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ON THE EXISTENCE OF UNIFORMLY DISTRIBUTED SEQUENCES IN COMPACT SPACES

by

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The theory of uniform distribution of sequences in compact Hausdorff spaces was developed by Hlawka [8], [9]. The notion of uniform distributivity of such sequences is defined relative to a given nonnegative regular normed Borel measure μ on the space. In many papers on the subject matter, the compact Hausdorff space was supposed to satisfy the second axiom of countability. Naturally, this offers advantages, especially when proving metric results ([2], [8], [11]). Furthermore, the existence problem for uniformly distributed sequences in those spaces can be easily settled. In fact, if X is a compact Hausdorff space with countable base, then, in a natural sense, almost all sequences in X are uniformly distributed with respect to the measure μ([8]). Nevertheless, a satisfactory theory of uniform distribution can also be developed to a certain extent for arbitrary compact Hausdorff spaces ([2]). But, strangely enough, nobody seems to have touched the existence problem in this general setting. The strongest constructive result still appears to be the one by Hedrlín [4] who showed that uniformly distributed sequences exist in every compact metric space, by using an explicit construction.

In this paper, we propose to study an arbitrary compact Hausdorff space X together with a (not necessarily regular) nonnegative normed Borel measure μ on X. We characterize all measures μ for which there exist uniformly distributed sequences. By using the general theory of weak convergence of measures as developed by Varadarajan [14] and Topsøe [13], the methods of the present paper yield similar results for completely regular spaces with τ-smooth probability measures.

I have not succeeded to prove a corresponding result for well distributed sequences, but at least a metric result can be given in this case (see Theorem 3). We show that, in a natural sense, almost no sequence is well distributed in X with respect to a regular μ. This metric theorem was given by Helmberg and Paalman-de Miranda [6] for compact Hausdorff spaces with countable base. For such spaces, the existence of well distributed sequences was shown by Baayen and Hedrlín [1]. No general existence theorem for well distributed sequences is yet known.

Let X be a compact Hausdorff space. Let $\mathscr{M}^+(X)$ be the set of all nonnegative normed Borel measures on X, let $\mathscr{R}(X)$ be the set of all signed regular Borel measures on X, and let $\mathscr{R}^+(X) = \mathscr{M}^+(X) \cap \mathscr{R}(X)$. We first extend some well-known definitions to our general case.

DEFINITION 1. Let $\mu \in \mathscr{M}^+(X)$. A subset M of X is called a *μ-continuity set* if its boundary is a μ-null set.

DEFINITION 2. Let $\mu \in \mathscr{M}^+(X)$, and let $\omega = (x_n)$, $n = 1, 2, \cdots$, be a sequence in X. For a subset M of X and a positive integer N, we define the *counting function* $A(M; N; \omega)$ by $A(M; N; \omega) = \sum_{n=1}^{N} c_M(x_n)$ where c_M denotes the characteristic function of M. The sequence (x_n) is called *μ-uniformly distributed in X* if

$$\lim_{N\to\infty} \frac{A(M; N; \omega)}{N} = \mu(M)$$

holds for every μ-continuity set M in X.

REMARK. If $\mu \in \mathscr{R}^+(X)$, then (x_n) is μ-uniformly distributed in X if and only if $\lim_{N\to\infty} 1/N \sum_{n=1}^{N} f(x_n) = \int_X f \, d\mu$ holds for every real-valued (complex-valued) continuous function f on X. This follows as in Helmberg [5], and could also be inferred from the Portmanteau Theorem [13, p. 40].

Although we do not have a notion of support for nonregular μ, the following definition is still meaningful. For a general concept of support see Topsøe [13, p. XIII].

DEFINITION 3. The measure $\mu \in \mathscr{M}^+(X)$ is called a *measure of finite support* if there exists a finite subset E of X with $\mu(E) = 1$. Let $\mathscr{F}(X)$ be the set of all measures of finite support on X.

COROLLARY 1. *If $\mu \in \mathscr{F}(X)$, then there exist finitely many points $x_1, \cdots, x_k$ in X such that $\mu(\{x_i\}) = \lambda_i > 0$ for all i, $1 \leqq i \leqq k$, and $\sum_{i=1}^{k} \lambda_i = 1$. The points $x_1, \cdots, x_k$ are uniquely determined by μ, and we put* $\operatorname{supp} \mu = \{x_1, \cdots, x_k\}$.

We have now all the concepts and the terminology available to enunciate our main result.

THEOREM 1. *Let $\mu \in \mathscr{M}^+(X)$. There exist μ-uniformly distributed sequences in X if and only if there exists a sequence (μ_j), $j = 1, 2, \cdots$, in $\mathscr{F}(X)$ such that*

$$\lim_{j\to\infty} \mu_j(M) = \mu(M) \tag{1}$$

for every μ-continuity set M in X.

Before we set out to prove this theorem, we draw some interesting conclusions. First of all, let us note that, by Corollary 1, every $\nu \in \mathscr{F}(X)$ is a convex linear combination of point measures. In particular, every $\nu \in \mathscr{F}(X)$ is regular. If $\mu \in \mathscr{R}^+(X)$, then condition (1) just means that the sequence (μ_j) converges weakly to μ ([13, p. 40]). Furnished with the topology of weak convergence, $\mathscr{R}(X)$ is a topological linear space. We can then reformulate Theorem 1 as follows.

COROLLARY 2. *Let $\mathscr{R}^+(X)$ satisfy the first axiom of countability, and let $\mu \in \mathscr{R}^+(X)$. There exist μ-uniformly distributed sequences in X if and only if μ lies in the closed convex hull in $\mathscr{R}(X)$ of the set of normed point measures.*

It is an important result in the theory of weak convergence in $\mathscr{R}(X)$ that the closed convex hull in $\mathscr{R}(X)$ of the set of normed point measures is, in fact, all of $\mathscr{R}^+(X)$ ([3, Ch. III, § 2, no. 4, Cor. 3], [13, p. 48]). Therefore we get:

THEOREM 2. *Let $\mathscr{R}^+(X)$ satisfy the first axiom of countability. Then for every $\mu \in \mathscr{R}^+(X)$ there exist μ-uniformly distributed sequences.*

REMARK. Theorem 2 generalizes the result of Hedrlín [4], since for compact metric X the space $\mathscr{R}^+(X)$ is metrizable ([14, Theorem 13]).

The basic idea in the sufficiency part of the proof of Theorem 1 is the construction of sequences in X which are 'very well' distributed with respect to the measures $\mu_j \in \mathscr{F}(X)$, $j = 1, 2, \cdots$. Those sequences will allow us to explicitly construct μ-uniformly distributed sequences in X.

LEMMA 1. *For $\mu \in \mathscr{F}(X)$, there exists a positive constant $C(\mu)$ and a sequence $\omega = (y_n)$ in X such that*

$$\left| \frac{A(M; N; \omega)}{N} - \mu(M) \right| \leqq \frac{C(\mu)}{N} \tag{2}$$

for all $N \geqq 1$ and for all subsets M of X. In particular,

$$C(\mu) = (k-1)\left[\frac{k}{2}\right]$$

will do, where $k =$ card (supp μ).

PROOF. Let supp $\mu = \{x_1, \cdots, x_k\}$. It will suffice to show that there exists a sequence $\omega = (y_n)$ in supp μ such that

$$\left| \frac{A(\{x_i\}; N; \omega)}{N} - \mu(\{x_i\}) \right| \leqq \frac{k-1}{N} \tag{3}$$

for all i, $1 \leqq i \leqq k$, and for all $N \geqq 1$. For then, the inequality (2) can be shown as follows. For any subset M of X, we have $A(M; N; \omega) = A(M \cap \operatorname{supp} \mu; N; \omega)$ and $\mu(M) = \mu(M \cap \operatorname{supp} \mu)$, therefore it suffices to consider a set $M \subset \operatorname{supp} \mu$. Moreover, for such a set M we have $A(\operatorname{supp} \mu \setminus M; N; \omega) = N - A(M; N; \omega)$ and $\mu(\operatorname{supp} \mu \setminus M) = 1 - \mu(M)$, hence we need only look at those sets M with card $M \leqq [k/2]$. But then

$$\left| \frac{A(M; N; \omega)}{N} - \mu(M) \right| \leqq \frac{k-1}{N} \operatorname{card} M \leqq \frac{1}{N}(k-1)\left[\frac{k}{2}\right],$$

and we are done.

To prove (3), we proceed by induction on card (supp μ). If card (supp μ) $= 1$, then μ is the normed point measure at $x_1 \in X$, and the constant sequence $\omega = (y_n)$ with $y_n = x_1$ for all $n \geqq 1$ will do. Assuming the proposition to be true for all $\nu \in \mathscr{F}(X)$ with card (supp ν) $= k$, we take $\mu \in \mathscr{F}(X)$ with $\mu(\{x_i\}) = \lambda_i > 0$ for $1 \leqq i \leqq k+1$ and $\sum_{i=1}^{k+1} \lambda_i = 1$. We define a measure $\nu \in \mathscr{F}(X)$ with supp $\nu = \{x_1, \cdots, x_k\}$, namely $\nu(\{x_i\}) = \lambda_i/(\lambda_1 + \cdots + \lambda_k)$ for $1 \leqq i \leqq k$. There exists a sequence $\xi = (z_n)$ in supp ν with

$$\left| \frac{A(\{x_i\}; N; \xi)}{N} - \nu(\{x_i\}) \right| \leqq \frac{k-1}{N}$$

for all i, $1 \leqq i \leqq k$, and all $N \geqq 1$. Now we define the sequence $\omega = (y_n)$ in the following way: $y_n = z_m$ if $n = [m/(\lambda_1 + \cdots + \lambda_k)]$ for some $m \geqq 1$ (m is unique because of $\lambda_1 + \cdots + \lambda_k < 1$), $y_n = x_{k+1}$ otherwise.

In order to show (3), we consider first a point x_i with $1 \leqq i \leqq k$. Then $A(\{x_i\}; N; \omega)$ will be equal to the number of positive integers m such that $[m/(\lambda_1 + \cdots + \lambda_k)] \leqq N$ and $z_m = x_i$. Therefore $A(\{x_i\}; N; \omega) = A(\{x_i\}; L; \xi)$ where $L = [(N+1)(\lambda_1 + \cdots + \lambda_k)] - \varepsilon$ and $\varepsilon = 1$ or 0 according as $(N+1)(\lambda_1 + \cdots + \lambda_k)$ is an integer or is not an integer. We get

$$\begin{aligned} \left| \frac{A(\{x_i\}; N; \omega)}{N} - \mu(\{x_i\}) \right| &= \left| \frac{A(\{x_i\}; L; \xi)}{L} \frac{L}{N} - (\lambda_1 + \cdots + \lambda_k)\nu(\{x_i\}) \right| \\ &\leqq \frac{L}{N} \left| \frac{A(\{x_i\}; L; \xi)}{L} - \nu(\{x_i\}) \right| + \nu(\{x_i\}) \left| \frac{L}{N} - (\lambda_1 + \cdots + \lambda_k) \right| \\ &\leqq \frac{k-1}{N} + \nu(\{x_i\}) \left| \frac{\lambda_1 + \cdots + \lambda_k - \{(N+1)(\lambda_1 + \cdots + \lambda_k)\} - \varepsilon}{N} \right|. \end{aligned}$$

Discussing the two possibilities for ε separately, it follows easily that the absolute value occurring in the last expression is at most $1/N$.

It remains to consider the point x_{k+1}. Since $A(\{x_{k+1}\}; N; \omega) = N - L$ where L is defined as above, we have

$$\left|\frac{A(\{x_{k+1}\}; N; \omega)}{N} - \mu(\{x_{k+1}\})\right| = \left|\lambda_1 + \cdots + \lambda_k - \frac{L}{N}\right| \leqq \frac{1}{N}$$

as above.

As the attentive reader will have observed, we did not attempt to find the best possible value for $C(\mu)$. The above lemma is of the same type as a result of Meijer [12, Theorem 1] who showed the existence of 'very well' distributed sequences in a different setting.

Proof of Theorem 1. Suppose there exists a μ-uniformly distributed sequence $\omega = (x_n)$ in X. Let ε_x denote the normed point measure at the point $x \in X$. Then the sequence (μ_j) in $\mathscr{F}(X)$ defined by $\mu_j = 1/j \sum_{n=1}^{j} \varepsilon_{x_n}$ satisfies the condition (1).

Conversely, suppose the condition of the theorem is satisfied by the sequence (μ_j) in $\mathscr{F}(X)$. We want to construct a μ-uniformly distributed sequence in X. By Lemma 1, for each μ_j there exists a positive constant $C(\mu_j) = C_j$ and a sequence $\omega_j = (x_{jn})$, $n = 1, 2, \cdots$, such that (2) holds. For each $j \geqq 1$, choose a positive integer r_j with $r_j \geqq j(C_1 + C_2 + \cdots + C_{j+1})$. As a matter of convenience, put $r_0 = 0$. We define a sequence $\omega = (x_n)$ in X in the following way. Every positive integer n has a unique representation of the form $n = r_0 + r_1 + \cdots + r_{j-1} + s$ with $j \geqq 1$ and $0 < s \leqq r_j$; we set $x_n = x_{js}$. Take an integer $N > r_1$; N can be written as $N = r_1 + r_2 + \cdots + r_k + s$ with $0 < s \leqq r_{k+1}$. For a μ-continuity set M in X, we get $A(M; N; \omega) = \sum_{j=1}^{k} A(M; r_j; \omega_j) + A(M; s; \omega_{k+1})$. Therefore

$$\begin{aligned}
\left|\frac{A(M; N; \omega)}{N} - \mu(M)\right| &= \Bigg| \sum_{j=1}^{k} \frac{r_j}{N}\left(\frac{A(M; r_j; \omega_j)}{r_j} - \mu_j(M)\right) \\
&\quad + \frac{s}{N}\left(\frac{A(M; s; \omega_{k+1})}{s} - \mu_{k+1}(M)\right) + \sum_{j=1}^{k} \frac{r_j}{N}\mu_j(M) + \frac{s}{N}\mu_{k+1}(M) \\
&\quad - \mu(M)\Bigg| \leqq \sum_{j=1}^{k} \frac{r_j}{N}\frac{C_j}{r_j} + \frac{s}{N}\frac{C_{k+1}}{s} + \Bigg|\frac{1}{N}\Big(\sum_{j=1}^{k} r_j \mu_j(M) + s\mu_{k+1}(M)\Big) \\
&\quad - \mu(M)\Bigg| \leqq \frac{1}{r_k}\sum_{j=1}^{k+1} C_j + \left|\frac{1}{N}\Big(\sum_{j=1}^{k} r_j \mu_j(M) + s\mu_{k+1}(M)\Big) - \mu(M)\right| \\
&\leqq \frac{1}{k} + \left|\frac{1}{N}\Big(\sum_{j=1}^{k} r_j \mu_j(M) + s\mu_{k+1}(M)\Big) - \mu(M)\right|.
\end{aligned}$$

If $N \to \infty$, then $k \to \infty$, and the first term in the above sum tends to zero. Moreover, by (1) and by Cauchy's Theorem, the second term tends to zero. The proof is complete.

We shall now prove the metric result for μ-well distributed sequences which we announced above. Since we will suppose $\mu \in \mathscr{R}^+(X)$, the known definition of μ-well distributivity in a compact Hausdorff space, as given by Baayen and Helmberg [2], can be employed. The sequence (x_n) in X is called *μ-well distributed in X* if, for every μ-continuity set M in X, we have $\lim_{N\to\infty} 1/N \sum_{n=1}^{N} c_M(x_{n+h}) = \mu(M)$ uniformly in $h = 0, 1, 2, \cdots$. By an argument which was first used by Hlawka [10, Satz 2], we can easily infer from this definition the following interesting property of μ-well distributed sequences.

LEMMA 2. *If (x_n) is μ-well distributed in X, then for every μ-continuity set M with $\mu(M) > 0$ there exists a positive integer $N_0 = N_0(M)$ such that at least one of any N_0 consecutive elements from (x_n) lies in M.*

Let X_∞ be the cartesian product of countably many copies of X, i.e. $X_\infty = \prod_{i=1}^{\infty} X_i$ with $X_i = X$ for all $i \geqq 1$. Let μ_∞ be the completion of the product measure on X_∞ induced by μ. We note that a sequence (x_n) in X can be viewed as a point in X_∞, and a family of sequences in X can be viewed as a subset of X_∞. Let M' stand for the complement of M in X.

THEOREM 3. *Let $\mu \in \mathscr{R}^+(X)$, and μ not a point measure. Then the family $\mathscr{W}$ of all μ-well distributed sequences in X, viewed as a subset of X_∞, satisfies $\mu_\infty(\mathscr{W}) = 0$.*

PROOF. As in the proof of Theorem 2 in [6], there exists a μ-continuity set M in X with $0 < \mu(M) < 1$. For $N \geqq 1$, let $\mathscr{W}_N$ be the family of all μ-well distributed sequences in X for which at least one of any N consecutive elements lies in M. By Lemma 2, we have $\mathscr{W} = \bigcup_{N=1}^{\infty} \mathscr{W}_N$. Thus it suffices to show that $\mu_\infty(\mathscr{W}_N) = 0$ for all $N \geqq 1$. For given $N \geqq 1$, let X_N be the cartesian product of N copies of X, and let μ_N be the product measure on X_N. Define F_N to be the set consisting of all points of X_N for which at least one coordinate belongs to M, i.e. $F_N = X_N \setminus \prod_{i=1}^{N} M_i'$ with $M_i = M$ for all i. We note that $\mu_N(F_N) = 1-(1-\alpha)^N$, where $\alpha = \mu(M)$. For $k \geqq 0$, put $\mathscr{F}_N^{(k)} = \{(x_1, x_2, \cdots) \in X_\infty / (x_{jN+1}, \cdots, \cdots, x_{jN+N}) \in F_N$ for $0 \leqq j \leqq k\}$. If follows from the definition of $\mathscr{W}_N$ that $\mathscr{W}_N \subset \bigcap_{k=0}^{\infty} \mathscr{F}_N^{(k)}$. Now $\mu_\infty(\mathscr{F}_N^{(k)}) = (1-(1-\alpha)^N)^{k+1}$, and so $1-(1-\alpha)^N < 1$ implies $\mu_\infty(\bigcap_{k=0}^{\infty} \mathscr{F}_N^{(k)}) = 0$. Thus, a fortiori, $\mu_\infty(\mathscr{W}_N) = 0$.

REMARK. If μ is the normed point measure at some point $x_0 \in X$, then Theorem 3 is not true. The constant sequence $x_0, x_0, \cdots$ is μ-well distributed in X and corresponds to a point in X_∞ having μ_∞-measure equal to one.

As a conclusion, I would like to pose the following problem which was not settled in this paper. It follows from both the zero-one law of Visser [15] and of Hewitt and Savage [7] that the family of μ-uniformly distributed sequences in X, viewed as a subset of X_∞, has either μ_∞-measure zero or one, if it is μ_∞-measurable at all. Characterize the measures $\mu \in \mathscr{M}^+(X)$, or at least those in $\mathscr{R}^+(X)$, for which this μ_∞-measure is equal to one!

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