$\mathscr{P} \xrightarrow{p_{M_1,M_2}} Z \xrightarrow{q_i} \boldsymbol{P}(M_i)$$

and since ad$-$bc $\neq 0$, Z is irreducible and $q_i^*R(\boldsymbol{P}(M_i)) = R(Z)$. Thus $p^*_{M_1}R(\boldsymbol{P}(M_1)) = p^*_{M_1,M_2}R(Z) = p^*_{M_2}R(\boldsymbol{P}(M_2))$. *Q.E.D.*

Step V. Choose once and for all an isomorphism:

(*) $\quad p^*_M R(\boldsymbol{P}(M)) \cong R(\boldsymbol{P}^1_K)$, the field of rational functions on $\boldsymbol{P}^1_K$.

The isomorphism of $p^*_M R(\boldsymbol{P}(M))$ with $R(\boldsymbol{P}^1_K)$ induces an isomorphism of the generic fibre $\boldsymbol{P}(M)_\eta$ with $\boldsymbol{P}^1_K$. Thus $\boldsymbol{P}(M)$ becomes a $\boldsymbol{P}^1$-bundle over S with generic fibre $\boldsymbol{P}^1_K$, i.e. $\boldsymbol{P}(M)$ define an element $\{M\} \in \Delta^{(0)}$. Thus we have associated an element of $\Delta^{(0)}$ to each component of $\mathscr{P}_0$. Let $\Delta^{(0)}_*$ be the set of elements of $\Delta^{(0)}$ that we get. By Prop. (4.16), the join Z of 2 $\boldsymbol{P}(M)$'s is flat over S with reducible closed fibre: therefore if $M_1 \neq M_2$ are 2 components of $\mathscr{P}_0$, the corresponding elements of $\Delta^{(0)}$ are distinct and compatible by (2.1). Since $\mathscr{P}_0$ has only double points, it follows from (2.3) that $\Delta^{(0)}_*$ is a linked subset. I claim that

$$\mathscr{P} \cong \mathscr{P}(\Delta_*).$$

In fact, it is easy to see that there is a formal morphism $\pi : \mathscr{P} \to \mathscr{P}(\Delta_*)$ which is an isomorphism on the closed fibre. Then apply the easy:

LEMMA (4.18). *Let $f : \mathscr{X} \to \mathscr{Y}$ be a formal morphism of formal schemes over S, whose topologies are defined by the ideals $m \cdot \mathcal{O}_{\mathscr{X}}$, $m \cdot \mathcal{O}_{\mathscr{Y}}$ respectively. If $f_0 : \mathscr{X}_0 \to \mathscr{Y}_0$ is an isomorphism and $\mathscr{X}$ is flat over S, then f is an isomorphism.*

Step VI. By Cor. (4.17), Γ leaves invariant the field of meromorphic functions $p_M^* R(\boldsymbol{P}(M))$ on $\mathscr{P}$, hence by our basic identification of this field with $R(\boldsymbol{P})_K^1$, Γ acts faithfully on $R(\boldsymbol{P}_K^1)$. This induces an embedding

$$\Gamma \hookrightarrow PGL(2, K).$$

Identifying Γ with its image here, then Γ maps $\Delta_*^{(0)}$ into itself and this induces an action of Γ on $\mathscr{P}(\Delta_*)$, and hence induces an action of Γ on $\mathscr{P}$ equal to the one we started with. It remains only to prove:

PROPOSITION (4.19). *Γ is a flat Schottky group and $\Delta_* = \Delta_\Gamma$.*

PROOF. Let $\gamma \in \Gamma$, $\gamma \neq e$. Since γ acts freely on the tree Δ_*, it leaves fixed 2 ends $x, y \in \partial\Delta_*$. Therefore γ leaves fixed $ix, iy \in K\boldsymbol{P}^1$. Let $\{M\}$ be a vertex on the line in Δ_* joining the end x to the end y. Then ix and iy are represented by homogeneous coordinates $u, v \in K+K$ such that $M = A \cdot u + A \cdot v$. Reordering x and y if necessary, we can assume that $\gamma\{M\}$ separates $\{M\}$ from x. Then $\gamma\{M\}$ is represented by a module N such that

$$M \not\supseteq N \ni u,$$
$$u \notin mM.$$

But if we lift γ to an element $\tilde{\gamma} \in GL(2, K)$, then

$$\begin{aligned} \tilde{\gamma}(M) &= \lambda \cdot N, \text{ some } \lambda \in K^*, \\ \tilde{\gamma}(u) &= \sigma \cdot u, \\ \tilde{\gamma}(v) &= \tau \cdot v. \end{aligned}$$

Then $N = \langle \sigma/\lambda u, \tau/\lambda v \rangle$, so $\sigma/\lambda \in A^*$, $\tau/\lambda \in m$. Then

$$t^{-1}(\gamma) = \frac{\sigma\tau}{(\sigma+\tau)^2} = \frac{\sigma/\lambda \cdot \tau/\lambda}{(\sigma/\lambda+\tau/\lambda)^2} = \text{unit} \cdot \frac{\tau}{\lambda} \in m.$$

Therefore γ is hyperbolic.

Next, the fixed points of the elements of Γ are contained in the set $i(\partial\Delta_*)$, hence by Prop. (1.22), any 4 of them have a cross-ratio in A or A^{-1}. Thus Γ is a flat Schottky group. Moreover, $\Delta_\Gamma^{(0)} \subset \Delta_*^{(0)}$ since by Prop. (1.20) $\{M(x, y, z)\} \in \Delta_*^{(0)}$ for any 3 fixed points x, y, z of Γ. Conversely, say v is a vertex of Δ_*. Since C_0 was a stable curve, $\mathscr{P}_0$ has the property that every component has at least 3 double points on it. Therefore every vertex of Δ_* is an endpoint of at least 3 edges. Take 3 edges meeting v. In Δ_*/Γ, choose 3 loops starting and ending at the image of v that start off on these edges. Let these loops define $\gamma_1, \gamma_2, \gamma_3 \in \Gamma \cong \pi_1(\Delta_*/\Gamma)$. Then the paths from v to $\gamma_i v$ start on these 3 edges. Let $\gamma_i^n v$ tend to an end $x_i \in \partial\Delta_*$. Then $i(x_i)$ is a fixed point of γ_i, hence

$$v = \{M(ix_1, ix_2, ix_3)\} \in \Delta_\Gamma^{(0)}. \qquad Q.E.D.$$

This completes the proof of:

THEOREM (4.20): *Every stable curve over S with non-singular generic fibre and k-split degenerate closed fibre is isomorphic to P_Γ for a unique flat Schottky group $\Gamma \subset PGL(2, K)$.*

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