COMPOSITIO MATHEMATICA

RALPH K. AMAYO

# Soluble subideals of Lie algebras

*Compositio Mathematica*, tome 25, nº 3 (1972), p. 221-232
<http://www.numdam.org/item?id=CM_1972__25_3_221_0>

COMPOSITIO MATHEMATICA, Vol. 25, Fasc. 3, 1972, pag. 221–232
Wolters-Noordhoff Publishing
Printed in the Netherlands

# SOLUBLE SUBIDEALS OF LIE ALGEBRAS

by

Ralph K. Amayo

## Introduction

The main object of this paper is to prove that the join of finitely many soluble subideals of a Lie algebra is soluble, answering question 5 of Stewart [4] p. 79; it is known that this join need not be a subideal. The Lie algebras considered are of finite or infinite dimension over fields of arbitrary characteristic.

A similar theorem for groups was first proved by Stonehewer [6] and in a different way by Roseblade [3]. The treatment here resembles Roseblade's and is based on it. It is possible to use Stonehewer's techniques, as is proved in Amayo [1]; these give in some ways less information than those of Roseblade, but in compensation provide far better bounds on the derived length of the join. However the treatment here enables us to prove certain coalescence results which we reserve for another paper.

Notation and terminology for Lie algebras will be the same as in Stewart [5] p. 291–292. The symbols $A, B, H, J, K, L, X, Y, \cdots$ will denote Lie algebras over some ground field $\mathfrak{k}$. Symbols $\lambda_i(m, n, p, \cdots)$ for $1 \leqq i$ will denote non-negative integers depending solely on the arguments explicitly shown in the brackets. If $A$ and $B$ are subalgebras of a Lie algebra $L$ then the sum $A+B$ is their vector space sum, which may or may not be a subalgebra of $L$.

In section 1 we derive some preliminary results and introduce the useful *circle product* whose properties are crucial to the proofs of the two major results, theorems 2.1 and 3.2. The circle product was suggested in conversation with Dr. J. E. Roseblade.

Section 2 deals with a special case of the main theorem. However since joins of soluble ideals are not in general subideals themselves we cannot use a direct induction argument to derive the main result. Also in this section we derive a few useful properties about the join of a pair of subideals.

Finally in section 3 the main theorem is proved (theorem 3.3) and a useful corollary is also mentioned.

I am grateful to Dr. J. E. Roseblade for many helpful conversations; my research supervisor Dr. I. N. Stewart; and the Association of Commonwealth Universities and the University of Ghana for financial support while this work was being done.

## 1. Preliminary results

PROPOSITION 1.1. *Suppose that* $J = \langle H_1, H_2, \cdots, H_n \rangle$ *and* $H_i \lhd J$ *for* $i = 1, 2, \cdots, n$. *If* $r_1, r_2, \cdots, r_n$ *are non-negative integers and* $r$ *is their sum then*

$$J^{(r)} \leqq H_1^{(r_1)} + \cdots + H_n^{(r_n)}.$$

PROOF. Induct on $r$. For $r = 0$ the result is trivial. Suppose $1 \leqq r$ and assume the result holds for $r-1$. As $1 \leqq r$, $1 \leqq r_i$ for some $i$. Define $K = \Sigma_{j \neq i} H_j^{(r_j)}$. By the inductive hypothesis

$$J^{(r-1)} \leqq K + H_i^{(r_i-1)}.$$

Since

$$K \lhd J, (K + H_i^{(r_i-1)})^2 \leqq K + H_i^{(r_i)}$$

and therefore

$$J^{(r)} = (J^{(r-1)})^2 \leqq (K + H_i^{(r_i-1)})^2 \leqq K + H_i^{(r_i)}.$$

This proves the inductive step and with it the required result.

PROPOSITION 1.2. *If* $L = H + K$, $H \lhd L$ *and* $K \lhd^m L$ *then* $L^{(mn)} \leqq H^{(n)} + K$ *for any non-negative integer* $n$.

PROOF. Trivial for $n = 0$. Suppose $1 \leqq n$ and assume inductively that

$$L^{(m\{n-1\})} \leqq H^{(n-1)} + K.$$

An easy second induction on $r$ yields

$$(H^{(n-1)} + K)^{(r)} \leqq H^{(n)} + K + [H^{(n-1)}, {}_r K].$$

Since $K \lhd^m L$, $[L, {}_m K] \leqq K$ and so for $r = m$,

$$L^{(mn)} = (L^{(m\{n-1\})})^{(m)} \leqq (H^{(n-1)} + K)^{(m)} \leqq H^{(n)} + K$$

and the result is proved.

DEFINITION. Let $H$ and $K$ be subalgebras of a Lie algebra $L$ and $J = \langle H, K \rangle$. The *circle product* $H \circ K$ of $H$ and $K$ is defined by

$$H \circ K = \langle [H, K]^J \rangle.$$

Define inductively

$$H \circ_1 K = H \circ K, H \circ_{m+1} K = (H \circ_m K) \circ K$$

for all positive integers $m$.

PROPOSITION 1.3. (a) *$H \circ K = K \circ H$ and $A \leqq B$ implies $A \circ C \leqq B \circ C$.*
(b) *If $J = \langle H, K\rangle$ then $\langle H^J\rangle = H + H \circ K = H + H \circ J$ and*

$$H_n = H + K \circ {}_n H$$

*where $H_n$ is the n-th ideal closure of $H$ in $J$.*

(c) *If $H \leqq L$ and $H_n$ denotes the n-th ideal closure of $H$ in $L$ then*

$$H_n = H + L \circ {}_n H.$$

(d) *Suppose $H_1, H_2, \cdots, H_m$ are subalgebras of $L$ and*

$$J = \langle H_1, H_2, \cdots, H_m\rangle.$$

*If $X \lhd L$ then*

$$X \circ H_i \lhd X \circ J \qquad \text{for } i = 1, 2, \cdots, m$$

*and*

$$X \circ J = X \circ H_1 + X \circ H_2 + \cdots + X \circ H_m.$$

PROOF. (a) First part follows from $[H, K] = [K, H]$ and the second part from $[A, C] \leqq [B, C] \leqq B \circ C$.

(b) By definition $H \circ K \lhd J$. Thus $H + H \circ K$ is idealised by $K$ and contains $H$ and so is an ideal of $J$ and therefore contains $\langle H^J\rangle$. But clearly $\langle H^J\rangle$ contains $H + H \circ K$ and the first equality follows. From (a), $H + H \circ K \leqq H + H \circ J \leqq \langle H^J\rangle$ and the second part follows. For the third part use induction on $n$. It is trivial for $n = 1$, since $H_1 = \langle H^J\rangle$. Suppose $1 \leqq n$ and $H_n = H + K \circ {}_n H$. By definition

$$H_{n+1} = \langle H)^{H_n}\rangle = H + (K \circ_n H) \circ H,$$

from the first part. But by definition $(K \circ {}_n H) \circ H = K \circ {}_{n+1} H$ and the inductive step is proved.

(c) Follows trivially on putting $K = L$ in (b).

(d) Since $X \lhd L$ then $X \circ J \leqq X$. By definition $X$ idealises $X \circ H_i$ and from (a), $X \circ H_i \leqq X \circ J$ and so $X \circ H_i \lhd X \circ J$ for each $i$.
Let

$$K = X \circ H_1 + \cdots + X \circ H_m.$$

Then for any $i, j$

$$[X \circ H_i, H_j] \leqq [X, H_j] \leqq X \circ H_j \leqq K.$$

Thus $K$ is idealised by all $H_j$ and so by $J$. By its definition $K$ is idealised by $X$. Now $[X, J]$ is generated by terms of the form

$$U = [X, {}_{r_1}H_{j_1}, {}_{r_2}H_{j_2}, \cdots, {}_{r_n}H_{j_n}]$$

where $r_1, r_2, \cdots, r_n$ are non-negative integers at least one of which is non-zero and $\{j_1, j_2, \cdots, j_n\} \subset \{1, 2, \cdots, m\}$. As $X \lhd L$,

$$U \leqq [X, H_{j_k}] \leqq X \circ H_{j_k} \leqq K,$$

where $r_k$ is the last non-zero integer in the sequence $r_1, \cdots, r_n$. Therefore

$$[X, J] \leqq K$$

and so

$$X \circ J \leqq K \leqq X \circ J,$$

and (d) is proved.

PROPOSITION 1.4. *Suppose* $J = \langle A, B \rangle \leqq L$, $H \leqq L$ and $H \lhd \langle H, A \rangle$. *If* $J = A + B$ *then*

$$\langle H^J \rangle = \langle H^B \rangle.$$

PROOF. Clearly it suffices to show that the vector space $H^B$ is idealised by $A$. Now $H^B$ is spanned by elements of the form

$$x_r = [h, b_1, b_2, \cdots, b_r]$$

where $h \in H$, $b_1, \cdots, b_r \in B$ and $r$ is a non-negative integer.

If $r = 0$ then $x_r = h \in H$ and $[x_r, a] \in H \leqq H^B$ for all $a \in A$, since $A$ idealises $H$. We note that any $x_{r+1}$ is of the form

$$x_{r+1} = [x_r, b_{r+1}]$$

for some $x_r$ and some $b_{r+1} \in B$. Let $a \in A$. Then as $J = A + B$ there exists $a_1 \in A$ and $b \in B$ with $[b_{r+1}, a] = a_1 + b$. Thus by the Jacobi identity

$$\begin{aligned}[x_{r+1}, a] &= [[x_r, b_{r+1}], a] \\ &= [[x_r, a], b_{r+1}] + [x_r, a_1] + [x_r, b].\end{aligned}$$

Hence if $[x_r, a] \in H^B$ for all $a \in A$ and all $x_r$ (fixed $r$) then the same is true for all $x_{r+1}$. This proves the required result.

COROLLARY 1.4.1. *If* $J = \langle A, B \rangle \leqq L$ *and* $H \leqq L$ *then* $J = A + B$ *implies that*

$$\langle H^J \rangle = \langle \langle H^A \rangle^B \rangle = \langle \langle H^B \rangle^A \rangle.$$

PROOF. By 1.4 and since $A$ idealises $\langle H^A \rangle$ and $B$ idealises $\langle H^B \rangle$.

*Note on notation.*

The *derived series* of a Lie algebra $L$ is defined inductively by $L^{(0)} = L$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$ for all $n \geqq 0$. $L^{(n)}$ is the *n-th derived term* and $L$ is said to be *soluble* if $L^{(n)} = 0$ for some $n$.

## 2. Joins of pairs of subideals

THEOREM 2.1. *Suppose that* $J = \langle H_1, H_2 \rangle$. *If* $H_1 \lhd^{h_1} J$ *and* $H_2 \lhd^{h_2} J$ *then there exists* $\lambda_1 = \lambda_2(h)$ *such that*

$$J^{(\lambda_1)} \leqq H_1 + H_2$$

*whenever* $h_1 + h_2 \leqq h$.

PROOF. Define $\lambda_1(h) = 0$ if $h = 0$ or $1$ and $\lambda_1(h) = 4^{h-2}\{(h-2)!\}$ for $2 \leqq h$. The theorem is obvious for $h \leqq 2$ and for $h_1 = 1$ or $h_2 = 1$. Assume that $h > 2$, $h_1 > 1$, $h_2 > 1$ and proceed by induction on $h$. For $i = 1, 2$ there exist subalgebras $K_i$ and $L_i$ of $J$ with $K_i \leqq L_i$ such that

$$H_i \lhd K_i \lhd^{h_i - 1} J$$

and

$$H_i \lhd^{h_i - 1} L_i \lhd J.$$

Let $m = \lambda_1(h-1)$, $\{i, j\} = \{1, 2\}$ and $X = (H_1 \circ H_2)^{(m)}$. Since $J = \langle H_j, K_i \rangle$ then the inductive hypothesis applied to the pair $K_i$, $H_j$ yields

$$J^{(m)} \leqq K_i + H_j,$$

and so

$$X \leqq J^{(m)} \cap L_i \leqq (K_i + H_j) \cap L_i = K_i + H_j \cap L_i.$$

Let $Y = \langle X, H_j \cap L_i \rangle = X + H_j \cap L_i$, since $X \lhd J$. Then

$$Y \leqq K_i + H_j \cap L_i$$

and so

$$Y = (K_i + H_j \cap L_i) \cap Y = K_i \cap Y + H_j \cap L_i.$$

By definition $K_i$ idealises $H_i$ and therefore from 1.4

$$\langle H_i^Y \rangle = \langle H_i^{\{H_j \cap L_i\}} \rangle \leqq \langle H_i, H_j \cap L_i \rangle.$$

This and the fact that $X \leqq Y$ give

$$X \circ H_i \leqq \langle H_i^X \rangle \leqq \langle H_i^Y \rangle \leqq \langle H_i, H_j \cap L_i \rangle.$$

Now $H_i \lhd^{h_i - 1} L_i$ and $H_j \cap L_i \lhd^{h_j} L_i$ and so the inductive hypothesis applied to the pair $H_i$, $H_j \cap L_i$ gives

$$(X \circ H_i)^{(m)} \leqq J_i^{(m)} \leqq H_i + H_j \cap L_i \subset H_1 + H_2$$

where $J_i = \langle H_i, H_j \cap L_i \rangle$. Thus

$$(X \circ H_1)^{(m)} + (X \circ H_2)^{(m)} \subset H_1 + H_2.$$

As $X \lhd J$ then by 1.3

$$X \circ H_i \lhd X \circ J = X \circ H_1 + X \circ H_2.$$

Therefore from 1.1

$$(X \circ J)^{(2m)} \leqq (X \circ H_1)^{(m)} + (X \circ H_2)^{(m)} \leqq H_1 + H_2.$$

Let $U = H_1 \circ H_2$. Then $X = U^{(m)}$ and so

$$U^{(1+3m)} \leqq [U^{(m)}, J]^{(2m)} \leqq (X \circ J)^{(2m)} \leqq H_1 + H_2$$

$U \lhd J$ and so $L_i$ can be taken to be $H_i + U$. Since $H_i \lhd^{h_i - 1} L_i$ then by 1.2

$$L_i^{(\{1+3m\}h_i - 1\})} \leqq U^{(1+3m)} + H_i \leqq H_1 + H_2 .$$

Finally since $L_1 \lhd J$, $L_2 \lhd J$ and $J = L_1 + L_2$ then by 1.1

$$\begin{aligned} J^{(\{1+3m\}\{h_1+h_2-2\})} &\leqq L_1^{(\{1+3m\}\{h_1-1\})} + L_2^{(\{1+3m\}\{h_2-1\})} \\ &\leqq H_1 + H_2 . \end{aligned}$$

Clearly $\{1+3m\}\{h_1+h_2-2\} \leqq 4m(h-2) = \lambda_1(h)$ and so

$$J^{(\lambda_1(h))} \leqq H_1 + H_2 .$$

This proves the inductive step and with it the theorem.

Definition. Suppose that $H \leqq L$, $K \leqq L$. The *permutizer* $P_H(K)$ of $K$ in $H$ is defined as the join of all subalgebras $M$ of $H$ such that

$$\langle M, K \rangle = M + K.$$

It is not hard to show that $P_H(K) + K = \langle P_H(K), K \rangle$ and so $P_H(K)$ is in fact the maximal subalgebra of $H$ satisfying the requirement that its join with $K$ equal its vector space sum with $K$.

Corollary 2.1.1. *Under the same hypothesis as theorem* 2.1,

$$H_1^{(\lambda_1)} \leqq P_{H_1}(H_2).$$

Proof. Let $K = \langle J^{(\lambda_1)}, H_2 \rangle = J^{(\lambda_1)} + H_2$, since $J^{(\lambda_1)} \lhd J$. From 2.1, $K \leqq H_1 + H_2$ and so $K = K \cap (H_1 + H_2) = K \cap H_1 + H_2$. This implies $K \cap H_1 \leqq P_{H_1}(H_2)$. But $H_1^{(\lambda_1)} \leqq J^{(\lambda_1)} \cap H_1 \leqq K \cap H_1$ and the result follows.

Definition. Suppose $A$ and $B$ are subalgebras of $L$. Then $A$ and $B$ are said to be *permutable* if $\langle A, B \rangle = A + B$, i.e. $P_A(B) = A$.

Lemma 2.2. *Suppose that* $J = \langle H_1, H_2 \rangle$, $H_1 \lhd^{h_1} J$ *and* $H_2 \lhd^{h_2} J$. *If* $H_1$ *and* $H_2$ *are permutable then there exists* $\lambda_2 = \lambda_2(h, r)$ *such that*

$$J^{(\lambda_2)} \leqq H_1^{(r_1)} + H_2^{(r_2)}$$

*whenever* $h_1 + h_2 \leqq h$ *and* $r_1 + r_2 \leqq r$.

Proof. Define $\lambda_2(h, r) = r$ if $h = 0$ or $1$ and $\lambda_2(h, r) = 2^{h-2}r$ otherwise. For $h \leqq 2$, $H_1 \lhd J$ and $H_2 \lhd J$ and the result follows from 1.1. Suppose $h > 2$ and assume inductively that the result is true for $h-1$. Let $m = \lambda_2(h-1, r)$ and $\{i, j\} = \{1, 2\}$. There exists $L_i$ such that

$$H_i \lhd^{h_i - 1} L_i \lhd J.$$

By hypothesis $J = H_i + H_j$ and so $L_i = H_j \cap L_i$ which implies $H_i$ and $H_j \cap L_i$ are permutable. Since also $H_j \cap L_i \lhd^{h_j} L_i$ then the inductive hypothesis applied to $H_i$ and $H_j \cap L_i$ gives

$$L_i^{(m)} \leqq H_i^{(r_i)} + (H_j \cap L_i)^{(r_j)} \leqq H_1^{(r_1)} + H_2^{(r_2)}.$$

Finally $J = L_1 + L_2$, $L_1 \lhd J$, $L_2 \lhd J$ and so by 1.1

$$J^{(2m)} \leqq L_1^{(m)} + L_2^{(m)} \leqq H_1^{(r_1)} + H_2^{(r_2)}.$$

Clearly $2m = \lambda_2(h, r)$ and this proves the inductive step and the lemma.

LEMMA 2.3. *Let $H \lhd^m L$, $K \lhd^n L$ and $J = \langle H, K \rangle$. If $J$ and $K$ are permutable and $H \lhd^r J$ then*

$$J \lhd^{mn(n+1)\cdots(n+r-1)} L.$$

PROOF. By induction on $r$. For $r = 1$ $H \lhd J$. Let

$$H = H_m \lhd H_{m-1} \lhd \cdots \lhd H_1 \lhd H_0 = L$$

be the ideal closure series of $H$ in $L$; thus $H_{i+1} = \langle (H)^{H_i} \rangle$ for $i = 0, 1, \cdots, m-1$. Then it follows by easy induction on $i$ that $K$ idealises each $H_i$. Thus $H_{i+1} \lhd H_i + K$ and so by lemma 5 of [2]

$$H_{i+1} + K \lhd^n H_i + K \ \{\text{since } K \lhd^n H_i + K\},$$

for $i = 0, 1, \cdots, m-1$, and so $J = H_m + K \lhd^{mn} L$. This is the result for $r = 1$. Assume $r > 1$ and the lemma true for $r-1$ in place of $r$. There exists a series

$$H = A_r \lhd A_{r-1} \lhd \cdots A_1 \lhd A_0 = J.$$

Let $K_1 = A_1 \cap K$, $J = H + K$, $H \leqq A_1$ and so $A_1 = A_1 \cap (H+K) = H + K_1$ which implies $H$ and $K_1$ are permutable. Furthermore $H \lhd^{r-1} A_1$ and $K_1 \lhd K \lhd^n L$ and so by the inductive hypothesis

$$A_1 \lhd^p L \ \{\text{since } K_1 \lhd^{n+1} A_1\},$$

where $p = m(n+1)(n+2) \cdots ((n+1)+(r-1)-1)$. Now by definition $A_1 \lhd J$ and $J = A_1 + K$ and so applying the first part of the proof,

$$J \lhd^{pn} L$$

and the induction is complete.

THEOREM 2.4. *Suppose that $J = \langle H_1, H_2 \rangle$ with $H_1 \lhd^{h_1} L$ and $H_2 \lhd^{h_2} L$. Then there exists $\lambda_3 = \lambda_3(h, r)$ such that*

$$J^{(\lambda_3)} \lhd^{\lambda_3} L$$
$$J^{(\lambda_3)} \leqq H_1^{(r_1)} + H_2^{(r_2)}$$

*whenever $h_1 + h_2 \leqq h$ and $r_1 + r_2 \leqq r$.*

PROOF. Define $\lambda_3(h, r) = (2h)!+\lambda_1(h)+\lambda_2(2h, r)$. Let $M = J^{(\lambda_1)}$. By 2.1, $M \leqq H_1+H_2$. Since $M \lhd J$, $\langle M, H_2\rangle = M+H_2 \leqq H_1+H_2$. Therefore

$$M+H_2 = (M+H_2) \cap (H_1+H_2) = U+H_2$$

where $U = (M+H_2) \cap H_1$. Thus $U$ and $H_2$ are permutable. Since $M \lhd J$ then by 2.3 $M+H_2 \lhd^{h_2} J$ and so $U \lhd^{h_2} H_1$ which implies $U \lhd^{h_1+h_2} L$. From above $U$ and $H_2$ are permutable and so by 2.3

$$U+H_2 \lhd^{(2h_2+h_1)!} L.$$

Clearly for any integer $n$, $\lambda_1 \leqq n$, $J^{(n)} \leqq M \leqq U+H_2$. Now $J^{(n)}$ is a characteristic ideal of $J$; $2h_2+h_1 \leqq h$ and so

$$J^{(n)} \lhd^{(2h)!} L.$$

Let $m = \lambda_2(2h, r)$. $U \lhd^{h_1+h_2} L$, $H_2 \lhd^{h_2} L$, and $U$ and $H_2$ are permutable. Therefore by 2.2

$$(U+H_2)^{(m)} \leqq U^{(r_1)}+H_2^{(r_2)}.$$

Since $J^{(\lambda_1)} \leqq U+H_2$ and $U^{(r_1)} \leqq H_1^{(r_1)})$ it follows that

$$J^{(\lambda_1+m)} \leqq H_1^{(r_1)}+H_2^{(r_2)}$$

and the theorem is proved.

COROLLARY 2.4.1. *The join of a pair of soluble subideals is soluble.*

## 3. The main theorem

LEMMA 3.1. *Suppose that* $Y = \langle Y_1, Y_2, \cdots, Y_r\rangle$, $Y_i \lhd Y$ *and* $Y_i \lhd^{n_i} L$ *for* $i = 1, 2, \cdots, r$. *Then*

$$Y \lhd^{n_1 n_2 \cdots n_r} L.$$

PROOF. By 2.3 and induction on $r$.

THEOREM 3.2. *Suppose that* $J = \langle H_1, H_2, \cdots, H_n\rangle$ *and* $H_i \lhd^{h_i} L$ *for* $i = 1, 2, \cdots, n$. *Then there exists* $\lambda_4 = \lambda_4(h, r)$ *such that*

$$J^{(\lambda_4)} \leqq H_1^{(r_1)}+H_2^{(r_2)}+ \cdots +H_n^{(r_n)}$$

*and*

$$J^{(\lambda_4)} \lhd^{\lambda_4} L$$

*whenever* $h_1+h_2+\cdots+h_n \leqq h$ *and* $r_1+r_2+\cdots+r_n \leqq r$.

PROOF. The case $n = 2$ is theorem 2.4. Assume then that $n > 2$ and let

$$U = H_1^{(r_1)}+H_1^{(r_2)}+ \cdots +H_n^{(r_n)}.$$

If $h = 0$ then $H_i = L$ for all $i$ and so define $\lambda_4(0, r) = r$. If $h_i = 0$ for some $i$ then $H_i = L$ and so $L^{(r)} \leqq U, L^{(r)} \lhd L$. Thus assume no $h_i$ is zero and that $\lambda_4(h-1, r)$ has been defined for all $r$ so as to satisfy all requirements.

Let

$$K_i = \langle H_j;\ j \neq i\rangle, \qquad 1 \leqq i \leqq n$$

and

$$l = \lambda_4(h-1, r(h-1)) \tag{1}$$

Since $1 \leqq h_i$, $(h_1+h_2+\cdots+h_n)-h_i \leqq h-1$ and so by the inductive hypothesis on $h$

$$K_i^{(l)} \leqq \sum_{j\neq i} H_j^{(r_j)} \tag{2}$$

and

$$K_i^{(l)} \lhd^l L \qquad 1 \leqq i \leqq n \tag{3}$$

$h_i \geqq 1$ and so there is an $L_i$ such that

$$H_i \lhd L_i \lhd^{h_i-1} L.$$

Put

$$J_i = \langle L_i, K_i\rangle = \langle L_i, \{H_j | j \neq i\}\rangle,$$

and

$$m = \lambda_4(h-1, 1+(h-1)l). \tag{4}$$

Consider $J_i$ as the join of $L_i$ and the $H_j$ for $j$ different from $i$ as shown above. Then by the induction on $h$

$$J_i^{(m)} \leqq L_i + \sum_{j\neq i} H_j^{(l)} \leqq L_i + K_i^{(l)}. \tag{5}$$

Let

$$M_i = \langle H_i, K_i^{(l)}\rangle$$

and

$$V = \langle J_i^{(m)}, K_i^{(l)}\rangle = J_i^{(m)} + K_i^{(l)},$$

since $J_i^{(m)} \lhd J_i$ and $V \leqq J_i$. Therefore from (5) $V \leqq L_i + K_i^{(l)}$ and so

$$V = L_i \cap V + K_i^{(l)}.$$

Now $H_i \lhd L_i$ and $J_i^{(m)} \leqq V$. Therefore by 1.4

$$J_i^{(m)} \circ H_i \leqq \langle H_i^V\rangle = \langle\{H_i\}K_i^{(l)}\rangle \leqq M_i. \tag{6}$$

Let

$$p = \lambda_3(h+l, r). \tag{7}$$

From (3) $K_i^{(l)} \lhd^l L$. $H_i \lhd^{h_i} L$ and $h_i \leqq h$, $r_i \leqq r$. So applying theorem 2.4 to the pair $H_i$, $K_i^{(l)}$ yields

$$M_i^{(p)} \leqq H_i^{(r_i)} + K_i^{(l)} \tag{8}$$

and

$$M_i^{(p)} \lhd^p L. \tag{9}$$

From (2) and (4) it follows that

$$M_i^{(p)} \leqq U. \tag{10}$$

Put

$$X_i = J_i^{(m)} \circ H_i,\ X = J_i^{(m)} \circ J.$$

Since $J_i^{(m)} \lhd J_i$ then $X = X_1 + \cdots + X_n$ by 1.4(d). Again $X_i \lhd X$ and so if $Y_i = X_i^{(p)}$ and $Y = \langle Y_1, \cdots, Y_n \rangle$ then $Y_i \lhd Y$ and $Y = Y_1 + Y_2 + \cdots + Y_n$. From (6) $X_i \leqq M_i$ and by definition $X_i \lhd J_i^{(m)} \lhd J_i$. This implies $Y_i \lhd^2 J_i$ and so $Y_i \lhd^2 M_i^{(p)}$. From (9) $M_i^{(p)} \lhd^p L$ and so $Y_i \lhd^{(2+p)} L$. Further as each $h_i$ is non-zero by assumption then $n \leqq h$. Therefore by 3.1

$$Y \lhd^{(2+p)^h} L \tag{11}$$

and from (10)

$$Y = \sum_{i=1}^{n} Y_i \leqq \sum_i M_i^{(p)} \leqq U. \tag{12}$$

Since $n \leqq h$ and $X = X_1 + \cdots + X_n$, $X_i \lhd X$ for all $i$ then by 1.1

$$X^{(hp)} \leqq X^{(np)} \leqq \sum_{i=1}^{n} X_i^{(p)} = Y. \tag{13}$$

Finally let

$$q = 1 + hp + m. \tag{14}$$

Clearly $J \leqq J_i = \langle L_i, \{H_j \mid j \neq i\} \rangle$ and so from (12), (13) and (14)

$$\begin{aligned} J^{(q)} &\leqq [J^{(m)}, J]^{(hp)} \\ &\leqq (J_i^{(m)} \circ J)^{(hp)} \\ &\leqq X^{(hp)} \\ &\leqq Y \end{aligned}$$

i.e.

$$J^{(q)} \leqq U = H_1^{(r_1)} + \cdots + H_n^{(r_n)}. \tag{15}$$

Now $J^{(q)} \lhd J$ and from above $J^{(q)} \leqq Y$ and so by (11)

$$J^{(q)} \lhd^{(1+(2+p)^h)} L. \tag{16}$$

Define

$$(17) \qquad \lambda_4(h, r) = q+1+(2+p)^h.$$

Then from (15), (16) and (17) it follows that

$$J^{(\lambda_4)} \leqq H_1^{(r_1)}+H_2^{(r_2)}+\cdots+H_n^{(r_n)}$$

and

$$J^{(\lambda_4)} \lhd^{\lambda_4} L.$$

This proves the inductive step and with it the theorem.

THEOREM 3.3. *The join of finitely many soluble subideals is soluble.*

PROOF. Immediate from 3.2.

COROLLARY 3.3.1. *Suppose $J$ is a Lie algebra such that every term of the derived series of $J$ is the join of finitely many nilpotent subideals. Then $J$ is nilpotent.*

PROOF. Let $J = \langle H, K, \cdots, T\rangle$ where $H, K, \cdots, T$ are nilpotent subideals. By 3.3 $J$ is soluble of derived length $d$, say. Induct on $d$. For $d = 1$ the result is trivial. Assume $d > 1$. $J^2$ satisfies the hypothesis and has derived length $d-1$ and so by the induction on $dJ^2$ is nilpotent. Since $H$ is a nilpotent subideal of $J$ then there is an integer $c$ such that $[J^2, {}_cH] = 0$. Further $(H+J^2)/J^2$ is nilpotent. Therefore by lemma 2.1 of Stewart [4] $H+J^2$ is nilpotent. Of course $H+J^2 \lhd J$. Similarly $K+J^2, \cdots, T+J^2$ are nilpotent ideals of $J$. Hence by lemma 1 of Hartley [2]

$$J = (H+J^2)+(K+J^2)+\cdots+(T+J^2)$$

is nilpotent.

REMARK. There exist non-nilpotent Lie algebras which are joins of finitely many nilpotent subideals (see for example section 7.2 of [2]). Thus 3.3.1 shows that the class of Lie algebras which are joins of finitely many nilpotent subideals is not closed under the taking of subalgebras or ideals.

## REFERENCES

R. K. AMAYO

[1] Infinite dimensional Lie algebras. M. Sc. Thesis (1970) University of Warwick.

B. HARTLEY

[2] Locally nilpotent ideals of a Lie algebra. Proc. Cambridge Philos. Soc. 63, 257–272 (1967).

J. E. ROSEBLADE

[3] The derived series of a join of subnormal subgroups. Math. Z. 117, 57–69 (1970).

I. N. STEWART

[4] Lie algebras. Lecture Notes in Mathematics 127, Springer. Berlin, Heidelberg, New York (1970).

I. N. STEWART

[5] An algebraic treatment of Malcev's theorems concerning nilpotent Lie groups and their Lie algebras. Compositio Mathematica, 22, 289–312 (1970).

S. E. STONEHEWER

[6] The join of finitely many subnormal subgroups. Bull. London Math. Soc. 2, 77–82 (1970).

(Oblatum 22–XI–1971)

Mathematics Institute
University of Warwick
Coventry CV4 7AL
England