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## PSEUDOCONCAVE LIE GROUPS

by

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### 1. Introduction

The purpose of this note is to prove that every connected pseudoconcave complex Lie group is a complex torus. We outline the proof here, leaving the details for later sections.

Let  $G$  be a connected pseudoconcave complex Lie group of dimension  $n$ . The adjoint representation,  $Ad$ , maps  $G$  holomorphically into  $C^{n^2}$ . If  $f$  is a function holomorphic on  $C^{n^2}$  then  $f \circ Ad$  is a holomorphic function on  $G$ . Every pseudoconcave complex manifold has only constant holomorphic functions. Thus, since the holomorphic functions on  $C^{n^2}$  separate points,  $Ad$  maps  $G$  onto the identity matrix. This implies that  $G$  is abelian. Hence there is a discrete subgroup of  $C^n$ ,  $\Gamma$  such that  $G$  is biholomorphically isomorphic to  $C^n/\Gamma$ . Since  $G$  has only constant holomorphic functions,  $\Gamma$  has rank  $n$  over  $C$ . Thus we may assume that  $\Gamma$  is generated as a  $Z$ -module by the  $R$ -linearly independent vectors  $v_1 \cdots, v_m, e_1, \cdots, e_n$ , where  $e_i$  is the  $i$ -th unit vector in  $C^n$  and  $m \leq n$ .

If we take vectors  $v'_i$  near enough to  $v_i$  then the vectors  $v'_1, \cdots, v'_m, e_1, \cdots, e_n$  will still be  $R$ -linearly independent. Let  $\Gamma'$  be the group generated by these vectors and  $G' = C^n/\Gamma'$ . Lemma 7, which uses some ideas of Morimoto [2], states that if  $m < n$  then we can find vectors  $v'_i$  arbitrarily close to  $v_i$  so that there is a non-constant function which is holomorphic on  $G'$ . But Lemma 8 states that if the vectors  $v'_i$  are near enough to  $v_i$  then  $G'$  is pseudoconcave. The only alternative is that  $m = n$  and therefore  $G$  is a complex torus.

### 2. Preliminary definitions and lemmata

For our purposes complex manifolds are assumed to be connected. Let  $X$  be an  $n$ -dimensional complex manifold and  $Y$  an open subset of  $X$ .  $Y$  is said to have smooth boundary if for every  $p \in \partial Y$  there is an open neighborhood  $U = U(p)$  and a real-valued  $C^\infty$ -function  $\varphi$  with nowhere vanishing gradient defined on  $U$  such that  $\{u \in U : \varphi(u) < 0\} = Y \cap U$ . The analytic tangent plane to  $\partial Y$  at  $p$  is the unique  $(n-1)$ -dimensional

complex vector space contained in the real tangent space at  $p$ . The Levi form of  $\varphi$  at  $p$ ,  $L_p(\varphi)$ , is the hermitian form determined by the matrix  $(\partial^2\varphi(p))/(\partial z_\alpha\partial\bar{z}_\beta)$ . The signature of  $L_p(\varphi)$  restricted to the analytic tangent plane at  $p$  is a biholomorphic invariant and is independent of the defining function  $\varphi$ . We say  $\partial Y$  is pseudoconcave at  $p$  if  $L_p(\varphi)$  has at least one negative eigenvalue when restricted to the analytic tangent plane at  $p$ .

Let  $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ . A closed disc in  $X$  with center  $x$  is the image of  $\bar{D}$  under a biholomorphic mapping into  $X$  with  $x$  corresponding to the origin. If  $\partial Y$  is pseudoconcave at  $p$  then there is a closed disc  $\Delta \subset \bar{Y}$  with center  $p$  such that  $\Delta \cap \partial Y = \{p\}$ .

**DEFINITION.** Let  $X$  be a complex manifold of complex dimension  $\geq 2$ . We say  $X$  is pseudoconcave if there is an open, non-empty, relatively compact subset  $Y \Subset X$  such that  $\partial Y$  is smooth and everywhere pseudoconcave.

**LEMMA 1.** *Every function  $f$  holomorphic on a pseudoconcave manifold  $X$  is constant.*

**PROOF.** Let  $Y$  display the pseudoconcavity of  $X$ . Then  $|f|$  takes its maximum on  $\bar{Y}$  at  $p \in \partial Y$ . There is a closed disc  $\Delta \subset \bar{Y}$  with center at  $p$  such that  $\Delta \cap \partial Y = \{p\}$ .  $f|_\Delta$  is holomorphic on  $\Delta$  and assumes its maximum modulus at the center of  $\Delta$ . Thus  $f|_\Delta$  is constant. Therefore  $|f|$  takes its maximum on  $\bar{Y}$  at points of  $Y$ . Thus the restriction of  $f$  to the connected component of  $Y$  containing  $\Delta \sim \{p\}$  is constant. Thus  $f$  is constant.

Let  $G$  be an  $n$ -dimensional complex Lie group. For every  $x \in G$  define the holomorphic automorphism  $A_x$  on  $G$  by  $A_x(g) = xgx^{-1}$ . The Lie algebra of  $G$ ,  $L(G)$ , is defined to be the tangent space of  $G$  at the identity. Thus, for every  $x \in G$ ,  $dA_x : L(G) \rightarrow L(G)$  is an automorphism of  $L(G)$ . We have, therefore, the holomorphic mapping  $Ad : G \rightarrow \text{Aut}(L(G))$  defined by  $Ad(x) = dA_x$ . Since  $\text{Aut}(L(G)) \subset \mathbb{C}^{n^2}$ ,  $Ad : G \rightarrow \mathbb{C}^{n^2}$ .

**LEMMA 2.** *Let  $G$  be an  $n$ -dimensional pseudoconcave complex Lie group. There is a discrete subgroup of  $\mathbb{C}^n$  of rank  $n$  over  $\mathbb{C}$ ,  $\Gamma$ , such that  $G$  is biholomorphically isomorphic to  $\mathbb{C}^n/\Gamma$ .*

**PROOF.** Every function  $f$  holomorphic on  $\mathbb{C}^{n^2}$  gives us a function  $f \circ Ad$  holomorphic on  $G$ .  $f \circ Ad$  is constant by Lemma 1. The holomorphic functions on  $\mathbb{C}^{n^2}$  separate points. Thus  $Ad[G] = \{I\}$ , where  $I$  is the  $n \times n$  identity matrix. Therefore  $G$  is abelian.

By standard results in Lie theory [1],  $G$  is bihomorphically isomorphic to  $\mathbb{C}^n/\Gamma$ , where  $\Gamma$  is a discrete subgroup of  $\mathbb{C}^n$  of rank  $l$  over  $\mathbb{C}$ . After making a  $\mathbb{C}$ -linear change of coordinates in  $\mathbb{C}^n$  we may assume that  $\Gamma$

is generated as a  $\mathbf{Z}$ -module by the  $\mathbf{R}$ -linearly independent vectors  $v_1, \dots, v_m, e_1, \dots, e_l$ , where  $e_i$  is the  $i$ -th unit vector in  $\mathbf{C}^n$ . Thus  $G \cong \mathbf{C}^l/\Gamma \times \mathbf{C}^{n-l}$ . By Lemma 1,  $l = n$  and thus  $\Gamma$  has rank  $n$  over  $\mathbf{C}$ .

If  $\Gamma$  has rank  $n$  over  $\mathbf{C}$  and is generated over  $\mathbf{Z}$  by the  $\mathbf{R}$ -linearly independent vectors  $v_1 \cdots, v_m, e_1, \dots, e_n$  where  $e_i$  is the  $i$ -th unit vector in  $\mathbf{C}^n$ , then we follow the notation of [2] by writing  $V = (v_1, \dots, v_m)$ ,  $\Gamma = \Gamma(V)$ , and  $M^*(n, m; \mathbf{C})$  as the set of such matrices  $V$ .

A proof of the following classical lemma can be found in [2].

LEMMA 3. *Suppose  $V, V' \in M^*(n, m; \mathbf{C})$ . Then  $\mathbf{C}^n/\Gamma(V)$  and  $\mathbf{C}^n/\Gamma(V')$  are biholomorphically isomorphic if and only if there is a matrix  $\begin{pmatrix} A & C \\ B & D \end{pmatrix} \in GL(n+m, \mathbf{Z})$ , with  $A$  an  $n \times n$  matrix, such that  $(A+V'B)V = (C+V'D)$ . Further, if only  $V'$  (resp.  $V$ ) is known to be in  $M^*(n, m; \mathbf{C})$  and such a matrix exists then  $V$  (resp.  $V'$ )  $\in M^*(n, m; \mathbf{C})$ .*

### 3. The main result

$V \in M^*(n, m; \mathbf{C})$  is said to satisfy the Morimoto condition if there exist non-zero vectors  $a \in \mathbf{Z}^n$  and  $c \in \mathbf{Z}^m$  such that  $aV = c$ . The following lemma is due to Morimoto [2], but the proof given here is somewhat simpler.

LEMMA 4. *Let  $G = \mathbf{C}^n/\Gamma(V)$ , where  $V \in M^*(n, m; \mathbf{C})$  satisfies the Morimoto condition. Then there exists  $V' \in M^*(n, m; \mathbf{C})$  such that  $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$  and  $G$  is biholomorphically isomorphic to  $\mathbf{C}^n/\Gamma(V')$ .*

PROOF. Assume that we have constructed a matrix

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \in GL(n+m, \mathbf{Z})$$

such that  $BV-D$  is invertible and the last row of  $C-AV$  is identically zero. Defining  $V' = (C-AV)(BV-D)^{-1}$ , we have  $(A+V'B)V = C+V'D$ . Thus, by Lemma 3,  $V' \in M(n, m; \mathbf{C})$  and  $\mathbf{C}^n/\Gamma(V') \cong \mathbf{C}^n/\Gamma(V)$ . Since the last row of  $C-AV$  is identically zero,  $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$ . It remains to construct such a matrix.

$V$  satisfies the Morimoto condition. Thus there are nonzero vectors  $a = (a_1, \dots, a_n) \in \mathbf{Z}^n$  and  $c = (a_{n+1}, \dots, a_{n+m}) \in \mathbf{Z}^m$  such that  $aV = c$ . We may assume by renumbering the vectors  $v_i$  that  $a_1 \neq 0$  and that  $\gcd(a_1, \dots, a_{n+m}) = 1$ . Set  $\gcd(a_1, \dots, a_k) = p_k$  for  $k = 2, \dots, n+m$  and  $p_1 = a_1$ . We can find relatively prime integers  $\alpha_k$  and  $\gamma_k$  such that  $p_{k-1}\alpha_k + a_k\gamma_k = p_k, k = 2, \dots, n+m$ . For  $j = 1, \dots, i-1$  and  $i = 2, \dots, n$ , the numbers  $\beta_{ij} \equiv -a_j\gamma_i/p_{i-1}$  are integers, since  $p_{i-1}$  divides  $a_1, a_2, \dots, a_{i-1}$ . Define  $D_k \equiv \det M_k$ , where

$$M_k = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_k \\ \beta_{21} & \alpha_2 & & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & & & 0 \\ \beta_{k1} & \cdots & \beta_{k,k-1} & & & \alpha_k \end{pmatrix}.$$

Let  $r_i = -\gamma_i/p_{i-1}$ . Subtract the first row of  $M_k$  multiplied by  $r_i$  from its  $i$ -th row,  $i = 2, \dots, k$ . The resulting matrix is triangular with determinant

$$D_k = a_1 \prod_{i=2}^k (\alpha_i - r_i a_i) = p_1 \prod_{i=2}^k ((\alpha_i p_{i-1} + a_i \gamma_i)/p_{i-1}) = \prod_{i=1}^k p_i / \prod_{i=1}^{k-1} p_i = p_k,$$

$k = 2, \dots, m+n$ . Let  $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ , with  $A$  an  $n \times n$ -matrix, be  $M_{n+m}$  with first and  $n$ -th rows interchanged. Since  $p_{n+m} = 1$ ,  $M \in GL(n+m, \mathbb{Z})$ .

Now

$$B = \begin{pmatrix} r_{n+1} a \\ \vdots \\ r_{n+m} a \end{pmatrix} \quad \text{and} \quad BV = \begin{pmatrix} r_{n+1} c \\ \vdots \\ r_{n+m} c \end{pmatrix}.$$

Let  $c(k) = (a_{n+1}, \dots, a_{n+k}, 0, \dots, 0)$ . Therefore

$$D = \begin{pmatrix} \alpha_{n+1} {}^t e_1 \\ \alpha_{n+2} {}^t e_2 + r_{n+2} c(1) \\ \vdots \\ \alpha_{n+m} {}^t e_m + r_{n+m} c(m-1) \end{pmatrix}.$$

where  $e_i$  is the  $i$ -th unit vector in  $\mathbb{C}^m$ .

Thus

$$BV - D = \begin{pmatrix} r_{n+1} a_{n+1} - \alpha_{n+1} & & & * \\ \cdot & \cdot & & \\ & & \cdot & \\ 0 & & r_{n+m} a_{n+m} - \alpha_{n+m} & \end{pmatrix}.$$

Using the same argument as in the case of  $M_k$ , we see that  $BV - D$  is invertible. Since  $aV = c$ , the last row of  $C - AV$  is identically zero.

**COROLLARY 5** *If  $V \in M^*(n, m; \mathbb{C})$  and satisfies the Morimoto condition then there is a non-constant function holomorphic on  $G = \mathbb{C}^n/\Gamma(V)$ .*

**PROOF.** By Lemma 4 there exists  $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$  such that  $G$  is biholomorphically isomorphic to  $\mathbb{C}^n/\Gamma(V')$ . But  $\mathbb{C}^n/\Gamma(V') \cong \mathbb{C}^{n-1}/\Gamma(V'') \times \mathbb{C}^*$ .

Since there are non-constant functions holomorphic on  $C^*$ , there are non-constant functions holomorphic on  $G$ .

**COROLLARY 6.** *Let  $G$  be a pseudoconcave, complex Lie group. Then  $G$  is biholomorphically isomorphic to  $C^n/\Gamma(V)$  where  $V \in M^*(n, m; C)$  does not satisfy the Morimoto condition.*

**PROOF.** By Lemma 2 there exists  $V \in M^*(n, m; C)$  such that  $G$  is biholomorphically isomorphic to  $C^n/\Gamma(V)$ . Suppose  $V$  satisfies the Morimoto condition. Then, by Corollary 5, there is a non-constant function holomorphic on  $G$ . But, by Lemma 1, every function holomorphic on a pseudoconcave manifold is constant. Thus  $V$  does not satisfy the Morimoto condition.

For  $n \times m$ -matrices  $V$  and  $V'$  we define  $d(V, V')$  as the euclidean distance between  $V$  and  $V'$  induced from  $C^{mn}$ . We topologize  $M^*(n, m; C)$  via the metric  $d$ .  $V$  is said to have rational coordinates if  $V = (v_{\alpha\beta}) = (x_{\alpha\beta} + iy_{\alpha\beta})$  with  $x_{\alpha\beta}, y_{\alpha\beta} \in Q$ . Define  $M_Q^*(n, m; C)$  as the set of matrices in  $M^*(n, m; C)$  with rational coordinates.  $M_Q^*(n, m; C)$  is dense in  $M^*(n, m; C)$ .

**LEMMA 7.** *Suppose  $m < n$ . Then the matrices in  $M_Q^*(n, m; C)$  satisfy the Morimoto condition. These are therefore dense in  $M^*(n, m; C)$ .*

**PROOF.** Suppose  $V \in M_Q^*(n, m; C)$ . Then  $V = X + iY$ , where  $X = (x_{\alpha\beta})$ ,  $Y = (y_{\alpha\beta})$  and  $x_{\alpha\beta}, y_{\alpha\beta} \in Q$ . Since  $m < n$  there exists a non-zero row vector  $a \in Q^n$  such that  $aY = 0$ . Define  $c \equiv aX \in Q^m$ . Thus  $aV = c$ . By clearing denominators we may take  $a \in Z^n$  and  $c \in Z^m$ . Thus every matrix  $V \in M_Q^*(n, m; C)$  satisfies the Morimoto condition.

The remaining step for the proof of the main theorem is to show that pseudoconcavity is invariant under small changes of the group.

**LEMMA 8.** *Suppose  $V^0 \in M^*(n, m; C)$  and  $G^0 = C^n/\Gamma(V^0)$  is a pseudoconcave Lie group. There exists  $\varepsilon > 0$  such that if  $V \in M^*(n, m; C)$  and  $d(V, V^0) < \varepsilon$  then  $G = C^n/\Gamma(V)$  is a pseudoconcave Lie group.*

**PROOF.** For every  $V \in M^*(n, m; C)$  choose  $v_{m+1}, \dots, v_n$  such that  $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n, e_1, \dots, e_n\}$  is a basis for  $R^{2n}$ . Define the real linear maps  $S_V : R^{2n} \rightarrow R^{2n}$  by  $v_i \rightarrow v_i^0$  and  $e_i \rightarrow e_i$  for  $i = 1, \dots, n$ . Thus  $S_V : C^n \rightarrow C^n$  is a diffeomorphism such that  $S_V(V, E_n) = (V^0, E_n)$ . Hence  $S_V$  induces a diffeomorphism  $T_V : G \rightarrow G^0$  such that

$$\begin{array}{ccc}
 C^n & \xrightarrow{S_V} & C^n \\
 \pi_V \downarrow & & \downarrow \pi_{V^0} \\
 G & \xrightarrow{T_V} & G^0
 \end{array}$$

is commutative, where  $\pi_V$  (resp.  $\pi_{V^0}$ ) is the quotient map which divides  $C^n$  by the group  $\Gamma(V)$  (resp.  $\Gamma(V^0)$ ).

If  $d(V, V^0) < \varepsilon$  and  $\varepsilon$  is sufficiently small, then we may assume that  $v_{m+1} = v_{m+1}^0, \dots, v_n = v_n^0$ . If  $\|\cdot\|$  is any norm on  $C^n$ , then there exists a constant  $c > 0$  such that

$$\|Sz - z\| \leq c\varepsilon\|z\|$$

for any  $z \in C^n$ . This means that  $\|S - I\| \leq c\varepsilon$  where  $I$  is the identity map of  $C^n$  onto itself.

Let  $\Omega$  display the pseudoconcavity of  $G^0$ . It is enough to show that there exists  $\varepsilon > 0$  such that  $T_V^{-1}[\Omega]$  is pseudoconcave, open subset of  $G$  when  $d(V^0, V) < \varepsilon$ . Since  $\partial T_V^{-1}[\Omega]$  is compact, it is enough to prove that  $\partial T_V^{-1}[\Omega]$  is pseudoconcave at each  $p \in \partial T_V^{-1}[\Omega]$  when  $d(V^0, V) < \varepsilon(p)$ . Now  $\pi_V$  and  $\pi_{V^0}$  are locally biholomorphic. Hence we only need to prove that  $\partial(S_V^{-1} \circ \pi_V^{-1}[\Omega])$  is pseudoconcave at some point  $q \in \pi_V^{-1}(p)$ . We may assume that  $q = 0$ , because translation acts biholomorphically on  $G$  (resp.  $G^0$ ). Thus we have reduced the proof of this lemma to the following:

LEMMA 9. *Let  $S_V : C^n \rightarrow C^n$  be the real linear transformation defined above. Let  $U$  be a neighborhood of 0 and  $\varphi$  a real valued  $C^\infty$ -function defined on  $U$  with nowhere vanishing gradient and  $\varphi(0) = 0$ . Assume that 0 is a pseudoconcave boundary point of  $U^- = \{u \in U : \varphi(u) < 0\}$ . Then there exists  $\varepsilon > 0$  such that  $d(V^0, V) < \varepsilon$  implies that 0 is a pseudoconcave boundary point of  $S_V^{-1}[U^-]$ .*

PROOF. Set  $\zeta = Sz$ . By a suitable choice of the basis for coordinates in  $C^n$ , we may assume that

$$\varphi(\zeta_1, \dots, \zeta_n) = 2\text{Re}(\zeta_n + \sum \alpha_{ij} \zeta_i \zeta_j) + \sum \beta_{ij} \zeta_i \bar{\zeta}_j + 0(\|\zeta\|^3).$$

Since the origin is a pseudoconcave boundary point of  $U^-$ , we may as well assume that  $\beta_{11} < 0$ .

Let  $\zeta = Sz$  be given by the equations

$$\zeta_i = \sum_{j=1}^n a_{ij} z_j + \sum_{j=1}^n b_{ij} \bar{z}_j. \quad 1 \leq i \leq n$$

Because of the assumption  $d(V, V^0) < \varepsilon$ , we must have  $a_{ij} - \delta_{ij} = 0(\varepsilon)$  and  $b_{ij} = 0(\varepsilon)$ , where  $\delta_{ij} = 0$  if  $i \neq j$  and  $\delta_{ii} = 1$ , as we have observed before.

Substituting the  $\zeta_i$  by the above expressions, we obtain for the function  $\varphi \circ S_V$  that the analytic tangent plane to  $\varphi \circ S_V(z) = 0$  at the origin is given by the equations

$$\sum_{j=1}^n (a_{nj} + \bar{b}_{nj}) z_j = 0.$$

Let  $L$  be the complex line defined by the equations of the analytic tangent plane and  $\{z_2 = \cdots = z_{n-1} = 0\}$ . On  $L$  we have  $(a_{n1} + \bar{b}_{n1})z_1 + (a_{nn} + b_{nn})z_n = 0$ . Thus  $z_n = 0(\varepsilon)z_1$  on  $L$ . Moreover the Levi form of  $\varphi \circ S_V$  at the origin restricted to  $L$  reduces to

$$(\beta_{11} + 0(\varepsilon))|z_1|^2.$$

Recall that  $\beta_{11} < 0$ . Hence if  $\varepsilon$  is sufficiently small, the Levi form of  $\varphi \circ S_V$  at the origin restricted to  $L$  has one negative eigenvalue.

**THEOREM.** *Every pseudoconcave, complex Lie group  $G^0$  is a complex torus.*

**PROOF.** We have already shown that there exists  $V^0 \in M^*(m, n; \mathbb{C})$  such that  $G^0$  is biholomorphically isomorphic to  $\mathbb{C}^n/\Gamma(V^0)$ . By Lemma 8, there exists  $\varepsilon > 0$  such that if  $d(V^0, V) < \varepsilon$  then  $G = \mathbb{C}^n/\Gamma(V)$  is pseudoconcave. If  $m > n$  then, by Lemma 7, there exists  $V \in M^*(n, m; \mathbb{C})$  satisfying  $d(V^0, V) < \varepsilon$  and the Morimoto condition. But, by Corollary 6, this is absurd. Thus  $m = n$  and  $G^0$  is a complex torus.

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