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ON A DISTRIBUTION PROBLEM IN FINITE SETS

by

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1.

In [2] the following problem emerged which deserves some interest of its own. Let $X = \{x_1, \cdots, x_k\}$ be a nonvoid finite set and let μ be a measure on X with $\mu(x_i) = \lambda_i > 0$ for $1 \leqq i \leqq k$ and $\sum_{i=1}^{k} \lambda_i = 1$.

Without loss of generality we may suppose that the x_i are arranged in such a way that $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k$. For an infinite sequence ω in X, let $A(i; N; \omega)$ denote the number of occurrences of the element x_i among the first N terms of ω and let $D(\omega) = \sup_{i,N}|A(i; N; \omega) - \lambda_i N|$ (the supremum is taken over $i = 1, 2, \cdots, k$; $N = 1, 2, \cdots$). We pose the problem: *how small can* $D(\omega)$ *be*?

Similarly, define $A(M; N; \omega)$ for a subset M of X to be the number of occurrences of elements from M among the first N terms of ω and put $C(\omega) = \sup_{M,N}|A(M; N; \omega) - \mu(M)N|$ (the supremum is taken over all subsets $M \subset X$ and $N = 1, 2, \cdots$). Then we may ask: *how small can* $C(\omega)$ *be*?

These problems are similar to the well-known problem of constructing a sequence with small discrepancy in the unit interval [0,1] (see e.g. v.d. Corput [1]).

It was shown in [2] that a 'very well' distributed sequence ω in X can be found with

$$D(\omega) \leqq k-1, \qquad C(\omega) \leqq (k-1)\left[\frac{k}{2}\right].$$

Those values, however, are far from being optimal. In section 2 of this paper we shall construct a sequence ω in X with

$$(1) \qquad D(\omega) \leqq \tfrac{1}{2} + \tfrac{1}{2}\sum_{n=1}^{k-2}\frac{1}{n} \quad \text{and} \quad C(\omega) \leqq \tfrac{1}{2}(k-1) \quad \text{for } k \geqq 2.$$

If $k = 1$ then, trivially, $C(\omega) = D(\omega) = 0$.

For some special measures μ on X better results can be obtained. If e.g. $\lambda_1 = \cdots = \lambda_k = 1/k$ then one easily verifies that the sequence $\omega = (y_n)_{n=1}^{\infty}$ defined by $y_n = x_i$ if $n \equiv i$ (mod. k) satisfies $D(\omega) = 1 - 1/k$.

In section 3 we construct a sequence η in X which gives a better result than (1) if λ_k is sufficiently small and $k \geqq 3$. In fact we prove

$$D(\eta) \leqq \begin{cases} \frac{1}{2}+\frac{1}{2}\lambda_k\,(k-2) \text{ if } k \text{ is even} \\ \frac{1}{2}+\frac{1}{2}\lambda_k\,(k-1) \text{ if } k \text{ is odd} \end{cases}$$

$$C(\eta)\begin{cases} = D(\eta) & \text{for } k = 2, 3 \\ \leqq \max\,(D(\eta), \frac{5}{4}) & \text{for } k = 4 \\ \leqq \max\,(D(\eta), \frac{25}{16}) & \text{for } k = 5 \\ \leqq \max\,(D(\eta), \frac{1}{2}(k-2)) & \text{for } k \geqq 6. \end{cases}$$

We remark that always $\lambda_k \geqq 1/k$.

Added in proof: Recently Tijdeman [3] found by an entirely different method: if $D_k = \sup_\mu \inf_\omega D(\omega)$, then it holds

$$1-\frac{1}{2(k-1)} \leqq D_k \leqq 1.$$

Moreover he generalized the results to countable sets.

A refinement of this method gives

$$D_k = 1-\frac{1}{2(k-1)}$$

(see [4]).

2.

By using some refinements of the method employed in [2], we can prove the following result.

THEOREM 1. *For any nonvoid finite set* $X = \{x_1, \cdots, x_k\}$ *and every measure* μ on X with $\mu(x_i) = \lambda_i > 0$ $(i = 1, \cdots, k)$, $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k$ *and* $\sum_{i=1}^{k} \lambda_i = 1$, *there is a sequence* ω *in* X *such that*

$$(2)\qquad \begin{aligned} |A(i; N; \omega)-\lambda_i N| &\leqq \tfrac{1}{2}+\tfrac{1}{2}\sum_{n=1}^{k-i}\frac{1}{n} \quad \text{if} \quad 2 \leqq i \leqq k \\ |A(1; N; \omega)-\lambda_1 N| &\leqq \tfrac{1}{2}\sum_{n=1}^{k-1}\frac{1}{n}, \end{aligned}$$

therefore

$$D(\omega) = 0 \quad \text{if } k = 1,$$

$$D(\omega) \leqq \tfrac{1}{2}+\tfrac{1}{2}\sum_{n=1}^{k-2}\frac{1}{n} \quad \text{if} \quad k \geqq 2;$$

moreover

$$C(\omega) \leqq \tfrac{1}{2}(k-1).$$

PROOF. We proceed by induction on k. Obviously the case $k = 1$ is trivial. Assuming the proposition to be true for an integer $k \geqq 1$, we shall prove that it also holds for $k+1$.

We consider the set $X = \{x_1, \cdots, x_{k+1}\}$ and a measure μ on X with

$$\mu(x_i) = \lambda_i > 0, \qquad \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{k+1}, \sum_{i=1}^{k+1} \lambda_i = 1.$$

On the subset $Y = \{x_1, \cdots, x_k\}$ of X, introduce a measure v by

$$v(x_i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_k} = \alpha_i.$$

Since $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k$ it follows that

$$\alpha_i \leqq \frac{1}{k-i+1} \quad \text{for} \quad 1 \leqq i \leqq k. \tag{3}$$

By induction hypothesis, there exists a sequence $\tau = (y_n)_{n=1}^{\infty}$ in Y with

$$\begin{aligned} |A(i; N; \tau) - \alpha_i N| &\leqq \tfrac{1}{2} + \tfrac{1}{2} \sum_{n=1}^{k-i} \frac{1}{n} \quad \text{if } 2 \leqq i \leqq k, \\ |A(1; N; \tau) - \alpha_1 N| &\leqq \tfrac{1}{2} \sum_{n=1}^{k-1} \frac{1}{n} \end{aligned} \tag{4}$$

for all $N \geqq 1$, and with

$$C(\tau) \leqq \tfrac{1}{2}(k-1). \tag{5}$$

We introduce the following notation: for a real number a let $||a|| = [a+\frac{1}{2}]$, i.e. the integer nearest to a. For $n \geqq 1$, put $R(n) = n - ||\lambda_{k+1} n||$. We define a sequence $\omega = (z_n)_{n=1}^{\infty}$ in X by setting

$$\left.\begin{aligned} z_n &= x_{k+1} \quad \text{if} \quad ||\lambda_{k+1} n|| > ||\lambda_{k+1}(n-1)||, \\ z_n &= y_{R(n)} \quad \text{if} \quad ||\lambda_{k+1} n|| = ||\lambda_{k+1}(n-1)||. \end{aligned}\right. \quad (n = 1, 2, \cdots)$$

We get then

$$A(k+1; N; \omega) = ||\lambda_{k+1} N|| = \lambda_{k+1} N + \varepsilon$$

with $|\varepsilon| \leqq \frac{1}{2}$, and therefore

$$|A(k+1; N; \omega) - \lambda_{k+1} N| \leqq \tfrac{1}{2}. \tag{6}$$

For $1 \leqq i \leqq k$, we have $A(i; N; \omega) = A(i; R(N); \tau)$ for all $N \geqq 1$ (if $R(N) = 0$, we had to read $A(i; R(N); \tau) = 0$). Now we write

$$|A(i; N; \omega) - \lambda_i N| \leqq |A(i; R(N); \tau) - \alpha_i R(N)| + |\alpha_i R(N) - \lambda_i N|. \tag{7}$$

Using the definitions of $R(N)$ of α_i and (3), we obtain

$$|\alpha_i R(N) - \lambda_i N| = |\alpha_i (N - \lambda_{k+1} N - \varepsilon) - \lambda_i N| = |\alpha_i \varepsilon| \leqq \frac{1}{2(k-i+1)}. \tag{8}$$

Hence by (7), (4) and (8) we get

$$|A(i;\,N;\,\omega)-\lambda_i N| \leqq \tfrac{1}{2}+\tfrac{1}{2}\sum_{n=1}^{k-i+1}\frac{1}{n} \quad \text{if } 2 \leqq i \leqq k,$$

$$|A(1;\,N;\,\omega)-\lambda_1 N| \leqq \tfrac{1}{2}\sum_{n=1}^{k}\frac{1}{n}.$$

Moreover (6) implies that the first inequality also holds for $i = k+1$. Therefore the relations (2) have been proved for $k+1$.

Furthermore we have to show that ω satisfies $C(\omega) \leqq k/2$. If M is a subset of X and M^c denotes its complement in X, then

$$(9) \qquad |A(M^c;\,N;\,\omega)-\mu(M^c)N| = |A(M;\,N;\,\omega)-\mu(M)N|.$$

Consequently, it suffices to consider subsets M of Y. Using (5) and the same type of arguments as above, we arrive at

$$\begin{aligned}|A(M;\,N;\,\omega)-\mu(M)N| &\leqq |A(M;\,R(N);\,\tau)-\nu(M)R(N)| \\ &\quad +|\nu(M)R(N)-\mu(M)N| \leqq \tfrac{1}{2}(k-1)+|\nu(M)\varepsilon| \leqq \tfrac{1}{2}k.\end{aligned}$$

3.

In this section we exhibit another construction principle which gives better results than the sequence of section 2 if $\lambda_k = \max \lambda_i$ is small and $k \geqq 3$. Since the case $k = 1$ is trivial we restrict ourselves to $k \geqq 2$. For a real number a we denote as above $||a|| = [a+\frac{1}{2}]$; moreover we define

$$(10) \qquad \{\{a\}\} = a-||a||.$$

Hence

$$(11) \qquad -\tfrac{1}{2} \leqq \{\{a\}\} < \tfrac{1}{2}.$$

We consider the following scheme consisting of an infinite number of rows and k columns.

x_1 λ_1	x_2 λ_2	$\cdots$ $\cdots$	x_k λ_k	
$\|\|\lambda_1\|\|$	$\|\|\lambda_2\|\|$	$\cdots$	$\|\|\lambda_k\|\|$	1st row
$\|\|2\lambda_1\|\|$	$\|\|2\lambda_2\|\|$	$\cdots$	$\|\|2\lambda_k\|\|$	2nd row
$\|\|n\lambda_1\|\|$	$\|\|n\lambda_2\|\|$	$\cdots$	$\|\|n\lambda_k\|\|$	nth row

The i^{th} column consists of $||\lambda_i|| \leqq ||2\lambda_i|| \leqq \cdots \leqq ||n\lambda_i|| \leqq \cdots$, where $||(n+1)\lambda_i|| = ||n\lambda_i||$ or $||(n+1)\lambda_i|| = ||n\lambda_i||+1$. Now we change this column in the following way.

If $||(n+1)\lambda_i|| = ||n\lambda_i||$ $(n = 0, 1, 2, \cdots)$ we omit $||(n+1)\lambda_i||$ such that we get a void place in the scheme.

If on the other hand $||(n+1)\lambda_i|| = ||n\lambda_i||+1$ $(n = 0, 1, 2, \cdots)$ we replace $||(n+1)\lambda_i||$ by x_i. We remark that in the last case

$$\{\{n\lambda_i\}\} \geqq \tfrac{1}{2}-\lambda_i, \tag{12}$$

$$\{\{(n+1)\lambda_i\}\} < -\tfrac{1}{2}+\lambda_i. \tag{13}$$

The i^{th} column now consists of places with x_i and void places. Up till the n^{th} row there are exactly $||n\lambda_i||$ places with x_i. We do so for $i = 1, 2, \cdots, k$. The sequence $\eta = (\eta_n)_{n=1}^{\infty}$ is the sequence which we get if we read the consecutive rows from the left to the right. After we have passed through the n^{th} row we have had $||n\lambda_i||$ times the element x_i and altogether $T(n) = \sum_{i=1}^{k}||n\lambda_i||$ elements of η. For this sequence η we will prove the following result.

THEOREM 2. *For the sequence* η *we have*

$$|A(i; N; \eta)-\lambda_i N| \leqq \tfrac{1}{2}+\tfrac{1}{2}\lambda_i(k-d), \tag{14}$$

where $d = 1$ *if* k *is odd and* $d = 2$ *if* k *is even. Therefore*

$$D(\eta) \leqq \tfrac{1}{2}+\tfrac{1}{2}\lambda_k(k-d).$$

Moreover

$$C(\eta)\begin{cases} = D(\eta) & \text{for } k = 2, 3 \\ \leqq \max\,(D(\eta), \tfrac{5}{4}) & \text{for } k = 4 \\ \leqq \max\,(D(\eta), \tfrac{25}{16}) & \text{for } k = 5 \\ \leqq \max\,(D(\eta), (k-2)/2) & \text{for } k \geqq 6. \end{cases}$$

PROOF. Since there is no risk of ambiguity we omit the η in $A(i; N; \eta)$ and $A(M, N; \eta)$.

First we remark that by (10)

$$\sum_{h=1}^{k}\{\{n\lambda_h\}\} = n-\sum_{h=1}^{k}||n\lambda_h||,$$

which implies that $\Sigma\{\{n\lambda_h\}\}$ has to be an integer. If we exclude the case k even, $(\{\{n\lambda_1\}\}, \cdots, \{\{n\lambda_k\}\}) = (-\tfrac{1}{2}, \cdots, -\tfrac{1}{2})$ we may conclude from (11)

$$|\sum_{h=1}^{k}\{\{n\lambda_h\}\}| \leqq \tfrac{1}{2}(k-d), \tag{15}$$

where $d = 1$ if k is odd, $d = 2$ if k is even. Using again (10) we get

$$\begin{aligned} A(i; T(n))-\lambda_i T(n) &= ||n\lambda_i||-\lambda_i\sum_{h=1}^{k}||n\lambda_h|| \\ &= -\{\{n\lambda_i\}\}+\lambda_i\sum_{h=1}^{k}\{\{n\lambda_h\}\}. \end{aligned} \tag{16}$$

Let N be an integer with $T(n) \leqq N \leqq T(n+1)$. Then $A(i; N) = A(i; T(n))$ or $A(i; N) = A(i; T(n))+1$. In the first case we have by (16), (11) and (15)

$$A(i; N)-\lambda_i N \leqq A(i; T(n))-\lambda_i T(n) = -\{\{n\lambda_i\}\}+\lambda_i \sum_{h=1}^{k} \{\{n\lambda_h\}\}$$
$$\leqq \tfrac{1}{2}+\tfrac{1}{2}\lambda_i(k-d).$$

In the second case x_i is an element of the $(n+1)^{\text{th}}$ row. Then by (12)

$$\{\{n\lambda_i\}\} \geqq \tfrac{1}{2}-\lambda_i. \tag{17}$$

Moreover $N \geqq T(n)+1$. Therefore using (16), (17) and (15) we arrive at

$$\begin{aligned} A(i; N)-\lambda_i N &\leqq A(i; T(n))-\lambda_i T(n)+1-\lambda_i \\ &= -\{\{n\lambda_i\}\}+\lambda_i \sum_{h=1}^{k} \{\{n\lambda_h\}\}+1-\lambda_i \leqq -\tfrac{1}{2}+\lambda_i+\tfrac{1}{2}\lambda_i(k-d)+1-\lambda_i \\ &= \tfrac{1}{2}+\tfrac{1}{2}\lambda_i(k-d). \end{aligned}$$

This upper bound trivially holds as well with $d = 2$ in the exceptional case excluded above.

In order to get a lower bound we proceed in a similar way. We have $A(i; N) = A(i; T(n+1))$ or $A(i; N) = A(i; T(n+1))-1$.

For the calculations we first exclude the case k even,

$$(\{\{(n+1)\lambda_1\}\}, \cdots, \{\{(n+1)\lambda_k\}\}) = (-\tfrac{1}{2}, \cdots, -\tfrac{1}{2}).$$

Then we obtain in the first case

$$\begin{aligned} A(i; N)-\lambda_i N &\geqq A(i; T(n+1))-\lambda_i T(n+1) \\ &= -\{\{(n+1)\lambda_i\}\}+\lambda_i \sum_{h=1}^{k} \{\{(n+1)\lambda_h\}\} \geqq -\tfrac{1}{2}-\tfrac{1}{2}\lambda_i(k-d). \end{aligned}$$

In the second case we have $N \leqq T(n+1)-1$. Moreover x_i occurs in the $(n+1)^{\text{th}}$ row and (13) gives

$$\{\{(n+1)\lambda_i\}\} < -\tfrac{1}{2}+\lambda_i.$$

Therefore

$$\begin{aligned} A(i; N)-\lambda_i N &\geqq A(i; T(n+1))-\lambda_i T(n+1)-1+\lambda_i \\ &= -\{\{(n+1)\lambda_i\}\}+\lambda_i \sum_{h=1}^{k} \{\{(n+1)\lambda_h\}\}-1+\lambda_i \\ &\geqq \tfrac{1}{2}-\lambda_i-\tfrac{1}{2}\lambda_i(k-d)-1+\lambda_i = -\tfrac{1}{2}-\tfrac{1}{2}\lambda_i(k-d). \end{aligned}$$

One easily verifies that these lower bounds also hold with $d = 2$ for the case k even,

$$(\{\{(n+1)\lambda_1\}\}, \cdots, \{\{(n+1)\lambda_k\}\}) = (-\tfrac{1}{2}, \cdots, -\tfrac{1}{2}).$$

Hence (14) has been proved.

In order to get an estimate for $C(\eta)$ we consider a nonvoid subset M of X. Put

$$M = \{x_{i_1}, \cdots, x_{i_j}\}, \quad \mu M = \lambda_{i_1} + \cdots + \lambda_{i_j} = \Lambda,$$
$$X \setminus M = \{x_{i_{j+1}}, \cdots, x_{i_k}\}.$$

Then

$$\begin{aligned} A(M; T(n)) - \Lambda T(n) &= \sum_{\nu=1}^{j} ||n\lambda_{i_\nu}|| - \Lambda \sum_{h=1}^{k} ||n\lambda_h|| \\ &= -\sum_{\nu=1}^{j} \{\{n\lambda_{i_\nu}\}\} + \Lambda \sum_{h=1}^{k} \{\{n\lambda_h\}\} \\ &= -(1-\Lambda)\sum_{\nu=1}^{j} \{\{n\lambda_{i_\nu}\}\} + \Lambda \sum_{\nu=j+1}^{k} \{\{n\lambda_{i_\nu}\}\}. \end{aligned}$$

Let N be an integer with $T(n) \leqq N \leqq T(n+1)$ and suppose

$$A(M; N) = A(M; T(n)) + t \text{ with } 0 \leqq t \leqq j.$$

Then $N \geqq T(n)+t$ and

$$\begin{aligned} A(M; N) - \Lambda N &\leqq A(M; T(n)) - \Lambda T(N) + t - \Lambda t \\ &= -(1-\Lambda)\sum_{\nu=1}^{j} \{\{n\lambda_{i_\nu}\}\} + \Lambda \sum_{\nu=j+1}^{k} \{\{n\lambda_{i_\nu}\}\} + t - \Lambda t. \end{aligned}$$

Suppose that $x_{u_1}, \cdots x_{u_t}$ are the elements of the $(n+1)^{\text{th}}$ row which are counted in $A(M; N)$ and not in $A(M; T(n))$.

Then by (12)

$$\{\{n\lambda_{u_\tau}\}\} \geqq \tfrac{1}{2} - \lambda_{u_\tau}. \qquad (\tau = 1, \cdots, t)$$

Therefore

$$\sum_{\nu=1}^{j} \{\{n\lambda_{i_\nu}\}\} \geqq \tfrac{1}{2}t - (\lambda_{u_1} + \cdots + \lambda_{u_t}) - \tfrac{1}{2}(j-t) \geqq t - \tfrac{1}{2}j - \Lambda.$$

Hence

$$\begin{aligned} A(M; N) - \Lambda N &\leqq -(1-\Lambda)(t - \tfrac{1}{2}j - \Lambda) + \tfrac{1}{2}\Lambda(k-j) + t - \Lambda t \\ &= \frac{j}{2} + \Lambda\left(\frac{k}{2} - j + 1\right) - \Lambda^2. \end{aligned}$$

In a similar way we find a lower bound for $A(M; N) - \Lambda N$ which has the same absolute value.

Hence

$$|A(M; N) - \Lambda N| \leqq \frac{j}{2} + \Lambda\left(\frac{k}{2} - j + 1\right) - \Lambda^2.$$

Since for $k = 2$, trivially, $C(\eta) = D(\eta)$, we suppose $k \geqq 3$. We observe that we can restrict ourselves to $\frac{1}{2}k \leqq j \leqq k-1$ (compare (9)). If $j = k-1$, the complement of M is a singleton which was dealt with in $D(\eta)$. In particular this implies $C(\eta) = D(\eta)$ for $k = 3$. If $\frac{1}{2}k+1 \leqq j \leqq k-2$, then clearly

$$\frac{j}{2} + \Lambda\left(\frac{k}{2} - j + 1\right) - \Lambda^2 \leqq \tfrac{1}{2}(k-2).$$

If $\frac{1}{2}k \leqq j \leqq \frac{1}{2}k + \frac{1}{2}$, then

$$\frac{j}{2} + \Lambda\left(\frac{k}{2} - j + 1\right) - \Lambda^2 \leqq \tfrac{1}{2}\left(\frac{k}{2} + \tfrac{1}{2}\right) + \Lambda - \Lambda^2 \leqq \frac{k}{4} + \tfrac{1}{2}.$$

For $k \geqq 6$, we have $\frac{1}{4}k + \frac{1}{2} \leqq (k-2)/2$ and so $C(\eta) \leqq \max\,(D(\eta), (k-2)/2)$. For $k = 4, 5$ one finds by separate discussion of the permissible values for $j : C(\eta) \leqq \max\,(D(\eta), \frac{5}{4})$ for $k = 4$, $C(\eta) \leqq \max\,(D(\eta), \frac{25}{16})$ for $k = 5$. This completes the proof.

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