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## ANOTHER PROPERTY OF THE SORGENFREY LINE

by

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### 1. Introduction

The Sorgenfrey line  $S$ , one of the most important counterexamples in general topology, is obtained by retopologizing the set of real numbers, taking half-open intervals of the form  $[a, b]$  to be a base for  $S$ . Recently, R. W. Heath and E. Michael proved that  $S^{\aleph_0}$ , the product of countably many copies of  $S$ , is perfect (= closed sets are  $G_\delta$ 's) [3]. At the Washington State Topology Conference in March, 1970, the question of whether  $S^{\aleph_0}$  is subparacompact was raised. In this paper, we give an affirmative answer to this question and, in the process, give some positive results concerning products of more general subparacompact spaces.

### 2. Definitions and preliminary results

(2.1) DEFINITION. A space  $X$  is *subparacompact* if every open cover of  $X$  has a  $\sigma$ -locally finite closed refinement.

The definitive study of subparacompact spaces is [2].

(2.2) DEFINITION [3]. A space  $X$  is *perfect* if every closed subset of  $X$  is a  $G_\delta$  in  $X$ . A space is *perfectly subparacompact* if it is perfect and subparacompact.

It is clear that if  $\mathcal{G}$  is any collection of open subsets of a perfectly subparacompact space  $X$ , then there is a collection  $\mathcal{F}$  of closed subsets of  $X$  which is  $\sigma$ -locally finite in  $X$ , refines  $\mathcal{G}$  and has  $\cup \mathcal{F} = \cup \mathcal{G}$ .

(2.3) DEFINITION (Morita, [4]).  $X$  is a  $P$ -space if for each open cover  $\{U(\alpha_1, \dots, \alpha_n) : \alpha_j \in A, n \geq 1\}$  of  $X$  which satisfies  $U(\alpha_1, \dots, \alpha_n) \subseteq U(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$  whenever  $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in A$  there is a closed cover  $\{H(\alpha_1, \dots, \alpha_n) : \alpha_j \in A, n \geq 1\}$  of  $X$  which satisfies:

- (i)  $H(\alpha_1, \dots, \alpha_n) \subseteq U(\alpha_1, \dots, \alpha_n)$  for each  $n$ -tuple  $(\alpha_1, \dots, \alpha_n)$  of elements of  $A$ ;
- (ii) if  $\langle \alpha_n \rangle$  is a sequence of elements of  $A$  such that  $\bigcup_{n=1}^{\infty} U(\alpha_1, \dots, \alpha_n) = X$ , then  $\bigcup_{n=1}^{\infty} H(\alpha_1, \dots, \alpha_n) = X$ .

It is easily seen that if  $X$  is perfect, then  $X$  is a  $P$ -space.

The following definition is equivalent to the definition in [5].

(2.4) DEFINITION. A space  $Y$  is a  $\Sigma$ -space if there is a sequence  $\langle \mathcal{F}(n) \rangle$  of locally finite closed covers of  $Y$  such that

- (i) each  $\mathcal{F}(n)$  is closed under finite intersections;
- (ii)  $\mathcal{F}(n) = \{F(\alpha_1, \dots, \alpha_n) : \alpha_j \in A\}$ ;
- (iii) each  $F(\alpha_1, \dots, \alpha_n)$  is the union of all  $F(\alpha_1, \dots, \alpha_n, \beta)$ ,  $\beta \in A$ ;
- (iv) for each  $y \in Y$ , the set  $C(y) = \bigcap_{n=1}^{\infty} [\bigcap \{F \in \mathcal{F}(n) : y \in F\}]$  is countably compact;
- (v) for each  $y \in Y$ , there is a sequence  $\langle \alpha_n \rangle$  of elements of  $A$  such that  $\{F(\alpha_1, \dots, \alpha_n) : n \geq 1\}$  is an *outer network* for  $C(y)$  in  $Y$ , i.e., if  $V$  is open in  $Y$  and  $C(y) \subseteq V$ , then for some  $n \geq 1$ ,  $C(y) \subseteq F(\alpha_1, \dots, \alpha_n) \subseteq V$ .

The sequence  $\langle \mathcal{F}(n) \rangle$  is called a *spectral  $\Sigma$ -network* for  $Y$ .

It is clear that if the space  $Y$  in (2.4) is subparacompact, then each of the sets  $C(y)$  will be compact. That the converse is also true is an unpublished result of E. Michael. Thus, any  $\Sigma$ -space  $Y$  in which closed, countably compact sets are compact is subparacompact. This is the case, for example, if  $Y$  is metacompact or  $\theta$ -refinable[6].

(2.5) PROPOSITION. Suppose that  $X$  and  $Y$  are regular subparacompact spaces. If  $X$  is a  $P$ -space and  $Y$  is a  $\Sigma$ -space, then  $X \times Y$  is subparacompact.

REMARK. The proof of (2.5) closely parallels the proof of Theorem 4.1 of [5]; we present only an outline.

PROOF. Since  $X \times Y$  is regular, it is enough to show that if  $\mathcal{W}$  is an open cover of  $X \times Y$  which is closed under finite unions, then  $\mathcal{W}$  has a  $\sigma$ -locally finite refinement. Let  $\mathcal{W}$  be such a cover. Let  $\langle \mathcal{F}(n) \rangle$  be a spectral  $\Sigma$ -network for  $Y$ , as in (2.4). For each  $n \geq 1$  and for each  $n$ -tuple  $(\alpha_1, \dots, \alpha_n)$  of elements of  $A$ , let  $\mathcal{U}(\alpha_1, \dots, \alpha_n) = \{R \subseteq X : R \text{ is open and } R \times F(\alpha_1, \dots, \alpha_n) \text{ is contained in some member of } \mathcal{W}\}$ . Let  $U(\alpha_1, \dots, \alpha_n) = \bigcup \mathcal{U}(\alpha_1, \dots, \alpha_n)$  and let  $\mathcal{U} = \{U(\alpha_1, \dots, \alpha_n) : n \geq 1 \text{ and } \alpha_i \in A\}$ . Then  $\mathcal{U}$  is an open cover of  $X$  as in (2.3) so  $\mathcal{U}$  has a closed refinement  $\mathcal{H} = \{H(\alpha_1, \dots, \alpha_n) : n \geq 1 \text{ and } \alpha_i \in A\}$  as in (2.3). Each  $H(\alpha_1, \dots, \alpha_n)$  is subparacompact, being closed in  $X$ , and is covered by  $\mathcal{U}(\alpha_1, \dots, \alpha_n)$ . Hence there are collections  $\mathcal{J}(\alpha_1, \dots, \alpha_n) = \bigcup_{m=1}^{\infty} \mathcal{J}_m(\alpha_1, \dots, \alpha_n)$  of closed subsets of  $H(\alpha_1, \dots, \alpha_n)$  which cover the sets  $H(\alpha_1, \dots, \alpha_n)$  and which are  $\sigma$ -locally finite in  $X$ . Let  $\mathcal{S}(m, n) = \{J \times F(\alpha_1, \dots, \alpha_n) : J \in \mathcal{J}_m(\alpha_1, \dots, \alpha_n) \text{ and } F(\alpha_1, \dots, \alpha_n) \in \mathcal{F}(n)\}$  and let  $\mathcal{S} = \bigcup_{m, n \geq 1} \mathcal{S}(m, n)$ . Then  $\mathcal{S}$  is  $\sigma$ -locally finite in  $X \times Y$

and refines  $\mathcal{W}$ . To show that  $\mathcal{S}$  covers  $X \times Y$ , let  $(x, y) \in X \times Y$  and let  $\langle \alpha_n \rangle$  be a sequence of elements of  $A$  such that  $\{F(\alpha_1, \dots, \alpha_n) : n \geq 1\}$  is an outer network for  $C(y)$  in  $Y$ . Then  $\bigcup_{n=1}^{\infty} U(\alpha_1, \dots, \alpha_n) = X$ , so  $\bigcup_{n=1}^{\infty} H(\alpha_1, \dots, \alpha_n) = X$ . Choose  $n$  such that  $x \in H(\alpha_1, \dots, \alpha_n)$  and  $m$  such that some  $J \in \mathcal{J}_m(\alpha_1, \dots, \alpha_n)$  has  $x \in J$ . Then  $(x, y) \in J \times F(\alpha_1, \dots, \alpha_n) \in \mathcal{S}(m, n) \subseteq \mathcal{S}$ .

(2.6) COROLLARY. If  $X$  is regular and perfectly subparacompact and if  $Y$  is metrizable, then  $X \times Y$  is subparacompact.

PROOF. Any perfect space is a  $P$ -space and any metrizable space is a subparacompact  $\Sigma$ -space.

Our proof that  $S^{\aleph_0}$  is subparacompact proceeds inductively to show that  $S^n$  is perfectly subparacompact for each  $n \geq 1$  and then invokes the following proposition.

(2.7) PROPOSITION. Suppose  $\{X(k) : k \geq 1\}$  is a collection of spaces such that each finite product  $\prod\{X(k) : 1 \leq k \leq n\}$  is perfectly subparacompact. Then so is  $\prod\{X(k) : k \geq 1\}$ .

PROOF. Let  $Y = \prod\{X(k) : k \geq 1\}$  and let  $Y(n) = \prod\{X(k) : 1 \leq k \leq n\}$ . Let  $\mathcal{W} = \{W(\alpha) : \alpha \in A\}$  be a basic open cover of  $Y$ . Thus, for each  $\alpha \in A$ , there is an integer  $N(\alpha)$  and open subsets  $U(k, \alpha)$  of  $X(k)$  for  $1 \leq k \leq N(\alpha)$  such that, if  $\pi_k : Y \rightarrow X(k)$  denotes the usual projection, then  $W(\alpha) = \bigcap \{\pi_k^{-1}[U(k, \alpha)] : 1 \leq k \leq N(\alpha)\}$ . Let  $A(n) = \{\alpha \in A : N(\alpha) = n\}$  and let

$$\mathcal{G}(n) = \{U(1, \alpha) \times \dots \times U(n, \alpha) : \alpha \in A(n)\}.$$

Since  $Y(n)$  is perfectly subparacompact, there is a  $\sigma$ -locally finite collection  $\mathcal{F}(n) = \bigcup_{m=1}^{\infty} \mathcal{F}(n, m)$  of closed subsets of  $Y(n)$  which refines  $\mathcal{G}(n)$  and covers  $\bigcup \mathcal{G}(n)$ . Let

$$\mathcal{H}(n, m) = \{F \times \prod_{k=n+1}^{\infty} X(k) : F \in \mathcal{F}(n, m)\},$$

Then  $\mathcal{H} = \bigcup_{m, n \geq 1} \mathcal{H}(n, m)$  is a  $\sigma$ -locally finite closed cover of  $Y$  which refines  $\mathcal{W}$ . Hence  $Y$  is subparacompact. That  $Y$  is also perfect follows from Proposition 2.1 of [3].

The following concept is the key to our proof that  $S^n$  is perfectly subparacompact for each finite  $n \geq 1$ .

(2.8) DEFINITION. A space  $X$  is *weakly  $\theta$ -refinable* if for each open cover  $\mathcal{U}$  of  $X$ , there is an open cover  $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}(n)$  of  $X$  which refines  $\mathcal{U}$  and which has the property that if  $p \in X$ , then there is an  $n$  such that  $p$  belongs to exactly  $k$  members of  $\mathcal{V}(n)$ , where  $k$  is some

finite, positive number. The collection  $\mathcal{V}$  is called a *weak  $\theta$ -refinement* of  $\mathcal{U}$ .

REMARK. In the language of [6], the refinement  $\mathcal{V}$  in (2.8) is  $\sigma$ -distributively point finite. As defined above, weak  $\theta$ -refinability is not the same as  $\theta$ -refinability as defined in [6]. Weak  $\theta$ -refinability and certain related properties will be studied in [1]; here we only summarize the properties which we will need in § 3.

(2.9) PROPOSITION. a) Any subparacompact space is weakly  $\theta$ -refinable; b) if  $X$  is perfect and weakly  $\theta$ -refinable, then  $X$  is subparacompact; c) if  $\mathcal{G}$  is any collection of open subsets of a perfect, weakly  $\theta$ -refinable space  $X$ , then there is an open cover  $\mathcal{H}$  of  $\cup(\mathcal{G})$  which is a weak  $\theta$ -refinement of  $\mathcal{G}$ .

To conclude this section, we restate the recent result of Heath and Michael which was mentioned in the Introduction.

(2.10) THEOREM [3].  $S^{\aleph_0}$  is perfect. Hence so is  $S^n$  for each finite  $n$ .

### 3. The Sorgenfrey line

(3.1) PROPOSITION. For any  $n \geq 1$ ,  $S^n$  is perfectly subparacompact.

PROOF. We argue inductively. The result is certainly true for  $n = 1$ , since  $S^1 = S$  is even perfectly paracompact. Let us assume, therefore, that  $S^n$  is known to be perfectly subparacompact and prove that  $S^{n+1}$  must also be perfectly subparacompact. Since  $S^{n+1}$  is perfect by (2.10), it suffices, by (2.9)b, to show that  $S^{n+1}$  is weakly  $\theta$ -refinable. Let  $Y = S^{n+1}$ .

Let  $\mathcal{W} = \{W(\alpha) : \alpha \in A\}$  be a basic open cover of the space  $Y = S^{n+1}$ , where  $W(\alpha) = U(1, \alpha) \times \cdots \times U(n+1, \alpha)$  with  $U(k, \alpha) = [a(k, \alpha), b(k, \alpha)[$  for each  $\alpha \in A$  and  $k \leq n+1$ . Let  $\hat{U}(k, \alpha) = ]a(k, \alpha), b(k, \alpha)[$  and let  $W(k, \alpha)$  be the set  $U(1, \alpha) \times \cdots \times \hat{U}(k, \alpha) \times \cdots \times U(n+1, \alpha)$ . Observe that each  $W(k, \alpha)$  is open in the space

$$Y(k) = S \times \cdots \times S \times R \times S \times \cdots \times S$$

where  $R$ , the real numbers with the usual topology, replaces  $S$  in the  $k^{\text{th}}$  coordinate. Furthermore, since  $Y(k)$  is homeomorphic to  $S^n \times R$ , our induction hypothesis together with (2.6) and the fact that the product of a perfect space with a metric space is again perfect, guarantees that  $Y(k)$  is perfectly subparacompact. Hence by (2.9)<sup>c</sup> the collection  $\mathcal{G}(k) = \{W(k, \alpha) : \alpha \in A\}$  has a weak  $\theta$ -refinement  $\mathcal{H}(k)$  which covers  $\cup \mathcal{G}(k)$  and which consists of open subsets of  $Y(k)$ . But then  $\mathcal{H}(k)$  is also a collection of open subsets of  $S^{n+1}$ .

Let

$$Z = \{x \in Y : \text{if } x \in W(\alpha) \in \mathcal{W}, \text{ then } x = (a(1, \alpha), \dots, a(n+1, \alpha))\}.$$

Then

$$Z = Y \setminus \cup \{[\cup \mathcal{H}(k)] : 1 \leq k \leq n+1\}.$$

For each  $x \in Z$ , choose  $\alpha(x) \in A$  such that  $x \in W(\alpha(x))$  and observe that if  $x$  and  $y$  are distinct elements of  $Z$ , then  $y \notin W(\alpha(x))$ . Letting

$$\mathcal{H}(0) = \{W(\alpha(x)) : x \in Z\},$$

we obtain a collection of open subsets of  $Y$  which covers  $Z$  in such a way that each point of  $Z$  belongs to exactly one member of  $\mathcal{H}(0)$ . Therefore,  $\mathcal{H} = \cup \{\mathcal{H}(k) : k \geq 0\}$  is a weak  $\theta$ -refinement of  $\mathcal{W}$ , as required to show that  $Y = S^{n+1}$  is weakly  $\theta$ -refinable.

(3.2) THEOREM.  $S^{\aleph_0}$  is perfectly subparacompact.

PROOF. Apply (3.1), (2.10) and (2.7).

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