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ON THE FAMILY RELATION FOR ARTINIAN RINGS

by

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In [4] (see also [5], Chapter III), Kruse and Price introduced the notion of two rings being in the same family. This was motivated by the relation of group isoclinism introduced by P. Hall in [3].

Rings R_1 and R_2 are said to be in the same *family*, denoted by $R_1 \overset{F}{\leftrightarrow} R_2$, if there exist isomorphisms $\phi : R_1/\mathfrak{A}(R_1) \to R_2/\mathfrak{A}(R_2)$ and $\psi : R_1^2 \to R_2^2$ such that

(1) if $(r_i + \mathfrak{A}(R_1))\phi = s_i + \mathfrak{A}(R_2)$ for $i = 1, 2$ then $(r_1 r_2)\psi = s_1 s_2$.

Here $\mathfrak{A}(R)$ denotes the annihilator of a ring R and is defined by

$$\mathfrak{A}(R) = \{x \in R |\ xr = rx = 0 \text{ for all } r \in R\}.$$

$\overset{F}{\leftrightarrow}$ is an equivalence relation which is identical with isomorphism if either $R_i^2 = R_i$ for $i = 1, 2$ or $\mathfrak{A}(R_1) = \mathfrak{A}(R_2) = 0$.

Also, if $R_1 \overset{F}{\leftrightarrow} R_2$, then R_1 is nilpotent if and only if R_2 is nilpotent. We show below how the family relation for commutative Artinian rings reduces to isomorphism between certain subrings equal to their own square and the family relation for certain nilpotent factor rings. We first need the following proposition.

PROPOSITION. *If $R_1 \overset{F}{\leftrightarrow} R_2$ then, for all integers $m \geqq 2$, $R_1^m \cong R_2^m$ and $R_1/R_1^m \overset{F}{\leftrightarrow} R_2/R_2^m$.*

PROOF. Let $\phi : R_1/\mathfrak{A}(R_1) \to R_2/\mathfrak{A}(R_2)$ and $\psi : R_1^2 \to R_2^2$ be isomorphisms such that (1) holds. Because of (1) it is clear that, for all $m \geqq 2$, the restriction of ψ to R_1^m is an isomorphism onto R_2^m.

Suppose $m \geqq 2$. For $i = 1, 2$ let $V_i = R_i/R_i^m$, $\alpha_i : R_i \to V_i$ be the canonical map and

$$L_i = \{x \in R_i | (xR_i) \cup (R_i x) \subseteq R_i^m\}.$$

Then $\mathfrak{A}(R_i) \subseteq L_i$ and $\mathfrak{A}(V_i) = L_i/R_i^m$. Because $\psi\alpha_2$ maps R_1^2 onto $R_2^2/R_2^m = V_2^2$ and has kernel $R_2^m\psi^{-1} = R_1^m$, ψ induces an isomorphism $\Psi : V_1^2 \to V_2^2$ given by

$$(x + R_1^m)\Psi = x\psi + R_2^m \qquad \text{for } x \in R_1^2.$$

For $i = 1, 2$ let β_i be the map from $R_i/\mathfrak{A}(R_i)$ onto R_i/L_i given by

$$(r_i+\mathfrak{A}(R_i))\beta_i = r_i+L_i \qquad r_i \in R_i$$

and let γ_i be the isomorphism from R_i/L_i onto $V_i/\mathfrak{A}(V_i)$ given by

$$(r_i+L_i)\gamma_i = (r_i+R_i^m)+\mathfrak{A}(V_i).$$

Then $\phi\beta_2\gamma_2$ maps $R_1/\mathfrak{A}(R_1)$ onto $V_2/\mathfrak{A}(V_2)$ and has kernel $(L_2/\mathfrak{A}(R_2))\phi^{-1}$. But because of (1) this equals $L_1/\mathfrak{A}(R_1)$ which is also the kernel of $\beta_1\gamma_1$. Hence ϕ induces an isomorphism Φ from $V_1/\mathfrak{A}(V_1)$ onto $V_2/\mathfrak{A}(V_2)$ given by

$$((r_1+L_1)\gamma^{-1})\Phi = (r_1+\mathfrak{A}(R_1))\phi\beta_2\gamma_2 \qquad r_1 \in R_1.$$

Finally it is easy to check that Φ and Ψ satisfy the compatability condition corresponding to (1) and hence $V_1 \overset{F}{\leftrightarrow} V_2$.

If R is a ring with D.C.C. on two-sided ideals (in particular, if R is Artinian), there is a least positive integer n such that $R^m = R^n$ for all $m \geqq n$. We denote R^n by $K(R)$. Then, of course, $K(R)^2 = K(R)$ and $R/K(R)$ is nilpotent.

Suppose R_1 and R_2 are two rings with D.C.C. on twosided ideals. If $R_1 \overset{F}{\leftrightarrow} R_2$ it follows from the proposition that $K(R_1) \cong K(R_2)$ and $R_1/K(R_1) \overset{F}{\leftrightarrow} R_2/K(R_2)$. That the converse is not true may be seen by considering the (non-commutative) 4 dimensional algebra R over the field with two elements and with basis e, a, b, c where multiplication is such that all products of the basis elements are zero except that $ee = e$, $ea = a$, $eb = b$ and $ac = b$. Then it is easy to check that R is associative, R^2 is the subspace with basis e, a, b, $K(R) = R^2$, $\mathfrak{A}(R) = 0 = \mathfrak{A}(R^2)$. Hence $K(R) = K(R^2)$, $R/K(R) \overset{F}{\leftrightarrow} R^2/K(R^2)$ but R and R^2 are not in the same family.

However, suppose in addition that R_1 and R_2 are commutative. Then, if J_i is the Jacobson radical of R_i, there exists an idempotent $e_i \in R_i$ such that e_i+J_i is the identity of R_i/J_i and, if $T_i = \{x-xe_i | x \in R_i\}$, $R_i = R_ie_i+T_i$ and $T_i \subseteq J_i$ is nilpotent ([1], Theorem 9.3 C). Since R_ie_i is a ring with identity e_i, $(R_ie_i)^m = R_ie_i$ for all $m \geqq 1$ and so, since T_i is nilpotent, $K(R_i) = R_ie_i$. Hence $R_i = K(R_i) \oplus T_i$ and $R_i/K(R_i) \cong T_i$. Thus if $K(R_1) \cong K(R_2)$ and $R_1/K(R_1) \overset{F}{\leftrightarrow} R_2/K(R_2)$ then also $T_1 \overset{F}{\leftrightarrow} T_2$ and so clearly $R_1 \overset{F}{\leftrightarrow} R_2$. This proves the following.

THEOREM. *Let R and S be rings with D.C.C. on two-sided ideals. If $R \overset{F}{\leftrightarrow} S$ then $K(R) \cong K(S)$ and $R/K(R) \overset{F}{\leftrightarrow} S/K(S)$. If, in addition, R and S are commutative, the converse is also true.*

Finally if $R_1 \overset{F}{\leftrightarrow} R_2$ then it is clear that $R_1/J_1 \cong R_2/J_2$. (Indeed, if $\mathscr{H}$ is any radical property (see [2], Chapter 1) such that every ring whose square is zero is an $\mathscr{H}$-ring and if $\mathscr{H}(R)$ denotes the $\mathscr{H}$-radical of a ring R then from $R_1/\mathfrak{A}(R_1) \cong R_2/\mathfrak{A}(R_2)$ it follows that $R_1/\mathscr{H}(R_1) \cong R_2/\mathscr{H}(R_2)$ since $\mathfrak{A}(R_i) \subseteq \mathscr{H}(R_i)$.) But if R_i is a commutative Artinian ring then J_i is the direct sum of the radical of $K(R_i)$ and T_i. Hence if $R_1 \overset{F}{\leftrightarrow} R_2$ and each R_i is commutative and Artinian, then $R_1/J_1 \cong R_2/J_2$ and $J_1 \overset{F}{\leftrightarrow} J_2$. That the converse is however false can be seen by considering R_1 as the ring of integers modulo 4 and R_2 the algebra over the field with two elements with basis 1 and x and with $x^2 = 0$.

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