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Compositio Mathematica, tome 25, n° 2 (1972), p. 123-134

<http://www.numdam.org/item?id=CM_1972__25_2_123_0>
SOME LIMIT THEOREMS FOR LOCAL TIME

by

R. K. Getoor\textsuperscript{1} and P. W. Millar\textsuperscript{2}

1. Introduction

Let $X = \{X(t), t \geq 0\}$ be a standard Markov process with state space $E$. Assume that for each $x \in E$, $x$ is regular for itself: i.e., if $T_x = \inf\{t > 0 : X(t) = x\}$, then $P_x\{T_x = 0\} = 1$. Then according to the theory of Blumenthal and Getoor, there is for each $x$, a continuous, unique (up to constant multiples), increasing additive functional $\{L^x_t, t \geq 0\}$, called the local time at $x$, which increases ‘only’ when the process $X$ is in the state $x$ (see [1], [2] for precise descriptions.) As such, $L^x_t$ is supposed to give some indication of how much time before $t$ the process $X$ spends in the vicinity of $x$. In special cases, $L^x_t$ admits representation as a limit of quantities that measure in some more direct way the amount of time spent near $x$. For example, if $B_n$ is a sequence of neighborhoods of $x$ with $\cap B_n = x$, then it often turns out that

$$L^x_t = \lim_{n \to \infty} [\mu(B_n)]^{-1} \int_0^t I_{B_n}[X(s)]ds$$

(see Griego, [6]) so that $L^x_t$ is a limit of ‘occupation times’ averaged over smaller and smaller neighborhoods of $x$. (Here $I_B$ is the indicator of $B$, $\mu$ is Lebesgue measure.) For certain diffusion processes, it is known that $L^x_t = \lim_{\epsilon \downarrow 0} \epsilon d^x_\epsilon(t)$, where $d^x_\epsilon(t)$ is the number of times the real valued process crosses down from $\epsilon > 0$ to 0 before time $t$ (this result, conjectured by Lévy [8], was proved by Ito and McKean [7]). Finally, Blumenthal and Getoor ([1], see also [5]) showed that under fairly general circumstances, if $X$ has a reference measure $\xi$, then an appropriate choice of the local time $L^\xi_t$ could serve as a density for occupation time, in the sense that for every Borel set $B$, $\int_B I_B[X(s)]ds = \int_B L^\xi_t \xi(dx)$ a.s. In this paper, a new description of $L^\xi_0$, valid for a wide class of real valued Markov processes, is found which describes $L^\xi_0$ more or less in terms of ‘the number of times’ the process ‘jumps across zero.’

\textsuperscript{1} Partially supported by the Air Force Office of Scientific Research under AFOSR Grant AF-AFOSR 67-1261 B.

\textsuperscript{2} Supported partly by NSF Grant GP-24490 and partly by a NSF postdoctoral fellowship.
While the basic theorem of this paper can be formulated for quite general Markov processes (see section 2), the character of this result is most easily illustrated by a few special cases. Let $X = \{X(t), t \geq 0\}$ be a real valued process with stationary independent increments and Lévy measure $\nu$. Let $J_n(t)$ be the number of jumps $j(X, s) = X(s) - X(s^-)$ before time $t$ for which $X(s^-) < 0 < X(s)$ and $2^{-n-1} < j(X, s) < 2^{-n}$. So $J_n(t)$ is the 'number of (upward) jumps across 0 having size in $(2^{-n-1}, 2^{-n})$'. Assume that $\nu(R) = \infty$, that 0 is regular for $\{0\}$, and that $L_t^x$ is jointly continuous in $(x, t)$. Then (theorem 3.2) there is a version $l_0^T$ of the local time at 0 (see section 2 for the precise description) such that for each $T$,

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |J_n(t)|/F_n = l_0^T = 0 \quad \text{a.s.}$$

provided $\Sigma[1/F_n] < \infty$, where $F_n = \int_{2^{-n-1}}^{2^{-n}} \nu(dx)$. This result is very close in spirit to the result of Ito and McKean mentioned above. An illustration is the case where $X$ is a stable process with index $\alpha$, $1 < \alpha < 2$, in which case $J_n(t)/2^{n(\alpha-1)}$ converges a.s. to $cl_0^T$ uniformly, where $c$ is a known constant. The basic result of section 2 also yields theorems of the following type. Let $X$ be again a stable process with index $\alpha$, $1 < \alpha < 2$, and set $Q_\varepsilon(t) = \sum_{s \leq t} |X(s) - X(s^-)|$, where the prime indicates that the sum is over all $s \leq t$ for which $X(s^-) < 0 < X(s)$ and $|X(s) - X(s^-)| < \varepsilon$. Thus $Q_\varepsilon(t)$ is the sum up to time $t$ of all the upward jumps across 0 which have magnitude at most $\varepsilon$. Then for stable $X$ with $1 < \alpha < 2$ (see theorem 3.1),

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |Q_\varepsilon(t)|/2^{2-\alpha} - cl_0^T = 0$$

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |Q_{2^{-n}}(t)|/2^{n(\alpha-2)} - cl_0^T = 0 \quad \text{a.s.}$$

for all $\delta > 0$, and $T > 0$.

Section 2 contains the statement and proof of the basic result, while section 3 presents a number of applications to processes with independent increments. The terminology referring to the theory of Markov processes will be that of [2].

### 2. Main result

Let $X = \{X(t), t \geq 0\}$ be a standard, real valued Markov process. Let $T_x = \inf\{t > 0 : X(t) = x\}$, and assume from now on that each $x$ is regular for itself. Define for $\alpha > 0$

$$\psi^\alpha(x, y) = E^x[\exp \{-\alpha T_y\}]$$

(2.1)

According to the theory of Blumenthal and Getoor [2], there is for each
x a continuous additive functional \( \{ L_t^x, t \geq 0 \} \), the local time at \( x \), satisfying

\[
E^x \int_0^\infty e^{-t} dL_t^x = \psi^1(x, y),
\]

so that in particular \( E^x \int_0^\infty e^{-t} dL_t^x = 1 \). Assume from now on that \( \psi^1(x, y) \) is jointly Borel measurable, and that there is a reference measure \( \xi \) for the process \( X \). If \( U^x \) is the usual operator \( U^x f(x) = E^x \int_0^\infty e^{-at} f[X(t)] dt \), then under the preceding assumptions Getoor and Kesten [5] have proved the following result which will be stated as a lemma for the convenience of the reader.

**Lemma 2.1** There is a strictly positive finite Borel function \( g \) on \( \mathbb{R} \) such that \( l_t^x(\omega) \) defined by \( l_t^x(\omega) = g(x)L_t^x(\omega) \) satisfies a.s.

\[
\int_0^t I_B[X(s)] ds = \int_B l_t^x(\omega) \xi(dx)
\]

for all \( t \geq 0 \) and Borel sets \( B \) simultaneously. Define for each \( \alpha > 0 \), \( u^\alpha(x, y) = g(y) E^x \int_0^\infty e^{-at} dL_t^x \). Then

\[
U^x f(x) = \int u^\alpha(x, y) f(y) \xi(dy) \quad \text{for all Borel } f \geq 0
\]

\[
u^1(y, y) = g(y)
\]

\[
u^\alpha(x, y) = E^x \int_0^\infty e^{-at} dL_t^x
\]

\[
\psi^1(x, y) = u^1(x, y)/u^1(y, y).
\]

Since a reference measure is assumed throughout this section, there exists, according to the theory of S. Watanabe [13], a Lévy system \( (N, A) \) for the process \( X \). Here \( N(x, dy) \) is a non-negative kernel such that for each \( x \in \mathbb{R}, N(x, \cdot) \) is a measure on the Borel sets of \( \mathbb{R} \) and for each Borel set \( B, N(\cdot, B) \) is a Borel function. \( A = \{ A(t), t \geq 0 \} \) is a finite, continuous additive functional having the following property: for every non-negative Borel function \( f \) on \( \mathbb{R} \times \mathbb{R} \) vanishing on the diagonal

\[
E^x \sum_{s \leq t} f[X(s-), X(s)] = E^x \int_0^t Nf[X(s)] dA(s)
\]

where \( Nf(x) = \int N(x, dy) f(x, y) \). For simplicity, assume from now on that \( A(t) = t \). This may always be achieved by a time change if necessary (see, for example [11]); in the case of processes with independent increments one may always take \( A(t) = t \). as can be verified directly.

For the remainder of this section assume
The following theorem is the main result of this section.

**Theorem 2.1.** Let \( X = \{X(t), t \geq 0\} \) be a standard, real valued Markov process. Assume that there is a reference measure \( \xi \), that each point of the state space is regular for itself, that \( \psi(x, y) \) is jointly Borel measurable, and that the additive functional \( A \) of the Lévy system \((N, A)\) satisfies \( A(t) = t \). For all sufficiently small \( \varepsilon \) or only for \( \varepsilon \downarrow 0 \) through a sequence, let \( f_\varepsilon(x, y) \) be a non-negative Borel function vanishing on the diagonal such that there exists a compact interval \( I(\varepsilon) = [a(\varepsilon), b(\varepsilon)] \) containing \( 0 \) and such that \( Nf_\varepsilon(x) = 0 \) outside \( I(\varepsilon) \). Assume \( a(\varepsilon) \to 0 \) and \( b(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), and that

\[
\gamma(x) = \left[1 - \psi(x, 0)\psi(x, x)^{-1}\right]^{+} \to 0 \quad \text{as} \quad x \to 0
\]

(2.9)

(2.10) \( u^1(x, x) \to u^1(0, 0) \) as \( x \to 0 \).

The following conclusions hold.

1. If \( \lim_{\varepsilon \to 0} G(\varepsilon)/[F(\varepsilon)]^2 = 0 \), then

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} |Q_\varepsilon(t)/F(\varepsilon) - l_0\varepsilon| > \delta = 0
\]

(2.11)

for each \( T \) and each \( \delta > 0 \).

2. If \( \{l_\varepsilon^x\} \) is jointly continuous in \((t, x)\) near \( 0 \) and if \( \{\varepsilon_n, n \geq 1\} \) is a sequence decreasing to \( 0 \) such that \( \sum_{n \geq 1} G(\varepsilon_n)/[F(\varepsilon)]^2 < \infty \), then

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |Q_{\varepsilon_n}(t)/F(\varepsilon_n) - l_\varepsilon^x| = 0 \quad \text{a.s.}
\]

(2.12)

for each \( T \).

**Remark.** If \( u^1(x, x) \) is continuous as a function of \( x \), then it follows from lemma 2.1 that \( l_\varepsilon^x \) will be jointly continuous in \((t, x)\) if and only if \( L_\varepsilon^x \) is jointly continuous. Conditions guaranteeing joint continuity of \( L_\varepsilon^x \) have been given recently by Getoor and Kesten [5].

Before proceeding with the proof, note that, under the assumptions of theorem 2.1, Lemma 2.1 permits the following conclusion.

**Lemma 2.2.** Let \( h(x) \) be a non-negative, finite Borel function. Then \( A(t) = \int_0^t h[X(s)]ds \) and \( B(t) = \int_{-\varepsilon}^t h(x)\xi(dx) \) are equivalent stochastic processes.
PROOF. It suffices to assume $h$ bounded. Let $u_A$ and $u_B$ be the potentials of $A$ and $B$ respectively. Then from (2.4),

$$u_A(x) \overset{\text{def}}{=} E_x \int_0^\infty e^{-t} dA(t)$$

$$= E_x \int_0^\infty e^{-t} h[X(s)] ds$$

$$= U^1 h(x) = \int u^1(x, y) h(y) \xi(dy),$$

and from (2.6)

$$u_B(x) = E_x \int_0^\infty e^{-t} dB(t)$$

$$= \int u^1(x, y) h(y) \xi(dy).$$

Since $A$ and $B$ are continuous additive functionals having the same bounded 1-potentials, the conclusion follows (see [2], p. 157).

The proof of theorem 2.1 has the following structure. In the terminology of Meyer [10], the process $Q_\epsilon(t)$ is increasing but not natural. By Meyer’s decomposition theorem for supermartingales, there is a unique natural increasing process $V_\epsilon(t)$ such that $Q_\epsilon(t) - V_\epsilon(t)$ is a martingale. The limit results on $Q_\epsilon(t)$ are then deduced by analysing $V_\epsilon(t)$ and the martingale separately.

**PROOF OF THEOREM 2.1.** First, observe that from 2.2

$$1 \geq E^y \int_0^T e^{-t} dL^x_t \geq e^{-T} E^y L^x_T,$$

so that

(2.13) \[ E^y L^x_T \leq e^T \] for every $T$,

and from lemma 2.1

(2.14) \[ E^y l^x_T \leq u^1(x, x) E^y L^x_T \leq u^1(x, x) e^T. \]

Let $V_\epsilon(t) = \int_0^t N f_\epsilon(x) \xi(dx)$. From lemma 2.2, $V_\epsilon(t) = \int l^x \times N f_\epsilon(x) \xi(dx)$, and from (2.14), $E^y V_\epsilon(t) = \int E^y(l^x) N f_\epsilon(x) \xi(dx) \leq e^t \int u^1(x, x) N f_\epsilon(x) \xi(dx)$. By assumption (2.10), $u^1(x, x) \to u^1(0, 0)$ as $x \to 0$, so $u^1(x, x)$ is bounded for $x$ near 0. Thus for $\epsilon$ small, and all $y$,

$$E^y V_\epsilon(t) \leq M e^t \int N f_\epsilon(x) \xi(dx) < \infty$$

for some constant $M$. Observe that $Q_\epsilon(t)$ is an additive functional and $V_\epsilon$ is a continuous additive functional. It then follows from (2.8) that
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\[ M_\varepsilon(t) = Q_\varepsilon(t) - V_\varepsilon(t) \]

is an additive functional with mean zero, and so must be a martingale relative to each \( P_\varepsilon \).

Define \( \mu^\varepsilon(dx) = Nf_\varepsilon(x)\xi(dx)/F(\varepsilon) \). By the assumptions of theorem 2.1, \( \mu^\varepsilon \) is a probability measure carried by \( I(\varepsilon) \), and \( \mu^\varepsilon \) converges weakly to unit mass at 0. Moreover,

\[ V_\varepsilon(t)/F(\varepsilon) = \int l_t^\varepsilon \mu^\varepsilon(dx), \]

and

\[ \sup_{0 \leq t \leq T} |V_\varepsilon(t)/F(\varepsilon) - l_0^\varepsilon| \leq \int \sup_{0 \leq t \leq T} |l_t^\varepsilon - l_0^\varepsilon| \mu^\varepsilon(dx). \]

If, as in case (2), \( l_t^\varepsilon \) is jointly continuous in \((t, x)\), then for almost all \( \omega \),

\[ \sup_{0 \leq t \leq T} |l_t^\varepsilon - l_0^\varepsilon| \to 0 \text{ as } x \to 0 \text{ by uniform continuity, implying that} \]

\[ \sup_{0 \leq t \leq T} |V_\varepsilon(t)/F(\varepsilon) - l_0^\varepsilon| \to 0 \text{ a.s. in this case. To treat case 1, recall}
\]

that according to a result of Meyer ([9], see also [2], V.3)

\[ P^0 \{ \sup_{0 \leq t \leq T} |L_t^\varepsilon - L_0^\varepsilon| > 2\delta \} \leq 2e^T e^{-\delta \gamma(x)}, \]

where \( \gamma(x) \) was defined in (2.9). By virtue of the formula

\[ E|Y| = \int_0^\infty P\{|Y| > s\} ds, \quad E^0 \sup_{0 \leq t \leq T} |L_t^\varepsilon - L_0^\varepsilon| \leq 4e^T \gamma(x). \]

Since (see lemma 2.1)

\[ l_t^\varepsilon - l_0^\varepsilon = u^\varepsilon(x, x)L_t^\varepsilon - u^\varepsilon(0, 0)L_0^\varepsilon \]

\[ = u^\varepsilon(x, x)[L_t^\varepsilon - L_0^\varepsilon] - L_0^\varepsilon[u^\varepsilon(x, x) - u^\varepsilon(0, 0)], \]

it follows from (2.13) and the calculation above that

\[ E^0 \sup_{0 \leq t \leq T} |l_t^\varepsilon - l_0^\varepsilon| \leq u^\varepsilon(x, x)4e^T \gamma(x) + e^T[u^\varepsilon(x, x) - u^\varepsilon(0, 0)]. \]

Hence

\[ E^0 \{ \sup_{0 \leq t \leq T} |V_\varepsilon(t)/F(\varepsilon) - l_0^\varepsilon| \} \leq E^0 \sup_{0 \leq t \leq T} |l_t^\varepsilon - l_0^\varepsilon| \mu^\varepsilon(dx) \]

\[ \leq 4e^T \int u^\varepsilon(x, x)\gamma(x)\mu^\varepsilon(dx) + e^T \int [u^\varepsilon(x, x) - u^\varepsilon(0, 0)] \mu_\varepsilon(dx). \]

By assumptions (2.9), (2.10) the integrands above are continuous at 0 and are equal to 0 there. Since \( \mu_\varepsilon(dx) \) converges weakly to unit mass at 0, this completes the treatment of \( V_\varepsilon(t) \) in both case 1 and case 2.

Turning next to the martingale \( M_\varepsilon(t) \), observe that a fundamental fact on Lévy systems (S. Watanabe [13], p. 63, eq. 3.11) implies that
Since $M_\varepsilon$ is a martingale, this together with the well-known martingale inequality of Doob ([4], p. 317) yields

$$E^0 \sup_{0 \leq t \leq T} \left[ M_\varepsilon(t) \right]^2 \leq 4E^0 \left[ M_\varepsilon(T) \right]^2$$

$$= 4E^0 \int_0^T Nf_\varepsilon^2(X(s))ds$$

$$= 4E^0 \int l_\varepsilon^T Nf_\varepsilon^2(x)\xi(dx)$$ (by lemma 2.2)

$$\leq Me^T \int Nf_\varepsilon^2(x)\xi(dx)$$  (if $\varepsilon$ is sufficiently small)

$$= Me^T G(\varepsilon).$$

Since by hypothesis $\lim_{\varepsilon \to 0} G(\varepsilon)/[F(\varepsilon)]^2 = 0$ in case 1, it follows that $\sup_{0 \leq t \leq T} M_\varepsilon(t)/[F(\varepsilon)]^2 \to 0$ in probability. In case 2, since

$$E^0 \sup_{0 \leq t \leq T} \left[ M_\varepsilon(t)/F(\varepsilon) \right]^2 < \text{const. } G(\varepsilon)/[F(\varepsilon)]^2$$

and $\Sigma G(\varepsilon_n)/[F(\varepsilon_n)]^2 < \infty$, a Borel Cantelli argument shows

$$\sup_{0 \leq t \leq T} M_\varepsilon(t)/F(\varepsilon_n) \to 0 \quad \text{a.s.}$$

This completes the proof of theorem 2.1.

3. Processes with stationary independent increments

Theorem 2.1 yields a number of interesting results when $X = \{X(t), t \geq 0\}$ is a process with independent increments. This section contains several of these applications.

Throughout this section, let $X = \{X(t), t \geq 0\}$ be a real valued process with stationary independent increments having right continuous paths with left limits. Of course, $E^0 e^{iux(t)} = \exp\{ -t\phi(u) \}$ where

$$\phi(u) = iau + (\frac{1}{2})Su^2 + \int_R \left\{ 1 - e^{iux} + iux/[1 + x^2] \right\}v(dx).$$

The measure $\nu$ is called the Lévy measure, and $\phi$ is called the exponent of $X$. Assume throughout that 0 is regular for itself; in the present circumstances this implies that each $x$ is regular for $\{x\}$. Precise conditions under which 0 is regular for $\{0\}$ may be found in [3]. Assume also from now on that $\nu(R) = \infty$. (If $\nu(R) < \infty$, then it is obvious that it is impossible to represent local time (when it exists) as a limit of quantities depending only on the jumps about 0).
Under the last two assumptions, it is known ([3], [12]) that for each \( a > 0 \) there exists a real, bounded, continuous function \( u^a(x) \) such that 
\[
U^a f(x) = \int f(y) u^a(y-x) dy
\]
for all bounded Borel \( f \), and satisfying 
\[
u^a(x) = u^a(0) \psi^a(0, x)
\]. It follows from this that Lebesgue measure is a reference measure and, in the notation of section 2, \( u^a(x, y) = u^a(y-x) \); see [5] for more detail on this point. Since there is a reference measure, a Lévy system \((N, A)\) exists which, in fact is given by \( N(x, dy) = \nu(dy-x) \), \( A(t) = t \). Actually, for the case of processes with independent increments one may show directly that this is a Lévy system for \( X \) even when there is no reference measure. It is clear that \( \psi^1(x, y) = \psi^1(0, y-x) \) is jointly Borel measurable in \((x, y)\) in the present case \((\psi^1(0, z)\) is continuous in \( z \)). Since \( \psi^1(x, y) = u^1(y-x)/u^1(0) \), it follows again from the continuity of \( u^1(\cdot) \) that \( \lim_{x \to 0} \psi^1(x, 0) = \lim_{x \to 0} \psi^1(0, x) = u^1(0)/u^1(0) = 1 \), so (2.9) holds. Finally, since \( u^1(x, x) = u^1(0) \), (2.10) holds trivially and so all assumptions of section 2 are satisfied by a real valued process with stationary independent increments having \( 0 \) regular for \( \{0\} \) and \( \nu(R) = \infty \).

For theorem 3.1, let \( F(\varepsilon) = F(g, \varepsilon) = \int_0^\varepsilon \int g(y) \nu(dy) d\varepsilon \), where \( g \) is a non-negative Borel function on \((0, \infty)\), bounded on finite intervals. If \( 0 < \delta < \varepsilon \), an integration by parts yields
\[
\int_\delta^\varepsilon x g(x) \nu(dx) = \delta \int_\delta^\varepsilon g(x) \nu(dx) + \int_\delta^\varepsilon \int_\delta^x g(y) \nu(dy) dx = \delta K(\delta) + \int_\delta^\varepsilon K(x) dx
\]
where \( K(x) = \int_\delta^x g(y) \nu(dy) \) is a function that increases as \( x \) decreases. If \( \delta \downarrow 0 \) and \( \int_0^\delta x g(x) \nu(dx) < \infty \), then it follows, since all terms above are positive, that \( \int_0^\delta K(x) dx < \infty \) and \( \lim_{\delta \to 0} \delta K(\delta) \) exists. Since
\[
\infty > \int_0^\delta K(x) dx \geq K(\delta) \delta,
\]
it follows that \( \lim_{\delta \to 0} \delta K(\delta) = 0 \). Hence, if \( \int_0^\delta x g(x) \nu(dx) < \infty \), then
\[
\infty > \int_0^\delta \int_\delta^x g(y) \nu(dy) dx = \int_0^\delta x g(x) \nu(dx).\]
Conversely it is not hard to see that if \( F(\varepsilon) < \infty \), then \( \infty > \int_0^\delta x g(x) \nu(dx) = F(\varepsilon) \).

**Theorem 3.1** Assume \( 0 < F(\varepsilon) = \int_0^\varepsilon x g(x) \nu(dx) < \infty \), and define \( G(\varepsilon) = \int_0^\varepsilon x[g(x)]^2 \nu(dx) \). Let \( Q_\varepsilon(t) = \sum_{s \leq t} g(|j(X, s)|) \) where the prime means that the sum is over only those jumps \( j(X, s) = X(s)-X(s-) \) for which \( X(s-) = 0 \leq X(s) \) and \( |j(X, s)| < \varepsilon \). Then:

(a) If \( \lim_{\varepsilon \to 0} G(\varepsilon)/[F(\varepsilon)]^2 = 0 \), then
\[
\lim_{\varepsilon \to 0} P\{ \sup_{0 \leq s \leq T} |Q_\varepsilon(t)/F(\varepsilon)| > \delta \} = 0
\]
for every \( T > 0 \) and \( \delta > 0 \).

(b) If \( l^T \) is jointly continuous in \((t, x)\) and if \( \{\varepsilon_n, n \geq 1\} \) is a positive sequence converging to zero such that \( \Sigma G(\varepsilon_n)/[F(\varepsilon_n)]^2 < \infty \), then
Conditions guaranteeing joint continuity of $l^x_t$ can be found in [5]. A large number of functions $g$ satisfy the hypothesis that $F(\epsilon) < \infty$; in particular $g(x) = |x|$ always works, since $\int_{|x| < 1} |x|^2 \nu(dx) < \infty$ for any Lévy measure $\nu$. As a special case, suppose $X$ is a stable process with index $\alpha$, $1 < \alpha < 2$. Then the exponent is of the form

$$\phi(u) = c_1 \int_{-\infty}^{\infty} [e^{iux} - 1 + iux/(1 + x^2)]x^{-\alpha - 1} dx$$

$$+ c_2 \int_{-\infty}^{0} [e^{iux} - 1 + iux/(1 + x^2)]|x|^{-\alpha - 1} dx.$$

where $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 > 0$. Suppose $c_1 > 0$ for convenience. If $g(x) = |x|$, then $F(\epsilon) = c^2 \epsilon^{2-\alpha}$, where $c = c_1(2-\alpha)^{-1}$, and $G(\epsilon)/[F(\epsilon)]^2 = \text{const.} \, \epsilon^{\alpha - 1}$.

Then, as mentioned in the introduction, $\lim_{\epsilon \to 0} Q_2(t)/\epsilon^{2-\alpha} = c l^0_t$ in probability, uniformly on compact intervals and

$$\lim_{n \to \infty} Q_{2-n}(t)/2^{-n(2-\alpha)} = c l^0_t \quad \text{a.s.,}$$

uniformly on compact intervals. (That $l^x_t$ is jointly continuous in the stable case is well-known - see [2] and the references there.) As another example, the asymmetric Cauchy processes are interesting to consider. Here the exponent $\phi$ is of the form (3.1) with $\alpha = 1$ and $c_1 \neq c_2$. Assume $c_1 > 0$ (if not, then one can establish the result below for $-X$ instead.) It was proved by Kesten and Getoor that no jointly continuous version of the local time exists for the asymmetric Cauchy processes ([5], example b, section 4). Choose $g$ of theorem 3.1 to be

$$g(u) = [(-\log |u|) \vee 0]^a.$$ 

Then for sufficiently small $\epsilon$,

$$F(\epsilon) = c_1 \int_{0}^{\epsilon} (-\log x)^a x^{-1} dx = \left(-c_1/(a+1)\right)(-\log \epsilon)^{a+1} < \infty$$

if $a < -1$

and

$$G(\epsilon) = c_1 \int_{0}^{\epsilon} (-\log x)^{2a} x^{-1} dx = -\left[c_1/(2a+1)\right](-\log \epsilon)^{2a+1}.$$ 

Thus $G(\epsilon)/[F(\epsilon)]^2 = [(a+1)^2/(2a+1)](1/\log \epsilon) \to 0$ as $\epsilon \to 0$, so a limit theorem continues to hold even in this singular case.
PROOF OF THEOREM 3.1. Let $f_\varepsilon(x, y) = g(|x - y|)I\{x < 0 < y; 0 < y - x < \varepsilon\}$, where $I\{A\}$ is the indicator of the set $A$. Then

$$Q_\varepsilon(t) = \sum_{s \leq t} f_\varepsilon[X(s-), X(s)].$$

In the notation of theorem 2.1,

$$Nf_\varepsilon(x) = \int_{-x}^x g(u)\nu(du)$$

if $x \in [-\varepsilon, 0]$ and $Nf_\varepsilon(x) = 0$, $x \notin [-\varepsilon, 0]$.

Also,

$$\int Nf_\varepsilon(x)dx = \int_0^0 dx \int_{-x}^x g(u)\nu(du) = \int_0^x dx \int_{x}^x g(u)\nu(du) = \int_0^x xg(x)\nu(dx) = F(\varepsilon).$$

Similarly,

$$\int Nf_\varepsilon^2(x)dx = \int_0^x x[g(x)]^2\nu(dx) = G(\varepsilon).$$

The result now follows from theorem 2.1.

Next, consider the following choice of $f_\varepsilon : f_\varepsilon(x, y) = I\{x < 0 < y; \lambda(\varepsilon)e < y - x < \varepsilon\}$, where $0 < \lambda(\varepsilon) < 1$. Then

$$J_\varepsilon(t) = \sum_{s \leq t} f_\varepsilon[X(s-), X(s)]$$

is equal to the number of jumps $j(X, s) = X(s) - X(s-)$ across 0 up to time $t$ for which $\varepsilon\lambda(\varepsilon) < j(X, s) < \varepsilon$. Here

$$Nf_\varepsilon(x) = \int v(dy) f_\varepsilon(x, y + x)$$

$$= \begin{cases} v[(-x, \varepsilon)], & -\varepsilon < x < -\varepsilon\lambda(\varepsilon) \\ v[(\varepsilon\lambda(\varepsilon), \varepsilon)], & -\varepsilon\lambda(\varepsilon) < x < 0 \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\int Nf_\varepsilon(x)dx = \int_0^0 Nf_\varepsilon(x)dx$$

$$= \int_{-\varepsilon}^\varepsilon v[(x, \varepsilon)]dx + \varepsilon\lambda(\varepsilon)v[(\varepsilon\lambda(\varepsilon), \varepsilon)]$$

$$= \int_{-\varepsilon}^\varepsilon xv(dx) = F(\varepsilon),$$

since $f_\varepsilon = f_\varepsilon^2$, $\int Nf_\varepsilon^2 = \int Nf_\varepsilon = F(\varepsilon)$ in this case. An application of theorem 2.1 to the preceding calculations then yields the following result.

THEOREM 3.2. Let $\lambda$ be a function such that $0 < \lambda(\varepsilon) < 1$ for all $\varepsilon$ and define $F(\varepsilon) = F(\lambda, \varepsilon) = \int_{\varepsilon\lambda(\varepsilon)} \nu(dx)$. Let $J_\varepsilon(t)$ be the number of jumps up to time $t$ for which $X(s-) < 0 < X(s)$ and $\varepsilon\lambda(\varepsilon) < X(s) - X(s-) < \varepsilon$. 

(b) If \( \lim_{\beta \to 0} F(\beta) = \infty \), then
\[
\lim_{\beta \to 0} P\left\{ \sup_{0 \leq t \leq T} |J_\beta(t)/F(\beta) - l_\beta^0| > \delta \right\} = 0.
\]

(b) If \( l_t^x \) is jointly continuous in \((x, t)\) and if \( \varepsilon_n \) is a positive sequence converging to zero such that \( \sum_{n \geq 1} |1/F(\varepsilon_n)| < \infty \) then
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |J_{\varepsilon_n}(t)/F(\varepsilon_n) - l_t^0| = 0 \quad a.s.
\]

Let \( X \) be a stable process with index \( \alpha, 1 < \alpha < 2, \) and exponent (3.1) with \( c_1 > 0, \) and let \( J_n(t) \) be the number of upward jumps across 0 up to time \( t \) having size \((2^{-n-1}, 2^{-n})\). According to theorem 3.2b,
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |J_n(t)/2^{(\alpha-1)} - c_1 l_t^0| = 0 \quad a.s.
\]
where \( c = c_1 (2^{\alpha-1} - 1)/(\alpha - 1), \) as mentioned in the introduction. (Take \( \varepsilon_n = 2^{-n} \) and \( \lambda(2^{-n}) = (\frac{1}{2}) \) for all \( n. \) If \( X \) is an asymmetric Cauchy process, \( F(\varepsilon) = c_1 \int_{y\varepsilon}^{\infty} x^{-1} dx = -c_1 \log \lambda(\varepsilon). \) If \( \lambda(\varepsilon) \to 0 \) as \( \varepsilon \to 0, \) then \( \lim_{\varepsilon \to 0} F(\varepsilon) = +\infty. \) Hence from theorem 3.2a,
\[
\sup_{0 \leq t \leq T} |J_\varepsilon(t)/[-\log \lambda(\varepsilon)] - c_1 l_t^0| \to 0
\]
in probability. Again a limit theorem continues to hold in the singular Cauchy case.

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