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SOME LIMIT THEOREMS FOR LOCAL TIME

by

R. K. Getoor 1 and P. W. Millar 2

1. Introduction

Let $X = \{X(t), t \geq 0\}$ be a standard Markov process with state space $E$. Assume that for each $x \in E$, $x$ is regular for itself: i.e., if $T_x = \inf\{t > 0 : X(t) = x\}$, then $P_x\{T_x = 0\} = 1$. Then according to the theory of Blumenthal and Getoor, there is for each $x$, a continuous, unique (up to constant multiples), increasing additive functional $\{L^x_t, t \geq 0\}$, called the local time at $x$, which increases ‘only’ when the process $X$ is in the state $x$ (see [1], [2] for precise descriptions.) As such, $L^x_t$ is supposed to give some indication of how much time before $t$ the process $X$ spends in the vicinity of $x$. In special cases, $L^x_t$ admits representation as a limit of quantities that measure in some more direct way the amount of time spent near $x$. For example, if $B_n$ is a sequence of neighborhoods of $x$ with $n B_n = x$, then it often turns out that

$$L^x_t = \lim_{n \to \infty} [\mu(B_n)]^{-1} \int_0^t I_{B_n}[X(s)]ds$$

(see Griego, [6]) so that $L^x_t$ is a limit of ‘occupation times’ averaged over smaller and smaller neighborhoods of $x$. (Here $I_B$ is the indicator of $B$, $\mu$ is Lebesgue measure.) For certain diffusion processes, it is known that $L^x_t = \lim_{\varepsilon \to 0} \varepsilon d_\varepsilon(t)$, where $d_\varepsilon(t)$ is the number of times the real valued process crosses down from $\varepsilon > 0$ to 0 before time $t$ (this result, conjectured by Lévy [8], was proved by Itô and McKean [7]). Finally, Blumenthal and Getoor ([1], see also [5]) showed that under fairly general circumstances, if $X$ has a reference measure $\xi$, then an appropriate choice of the local time $L^x_t$ could serve as a density for occupation time, in the sense that for every Borel set $B$, $\int_B I_B[X(s)]ds = \int_B L^x_\xi(dx)$ a.s. In this paper, a new description of $L^0_t$, valid for a wide class of real valued Markov processes, is found which describes $L^0_t$ more or less in terms of ‘the number of times’ the process ‘jumps across zero.’

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While the basic theorem of this paper can be formulated for quite general Markov processes (see section 2), the character of this result is most easily illustrated by a few special cases. Let \( X = \{X(t), \ t \geq 0\} \) be a real valued process with stationary independent increments and Lévy measure \( \nu \). Let \( J_n(t) \) be the number of jumps \( j(X, s) = X(s) - X(s-) \) before time \( t \) for which \( X(s-) < 0 < X(s) \) and \( 2^{-n-1} < j(X, s) < 2^{-n} \). So \( J_n(t) \) is the number of (upward) jumps across 0 having size in \((2^{-n-1}, 2^{-n})\). Assume that \( \nu(R) = \infty \), that 0 is regular for \( \{0\} \), and that \( L_t^x \) is jointly continuous in \((x, t)\). Then (theorem 3.2) there is a version \( \ell_t^0 \) of the local time at 0 (see section 2 for the precise description) such that for each \( T \),

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |J_n(t)/F_n - \ell_t^0| = 0 \quad \text{a.s.}
\]

provided \( \Sigma[1/F_n] < \infty \), where \( F_n = \int 2^{-n} \nu(dx) \). This result is very close in spirit to the result of Ito and McKean mentioned above. An illustration is the case where \( X \) is a stable process with index \( \alpha \), \( 1 < \alpha < 2 \), in which case \( J_n(t)/2^{n(\alpha - 1)} \) converges a.s. to \( c\ell_t^0 \) uniformly, where \( c \) is a known constant. The basic result of section 2 also yields theorems of the following type. Let \( X \) be again a stable process with index \( \alpha \), \( 1 < \alpha < 2 \), and set \( Q_\varepsilon(t) = \sum_{s \leq t} |X(s) - X(s-)| \), where the prime indicates that the sum is over all \( s \leq t \) for which \( X(s-) < 0 < X(s) \) and \( |X(s) - X(s-)| < \varepsilon \). Thus \( Q_\varepsilon(t) \) is the sum up to time \( t \) of all the upward jumps across 0 which have magnitude at most \( \varepsilon \). Then for stable \( X \) with \( 1 < \alpha < 2 \) (see theorem 3.1),

\[
\lim_{\varepsilon \to 0} \mathbb{P}\left\{ \sup_{0 \leq t \leq T} |Q_\varepsilon(t)/\varepsilon^{2-\alpha} - c\ell_t^0| > \delta \right\} = 0
\]

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |Q_{2^{-n}}(t)/2^{n(\alpha - 2)} - c\ell_t^0| = 0 \quad \text{a.s.}
\]

for all \( \delta > 0 \), and \( T > 0 \).

Section 2 contains the statement and proof of the basic result, while section 3 presents a number of applications to processes with independent increments. The terminology referring to the theory of Markov processes will be that of [2].

### 2. Main result

Let \( X = \{X(t), \ t \geq 0\} \) be a standard, real valued Markov process. Let \( T_x = \inf\{t > 0: X(t) = x\} \), and assume from now on that each \( x \) is regular for itself. Define for \( \alpha > 0 \)

\[
\psi^\alpha(x, y) = E^x[\exp \{-\alpha T_y\}]
\]

According to the theory of Blumenthal and Getoor [2], there is for each
a continuous additive functional \( \{L^x_t, t \geq 0\} \), the local time at \( x \), satisfying
\[ E^x \int_0^\infty e^{-t} dL^x_t = \psi^1(x, y), \]
so that in particular \( E^x \int_0^\infty e^{-t} dL^x_t = 1 \). Assume from now on that \( \psi^1(x, y) \)
is jointly Borel measurable, and that there is a reference measure \( \xi \) for the process \( X \). If \( U^x \) is the usual operator \( U^x f(x) = E^x \int_0^\infty e^{-at} f[X(t)] dt \),
then under the preceding assumptions Getoor and Kesten [5] have proved the following result which will be stated as a lemma for the convenience of the reader.

**Lemma 2.1** There is a strictly positive finite Borel function \( g \) on \( R \) such that \( I^x_t(\omega) \) defined by \( I^x_t(\omega) = g(x)L^x_t(\omega) \) satisfies a.s.
\[ \int_0^t I^x_B[X(s)] ds = \int_B I^x_t(\omega) \xi(dx) \]
for all \( t \geq 0 \) and Borel sets \( B \) simultaneously. Define for each \( \alpha > 0 \),
\( u^\alpha(x, y) = g(y) E^x \int_0^\infty e^{-at} dL^x_t \). Then
\[ U^xf(x) = \int u^\alpha(x, y)f(y) \xi(dy) \]
for all Borel \( f \geq 0 \)
\[ u^1(y, y) = g(y) \]
\[ u^\alpha(x, y) = E^x \int_0^\infty e^{-at} dL^x_t \]
\[ \psi^1(x, y) = u^1(x, y)/u^1(y, y). \]

Since a reference measure is assumed throughout this section, there exists, according to the theory of S. Watanabe [13], a Lévy system \( (N, A) \) for the process \( X \). Here \( N(x, dy) \) is a non-negative kernel such that for each \( x \in R, N(x, \cdot) \) is a measure on the Borel sets of \( R \) and for each Borel set \( B, N(\cdot, B) \) is a Borel function. \( A = \{A(t), t \geq 0\} \) is a finite, continuous additive functional having the following property: for every non-negative Borel function \( f \) on \( R \times R \) vanishing on the diagonal
\[ E^x \sum_{s \leq t} f[X(s-), X(s)] = E^x \int_0^t Nf[X(s)] dA(s) \]
where \( Nf(x) = \int N(x, dy)f(x, y) \). For simplicity, assume from now on that \( A(t) = t \). This may always be achieved by a time change if necessary (see, for example [11]); in the case of processes with independent increments one may always take \( A(t) = t \) as can be verified directly.

For the remainder of this section assume
Theorem 2.1. Let $X = \{X(t), t \geq 0\}$ be a standard, real valued Markov process. Assume that there is a reference measure $\xi$, that each point of the state space is regular for itself, that $\psi(x, y)$ is jointly Borel measurable, and that the additive functional $A$ of the Lévy system $(N, A)$ satisfies $A(t) = t$. For all sufficiently small $\varepsilon$ or only for a $\varepsilon$ through a sequence, let $f_\varepsilon(x, y)$ be a non-negative Borel function vanishing on the diagonal such that there exists a compact interval $I(\varepsilon) = [a(\varepsilon), b(\varepsilon)]$ containing 0 and such that $Nf_\varepsilon(x) = 0$ outside $I(\varepsilon)$. Assume $a(\varepsilon) \to 0$ and $b(\varepsilon) \to 0$ as $\varepsilon \to 0$, and that

$$F(\varepsilon) = \int Nf_\varepsilon(x)\xi(dx) = \int_{I(\varepsilon)} Nf_\varepsilon(x)\xi(dx)$$

satisfies $0 < F(\varepsilon) < \infty$. Assume (2.9) and (2.10). Finally, define

$$G(\varepsilon) = \int Nf_\varepsilon^2(x)\xi(dx) \quad \text{and} \quad Q_\varepsilon(t) = \sum_{s \leq t} f_\varepsilon[X(s-), X(s)].$$

Then the following conclusions hold.

1. If $\lim_{\varepsilon \to 0} G(\varepsilon)/[F(\varepsilon)]^2 = 0$, then

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \frac{|Q_\varepsilon(t)/F(\varepsilon) - I^0_t|}{\varepsilon} > \delta = 0$$

for each $T$ and each $\delta > 0$.

2. If $\{I^x_t\}$ is jointly continuous in $(t, x)$ near 0 and if $\{\varepsilon_n, n \geq 1\}$ is a sequence decreasing to 0 such that $\sum_{n \geq 1} G(\varepsilon_n)/[F(\varepsilon_n)]^2 < \infty$, then

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |Q_{\varepsilon_n}(t)/F(\varepsilon_n) - I^0_t| = 0 \quad \text{a.s.}$$

for each $T$.

Remark. If $u^x(x, x)$ is continuous as a function of x, then it follows from lemma 2.1 that $I^x_t$ will be jointly continuous in $(t, x)$ if and only if $L^x_t$ is jointly continuous. Conditions guaranteeing joint continuity of $L^x_t$ have been given recently by Getoor and Kesten [5].

Before proceeding with the proof, note that, under the assumptions of theorem 2.1, Lemma 2.1 permits the following conclusion.

Lemma 2.2. Let $h(x)$ be a non-negative, finite Borel function. Then $A(t) = \int_0^t h[X(s)]ds$ and $B(t) = \int_0^t h(x)\xi(dx)$ are equivalent stochastic processes.
PROOF. It suffices to assume $h$ bounded. Let $u^A_\alpha$ and $u^B_\alpha$ be the $\alpha$ potentials of $A$ and $B$ respectively. Then from (2.4),

$$u^A_\alpha(x) \overset{def}{=} \mathbb{E}^x \int_0^\infty e^{-t}dA(t) = \mathbb{E}^x \int_0^\infty e^{-t}h[X(s)]ds = U^1h(x) = \int u^1(x, y)h(y)\xi(dy),$$

and from (2.6)

$$u^B_\alpha(x) = \mathbb{E}^x \int_0^\infty e^{-t}dB(t) = \int u^1(x, y)h(y)\xi(dy).$$

Since $A$ and $B$ are continuous additive functionals having the same bounded 1-potentials, the conclusion follows (see [2], p. 157).

The proof of theorem 2.1 has the following structure. In the terminology of Meyer [10], the process $Q_\varepsilon(t)$ is increasing but not natural. By Meyer’s decomposition theorem for supermartingales, there is a unique natural increasing process $V_\varepsilon(t)$ such that $Q_\varepsilon(t) - V_\varepsilon(t)$ is a martingale. The limit results on $Q_\varepsilon(t)$ are then deduced by analysing $V_\varepsilon(t)$ and the martingale separately.

PROOF OF THEOREM 2.1. First, observe that from 2.2

$$1 \geq \mathbb{E}^y \int_0^T e^{-t}dL_\varepsilon^x \geq e^{-T}E^yL_\varepsilon^x,$$

so that

$$E^yL_\varepsilon^x \leq e^T \quad \text{for every } T,$$

and from lemma 2.1

$$E^yL_\varepsilon^x \leq u^1(x, x)E^yL_\varepsilon^x \leq u^1(x, x)e^T. \tag{2.14}$$

Let $V_\varepsilon(t) = \int_0^t Nf_\varepsilon[X(s)]ds$. From lemma 2.2, $V_\varepsilon(t) = \int l_\varepsilon^x Nf_\varepsilon(x)\xi(dx)$, and from (2.14), $E^yV_\varepsilon(t) = \int E^y(l_\varepsilon^x Nf_\varepsilon(x)\xi(dx) \leq e^t \int u^1(x, x)Nf_\varepsilon(x)\xi(dx)$. By assumption (2.10), $u^1(x, x) \to u^1(0, 0)$ as $x \to 0$, so $u^1(x, x)$ is bounded for $x$ near 0. Thus for $\varepsilon$ small, and all $y$,

$$E^y V_\varepsilon(t) \leq Me^t \int Nf_\varepsilon(x)\xi(dx) < \infty$$

for some constant $M$. Observe that $Q_\varepsilon(t)$ is an additive functional and $V_\varepsilon$ is a continuous additive functional. It then follows from (2.8) that
(2.15) \[ M_\varepsilon(t) = Q_\varepsilon(t) - V_\varepsilon(t) \]
is an additive functional with mean zero, and so must be a martingale relative to each \( P^\varepsilon \).

Define \( \mu^\varepsilon(dx) = Nf_\varepsilon(x)\xi(dx)/F(\varepsilon) \). By the assumptions of theorem 2.1, \( \mu^\varepsilon \) is a probability measure carried by \( I(\varepsilon) \), and \( \mu^\varepsilon \) converges weakly to unit mass at 0. Moreover,

\[ V_\varepsilon(t)/F(\varepsilon) = \int l^\varepsilon_t \mu^\varepsilon(dx), \]

and

\[ \sup_{0 \leq t \leq T} |V_\varepsilon(t)/F(\varepsilon) - l^0_t| \leq \int \sup_{0 \leq t \leq T} |l^\varepsilon_t - l^0_t| \mu^\varepsilon(dx). \]

If, as in case (2), \( l^\varepsilon_t \) is jointly continuous in \((t, x)\), then for almost all \( \omega \),

\[ \sup_{0 \leq t \leq T} |l^\varepsilon_t - l^0_t| \to 0 \quad \text{as} \quad x \to 0 \quad \text{by uniform continuity, implying that} \]

\[ \sup_{0 \leq t \leq T} |V_\varepsilon(t)/F(\varepsilon) - l^0_t| \to 0 \quad \text{a.s. in this case.} \]

To treat case 1, recall that according to a result of Meyer ([9], see also [2], V.3)

\[ P^0\{ \sup_{0 \leq t \leq T} |L^\varepsilon_t - L^0_t| > 2\delta \} \leq 2e^T e^{-\delta \gamma(x)}, \]

where \( \gamma(x) \) was defined in (2.9). By virtue of the formula

\[ E[Y] = \int_0^\infty P\{|Y| > s\} ds, \quad E^0\sup_{0 \leq t \leq T} |L^\varepsilon_t - L^0_t| \leq 4e^T \gamma(x). \]

Since (see lemma 2.1)

\[ l^\varepsilon_t - l^0_t = u^1(x, x)L^\varepsilon_t - u^1(0, 0)L^0_t \]

\[ = u^1(x, x)[L^\varepsilon_t - L^0_t] - L^0_t[u^1(x, x) - u^1(0, 0)], \]

it follows from (2.13) and the calculation above that

(2.16) \[ E^0\sup_{0 \leq t \leq T} |l^\varepsilon_t - l^0_t| \leq u^1(x, x)4e^T \gamma(x) + e^T[u^1(x, x) - u^1(0, 0)]. \]

Hence

\[ E^0\{ \sup_{0 \leq t \leq T} |V_\varepsilon(t)/F(\varepsilon) - l^0_t| \} \leq \int E^0\sup_{0 \leq t \leq T} |l^\varepsilon_t - l^0_t| \mu^\varepsilon(dx) \]

\[ \leq 4e^T \int u^1(x, x)\gamma(x)\mu^\varepsilon(dx) + e^T \int [u^1(x, x) - u^1(0, 0)] \mu_\varepsilon(dx). \]

By assumptions (2.9), (2.10) the integrands above are continuous at 0 and are equal to 0 there. Since \( \mu_\varepsilon(dx) \) converges weakly to unit mass at 0, this completes the treatment of \( V_\varepsilon(t) \) in both case 1 and case 2.

Turning next to the martingale \( M_\varepsilon(t) \), observe that a fundamental fact on Lévy systems (S. Watanabe [13], p. 63, eq. 3.11) implies that
Since \( M_\varepsilon \) is a martingale, this together with the well-known martingale inequality of Doob ([4], p. 317) yields

\[
E^0 \sup_{0 \leq t \leq T} [M_\varepsilon(t)]^2 \leq 4E^0[M_\varepsilon(T)]^2
\]

\[
= 4E^0 \int_0^T Nf_\varepsilon^2[X(s)]\,ds
\]

\[
= 4E^0 \int l_T^\varepsilon Nf_\varepsilon^2(x)\xi(dx) \quad \text{(by lemma 2.2)}
\]

\[
\leq Me^T \int Nf_\varepsilon^2(x)\xi(dx) \quad \text{(if \( \varepsilon \) is sufficiently small)}
\]

\[
= Me^T G(\varepsilon).
\]

Since by hypothesis \( \lim_{\varepsilon \to 0} G(\varepsilon)/[F(\varepsilon)]^2 = 0 \) in case 1, it follows that \( \sup_{0 \leq t \leq T} M_\varepsilon(t)/[F(\varepsilon)]^2 \to 0 \) in probability. In case 2, since

\[
E^0 \sup_{0 \leq t \leq T} [M_\varepsilon(t)/F(\varepsilon)]^2 < \text{const. } G(\varepsilon)/[F(\varepsilon)]^2
\]

and \( \Sigma G(\varepsilon_n)/[F(\varepsilon_n)]^2 < \infty \), a Borel Cantelli argument shows

\[
\sup_{0 \leq t \leq T} M_\varepsilon(t)/F(\varepsilon_n) \to 0 \quad \text{a.s.}
\]

This completes the proof of theorem 2.1.

### 3. Processes with stationary independent increments

Theorem 2.1 yields a number of interesting results when \( X = \{X(t), \ t \geq 0\} \) is a process with independent increments. This section contains several of these applications.

Throughout this section, let \( X = \{X(t), \ t \geq 0\} \) be a real valued process with stationary independent increments having right continuous paths with left limits. Of course, \( E^0 e^{iuX(t)} = \exp\{-t\phi(u)\} \) where

\[
\phi(u) = iau + (\frac{1}{2})Su^2 + \int R \{1 - e^{ix} + iux/[1 + x^2]\} \nu(dx).
\]

The measure \( \nu \) is called the Lévy measure, and \( \phi \) is called the exponent of \( X \). Assume throughout that 0 is regular for itself; in the present circumstances this implies that each \( x \) is regular for \( \{x\} \). Precise conditions under which 0 is regular for \( \{0\} \) may be found in [3]. Assume also from now on that \( \nu(R) = \infty \). (If \( \nu(R) < \infty \), then it is obvious that it is impossible to represent local time (when it exists) as a limit of quantities depending only on the jumps about 0).
Under the last two assumptions, it is known ([3], [12]) that for each 
\( a > 0 \) there exists a real, bounded, continuous function \( u^a(x) \) such that 
\( U^a f(x) = \int f(y) u^a(y-x)dy \) for all bounded Borel \( f \), and satisfying 
\( u^a(x) = u^a(0) \psi^a(0, x) \). It follows from this that Lebesgue measure is a 
reference measure and, in the notation of section 2, \( u^a(x, y) = u^a(y-x) \); 
see [5] for more detail on this point. Since there is a reference measure, 
a Lévy system \((N, A)\) exists which, in fact is given by \( N(x, dy) = \nu(dy-x) \), 
\( A(t) = t \). Actually, for the case of processes with independent increments 
one may show directly that this is a Lévy system for \( X \) even when there 
is no reference measure. It is clear that \( \psi^1(x, y) = \psi^1(0, y-x) \) is jointly 
Borel measurable in \((x, y)\) in the present case \((\psi^1(0, z) \) is continuous in \( z)\). 
Since \( \psi^1(x, y) = u^1(y-x)/u^1(0) \), it follows again from the continuity 
of \( u^1(\cdot) \) that \( \lim_{x \to 0} \psi^1(x, 0) = \lim_{x \to 0} \psi^1(0, x) = u^1(0)/u^1(0) = 1 \), so 
(2.9) holds. Finally, since \( u^1(x, x) = u^1(0) \), (2.10) holds trivially and so 
all assumptions of section 2 are satisfied by a real valued process with 
stationary independent increments having 0 regular for \( \{0\} \) and \( \nu(R) = \infty \).

For theorem 3.1, let \( F(\delta) = \int_0^\infty \int_0^\infty g(y)v(dy)dx \), where \( g \) is a 
non-negative Borel function on \((0, \infty)\), bounded on finite intervals. If 
\( 0 < \delta < \epsilon \), an integration by parts yields

\[
\int_0^\epsilon x g(x)v(dx) = \delta \int_0^\epsilon g(x)v(dx) + \int_0^\epsilon \int_x^\epsilon g(y)v(dy)dx = \delta K(\delta) + \int_0^\epsilon K(x)dx
\]

where \( K(x) = \int_x^\epsilon g(y)v(dy) \) is a function that increases as \( x \) decreases. 
If \( \delta \downarrow 0 \) and \( \int_0^\epsilon x g(x)v(dx) < \infty \), then it follows, since all terms above are 
positive, that \( \int_0^\epsilon K(x)dx < \infty \) and \( \lim_{\delta \to 0} \delta K(\delta) \) exists. Since 
\( \infty > \int_0^\epsilon K(x)dx \geq K(\delta)\delta \), it follows that \( \lim_{\delta \to 0} \delta K(\delta) = 0 \). Hence, if \( \int_0^\epsilon x g(x)v(dx) < \infty \), then 
\( \infty > \int_0^\epsilon \int_x^\epsilon g(y)v(dy)dx = \int_0^\epsilon x g(x)v(dx) \). Conversely it is not hard to see 
that if \( F(\epsilon) < \infty \), then \( \infty > \int_0^\epsilon x g(x)v(dx) = F(\epsilon) \).

**Theorem 3.1** Assume \( 0 < F(\epsilon) = \int_0^\epsilon x g(x)v(dx) < \infty \), and define 
\( G(\epsilon) = \int_0^\epsilon x [g(x)]^2v(dx) \). Let \( Q_\epsilon(t) = \sum_{s \leq t} \epsilon(|j(X, s)|) \) where the prime 
means that the sum is over only those jumps \( j(X, s) = X(s) - X(s-) \) for 
which \( X(s-) < 0 < X(s) \) and \( |j(X, s)| < \epsilon \). Then:

(a) If \( \lim_{\epsilon \to 0} G(\epsilon)/[F(\epsilon)]^2 = 0 \), then 
\[
\lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} \left\{ \left| Q_\epsilon(t)/F(\epsilon) - l_\epsilon^0 \right| > \delta \right\} = 0
\]
for every \( T > 0 \) and \( \delta > 0 \).

(b) If \( l_\epsilon^x \) is jointly continuous in \((t, x)\) and if \( \{\epsilon_n, n \geq 1\} \) is a positive 
sequence converging to zero such that \( \Sigma G(\epsilon_n)/[F(\epsilon_n)]^2 < \infty \), then
Conditions guaranteeing joint continuity of $l^x_t$ can be found in [5]. A large number of functions $g$ satisfy the hypothesis that $F(\varepsilon) < \infty$; in particular $g(x) = |x|$ always works, since $\int_{|x| < 1} |x|^2 v(dx) < \infty$ for any Lévy measure $v$. As a special case, suppose $X$ is a stable process with index $\alpha$, $1 < \alpha < 2$. Then the exponent is of the form

$$\phi(u) = c_1 \int_0^\infty \left[ e^{iux} - 1 + iux/(1 + x^2) \right] x^{-\alpha - 1} dx$$

$$+ c_2 \int_{-\infty}^0 \left[ e^{iux} - 1 + iux/(1 + x^2) \right] |x|^{-\alpha - 1} dx.$$

where $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 > 0$. Suppose $c_1 > 0$ for convenience.

If $g(x) = |x|$, then $F(\varepsilon) = c e^{2-\alpha}$, where $c = c_1 (2 - \alpha)^{-1}$, and

$$G(\varepsilon)/[F(\varepsilon)]^2 = \text{const.} \ v^{\alpha - 1}.$$

Then, as mentioned in the introduction, $\lim_{\varepsilon \to 0} Q_\varepsilon(t)/\varepsilon^{2-\alpha} = cl_t^0$ in probability, uniformly on compact intervals and

$$\lim_{n \to \infty} Q_{2^{-n}}(t)/2^{-n(2-\alpha)} = cl_t^0 \quad \text{a.s.,}$$

uniformly on compact intervals. (That $l^x_t$ is jointly continuous in the stable case is well-known - see [2] and the references there.) As another example, the asymmetric Cauchy processes are interesting to consider. Here the exponent $\phi$ is of the form (3.1) with $\alpha = 1$ and $c_1 \neq c_2$. Assume $c_1 > 0$ (if not, then one can establish the result below for $-X$ instead.) It was proved by Kesten and Getoor that no jointly continuous version of the local time exists for the asymmetric Cauchy processes ([5], example b, section 4). Choose $g$ of theorem 3.1 to be

$$g(u) = [(-\log |u|) \vee 0]^a.$$

Then for sufficiently small $\varepsilon$,

$$F(\varepsilon) = c_1 \int_0^\varepsilon (-\log x)^a x^{-1} dx = -c_1/(a+1)(-\log \varepsilon)^{a+1} < \infty$$

if $a < -1$ and

$$G(\varepsilon) = c_1 \int_{-\varepsilon}^0 (-\log x)^2 x^{-1} dx = -c_1/(2a+1)(-\log \varepsilon)^{2a+1}.$$

Thus $G(\varepsilon)/[F(\varepsilon)]^2 = [(a+1)^2/(2a+1)][1/(-\log \varepsilon) \to 0$ as $\varepsilon \to 0$, so a limit theorem continues to hold even in this singular case.
PROOF OF THEOREM 3.1. Let \( f_\varepsilon(x, y) = g(|x - y|)I\{x < 0 < y; 0 < y - x < \varepsilon\} \), where \( I\{A\} \) is the indicator of the set \( A \). Then \( Q_\varepsilon(t) = \sum_{s \leq t} f_\varepsilon[X(s-), X(s)] \). In the notation of theorem 2.1,

\[
Nf_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} g(u)\nu(du) \quad \text{if } x \in [-\varepsilon, 0) \text{ and } Nf_\varepsilon(x) = 0, x \notin [-\varepsilon, 0].
\]

Also,

\[
\int Nf_\varepsilon(x)dx = \int_{-\varepsilon}^{0} dx \int_{-\varepsilon}^{\varepsilon} g(u)\nu(du) = \int_{0}^{\varepsilon} dx \int_{-\varepsilon}^{\varepsilon} g(u)\nu(du) = \int_{0}^{\varepsilon} xg(x)\nu(dx) = F(\varepsilon).
\]

Similarly,

\[
\int Nf_\varepsilon^2(x)dx = \int_{0}^{\varepsilon} x[g(x)]^2\nu(dx) = G(\varepsilon).
\]

The result now follows from theorem 2.1.

Next, consider the following choice of \( f_\varepsilon : f_\varepsilon(x, y) = I\{x < 0 < y\}; \lambda(\varepsilon)\varepsilon < y - x < \varepsilon\} \), where \( 0 < \lambda(\varepsilon) < 1 \). Then

\[
J_\varepsilon(t) = \sum_{s \leq t}^\varepsilon [X(s-), X(s)]
\]

is equal to the number of jumps \( j(X, s) = X(s) - X(s-) \) across \( 0 \) up to time \( t \) for which \( \varepsilon \lambda(\varepsilon) < j(X, s) < \varepsilon \). Here

\[
Nf_\varepsilon(x) = \int v(dy)f_\varepsilon(x, y+x)
\]

\[
= \begin{cases} 
  v([-\varepsilon, \varepsilon]), & -\varepsilon < x < -\varepsilon\lambda(\varepsilon) \\
  v((\varepsilon\lambda(\varepsilon), \varepsilon]), & -\varepsilon\lambda(\varepsilon) < x < 0 \\
  0, & \text{otherwise}.
\end{cases}
\]

Then

\[
\int Nf_\varepsilon(x)dx = \int_{-\varepsilon}^{0} Nf_\varepsilon(x)dx
\]

\[
= \int_{-\varepsilon}^{\varepsilon} v[(x, \varepsilon)]dx + \varepsilon\lambda(\varepsilon)v[(\varepsilon\lambda(\varepsilon), \varepsilon)]
\]

\[
= \int_{-\varepsilon}^{\varepsilon} xv(dx) = F(\varepsilon),
\]

since \( f_\varepsilon = f_\varepsilon^2 \), \( \int Nf_\varepsilon^2 = \int Nf_\varepsilon = F(\varepsilon) \) in this case. An application of theorem 2.1 to the preceding calculations then yields the following result.

THEOREM 3.2. Let \( \lambda \) be a function such that \( 0 < \lambda(\varepsilon) < 1 \) for all \( \varepsilon \) and define \( F(\varepsilon) = F(\lambda, \varepsilon) = \int_{-\varepsilon\lambda(\varepsilon)}^{\varepsilon\lambda(\varepsilon)} xv(dx) \). Let \( J_\varepsilon(t) \) be the number of jumps up to time \( t \) for which \( X(s-) < 0 < X(s) \) and \( \varepsilon\lambda(\varepsilon) < X(s) - X(s-) < \varepsilon \).
(b) If $l_t^x$ is jointly continuous in $(x, t)$ and if $\varepsilon_n$ is a positive sequence converging to zero such that $\sum_{n \geq 1} [1/F(\varepsilon_n)] < \infty$ then

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |J_n(t)/F(\varepsilon_n) - l_t^0| = 0 \quad \text{a.s.}$$

Let $X$ be a stable process with index $\alpha$, $1 < \alpha < 2$, and exponent (3.1) with $c_1 > 0$, and let $J_n(t)$ be the number of upward jumps across 0 up to time $t$ having size $(2^{-n-1}, 2^{-n})$. According to theorem 3.2b,

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |J_n(t)/2^{n(\alpha-1)} - c l_t^0| = 0 \quad \text{a.s.}$$

where $c = c_1 (2^{n-1} - 1)/(\alpha - 1)$, as mentioned in the introduction. (Take $\varepsilon_n = 2^{-n}$ and $\lambda(2^{-n}) = (\frac{1}{2})$ for all $n$.) If $X$ is an asymmetric Cauchy process, $F(\varepsilon) = c_1 \int_{\varepsilon}^{\infty} x^{-1} dx = -c_1 \log \lambda(\varepsilon)$. If $\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$, then $\lim_{\varepsilon \to 0} F(\varepsilon) = +\infty$. Hence from theorem 3.2a,

$$\sup_{0 \leq t \leq T} |J_\varepsilon(t)/[-\log \lambda(\varepsilon)] - c_1 l_t^0| \to 0$$

in probability. Again a limit theorem continues to hold in the singular Cauchy case.

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