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ON FIRST ORDER ELLIPTIC EQUATIONS FOR SECTIONS OF COMPLEX LINE BUNDLES

by

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Introduction

Let M be a real 2-dimensional C^∞ manifold, E and F smooth vector-bundles over M with real 2-dimensional fibres. Then each linear first order elliptic partial differential operator L from C^∞ sections of E to C^∞ sections of F can locally be brought into a standard form, as follows.

THEOREM 1.

a) *For each $x_0 \in M$ there is a neighborhood U of x_0, a local coordinatization γ of U and local trivializations τ^E, resp. τ^F of E, resp. F over U in which L has the form:*

$$Lu = \tfrac{1}{2}(\partial u/\partial x_1 + i\partial u/\partial x_2) + b(x)\cdot \bar{u}. \tag{1}$$

Here $b(x)$ is a complex valued C^∞ function and the fiber $\mathbf{R}^2$ is identified with $\mathbf{C}$.

b) *If γ_j, τ_j^E, τ_j^F, $j = 1, 2$ are local coordinatizations, resp. local trivializations of E and F as in* a), *then either $\gamma_1 \circ \gamma_2^{-1}$ is holomorphic and $\tau_1^E \cdot (\tau_2^E)^{-1}$, $\tau_1^F \cdot (\tau_2^F)^{-1}$ are multiplications with complex numbers in the fibers, $\tau_1^E \cdot (\tau_2^E)^{-1}$ depending holomorphically on x, or $\gamma_1 \circ \gamma_2^{-1}$ is anti-holomorphic and $\tau_1^E \cdot (\tau_2^E)^{-1}$, $\tau_1^F \cdot (\tau_2^F)^{-1}$ are multiplications with complex numbers followed by complex conjugation.*

c) *If L is a complex linear operator for some given complex structures on E and F, then the trivialisations τ^E, τ^F in* a) *can be chosen complex linear.*

This theorem is classical, c.f. Vekua [13] or the supplement to Ch. IV in [4] of Bers. If M is orientable then this leads to a unique complex analytic structure on M, and an identification of E with a holomorphic complex line bundle ξ on M and of F with $\bar{\kappa}\cdot\xi$, such that:

$$Lu = \bar{\partial} u + b\cdot \bar{u} \text{ on sections } u \text{ of } \xi. \tag{2}$$

Here $b \in \Gamma(M, C^\infty(\bar{\kappa}\cdot(\bar{\xi})^{-1}\cdot \xi))$ and κ is the canonical bundle of M. If L is a complex linear operator then M is automatically orientable and

L is reduced to $\bar{\partial}$ acting on ξ. If M is not orientable one can study L by changing to the 2-fold orientable covering of M.

If M is not compact then the elliptic theory of Malgrange [9], Ch. 3, combined with the theorem of unique continuation of solutions of $\bar{\partial}u+a\cdot u+b\cdot\bar{u}=0$ of Carleman [3], implies that L is surjective: $\Gamma(M, C^\infty(E)) \to \Gamma(M, C^\infty(F))$. This can be generalized to the case that L is a first order operator on a higher dimensional manifold M, acting as an elliptic operator in the direction of the leaves of a 2-dimensional foliation in M. One obtains semi-global solvability for the equation $Lu = f$ if no leaf is contained in a compact subset of M, and global solvability if in addition a convexity condition for the leaves is satisfied as in [5], Theorem 7.1.6. Application to the Hamilton operator H_p leads to corresponding results for general pseudo-differential operators acting on real 2-dimensional bundles with 2-dimensional bicharacteristic strips. See [5], Ch. 7.

If $L = L_1 + iL_2$ is a complex vector field acting on a trivial line bundle then semi-global solvability conversely implies that no leaf is contained in a compact subset of M ([5], Th. 7.1.5). However, in general one can even have global solvability if M is a compact surface. If more generally M is fibered by compact surfaces on which L acts, then global solvability on the fibers leads to global solvability on M.

So assume from now on that M is a compact and orientable surface, L as in (0.2). Then

$$\text{index } L = \text{index } \bar{\partial} = c(\xi)+1-g. \tag{3}$$

The first identity follows from general elliptic theory and the second one is the theorem of Riemann-Roch. $c(\xi)$ is the Chern class of ξ and g is the genus of M. (See Gunning [6] for the theory of compact Riemann surfaces used here.) In particular L can only by surjective if $c(\xi) \geqq g-1$. Using the similarity principle of Bers [2], we obtain for each $v \in \Gamma(M, C^\infty(\kappa\xi^{-1}))$, ${}^tLv = 0$, $v \neq 0$, a non-zero holomorphic section v' of some holomorphic line bundle $\kappa\cdot(\xi')^{-1}$ with $c(\xi') = c(\xi)$. From the results below it therefore follows that L is surjective if $c(\xi) > 2(g-1)$. So there remains a gap between the necessary and sufficient condition for global solvability if $g \geqq 1$, $g-1 \leqq c(\xi) \leqq 2(g-1)$.

If L is complex linear then the reduction to $\bar{\partial}$ acting on ξ leads to a much more detailed description. In this case surjectivity is equivalent to the condition that $\kappa\cdot\xi^{-1}\cdot\zeta_q^{-n}$, considered as an element of the Jacobi-variety $J(M)$ of M, does not belong to the set W^n defined by:

$$\begin{aligned} &W^n = \emptyset \text{ if } n < 0,\ W^0 = \{0\}, \text{ and for } n \geqq 1: \\ &W^n = \{\zeta_{p_1}\cdot\zeta_{p_2}\cdots\zeta_{p_n}\cdot\zeta_q^{-n} \in J(M);\ p_1,\cdots,p_n \in M\}. \end{aligned} \tag{4}$$

Here $n = c(\kappa \cdot \xi^{-1}) = 2(g-1)-c(\xi)$, ζ_p is the point bundle of $p \in M$. The point $q \in M$ is arbitrary but fixed. The W^n are known in algebraic geometry as the varieties of special divisors of M. They are algebraic subvarieties of $J(M)$ of complex dimension n if $1 \leqq n \leqq g$. $W^n = J(M)$ for $n \geqq g$ because of Riemann-Roch.

In Section 1 we give an elementary proof of Theorem 1, followed by a discussion in more detail of the identification of the operator $\bar{\partial}+a$ acting on the holomorphic line bundle ξ_0 (ξ_0 fixed, a varying) with $\bar{\partial}$ acting on the holomorphic line bundle ξ depending on a. In Section 2 we discuss the relation between the surjectivity of $\bar{\partial}$ and the algebraic varieties W^n mentioned above. Although this is only a standard application of the classical theory of Riemann surfaces, we like to present this here as an example of an elliptic equation on a compact manifold with a rather intricate global solvability condition on the lower order term a. We conclude by mentioning what is known about the singularities of the varieties W^n.

I am indebted to Lars Hörmander for the suggestion that [5], Ch. 7 should be generalized to to operators on line bundles, and to Lipman Bers and Frans Oort for helping me with the literature.

1. Reduction to $\bar{\partial}$ acting on a holomorphic line bundle

For arbitrary local trivializations of E and F over U, the principal symbol of L is a C^∞ mapping: $(x, \xi) \mapsto A(x, \xi)$ from $T^*(U)$ to the space of real 2×2-matrices, the mapping is linear in ξ. Here the principal symbol is defined such that $L = A(x, \partial/\partial x)+$zero order terms, on local coordinates.

Ellipticity means that $\det A(x, \xi) \neq 0$ for $\xi \neq 0$, so $\det A(x, \xi)$ is the principal symbol of a real second order elliptic operator P on U. According to a classical theorem on normal forms of such operators we can find local coordinates such that the second order part of P is equal to $c(x) \cdot \Delta$ for a smooth function $c(x) \neq 0$. Here $\Delta = \partial^2/\partial x_1^2+\partial^2/\partial x_2^2$ on $\boldsymbol{R}^2$. (See Courant and Hilbert [4], Ch. III, § 1.) So on these coordinates:

$$Lu = A_1(x) \cdot \partial u/\partial x_1 + A_2(x) \cdot \partial u/\partial x_2 + B(x) \cdot u, \tag{1.1}$$

where $A_1(x)$, $A_2(x)$, $B(x)$ are real 2×2-matrices depending smoothly on x, and $\det (A_1(x)\xi_1+A_2(x)\xi_2) = c(x) \cdot (\xi_1^2+\xi_2^2)$.

Now we retrivialize E and F, that is we write $u(x) = S(x) \cdot v(x)$, $f(x) = T(x) \cdot g(x)$ for some real 2×2-matrices $S(x)$, $T(x)$ depending smoothly on x. Then $Lu = f$ becomes

$$g = T^{-1}A_1 S\partial v/\partial x_1 + T^{-1}A_2 S\partial v/\partial x_2 + \text{zero order terms}. \tag{1.2}$$

So we try to choose S, T such that $T^{-1}A_1S = \frac{1}{2}I$, $T^{-1}A_2S = \frac{1}{2}i$, here $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This means that $T = \frac{1}{2}S^{-1}A_1^{-1}$ and

$$(1.3) \qquad S^{-1}\, A_1(x)^{-1}A_2(x)S = i.$$

The equation (1.3) is solvable if and only if $A_1^{-1}A_2$ has eigenvalues $\pm i$. Now

$$\det(A_1^{-1}A_2-\lambda I) = \det A_1^{-1}\cdot\det(A_2-\lambda A_1) = \det A_1^{-1}\cdot c(x)\cdot(1+\lambda^2),$$

so for each $x\in U$ the equation (1.3) has a solution. The mapping $S\mapsto SiS^{-1}$ is a smooth fibration of $GL(2, \boldsymbol{R})$ over the manifold of real 2×2 matrices with eigenvalues $\pm i$, so the solution S can locally be chosen to depend smoothly on x.

We now have local coordinatizations and trivializations in which $Lu = \partial u/\partial\bar{z}+a(z)\cdot u+b(z)\cdot\bar{u}$, $z = x_1+ix_2$. Here a, b are complex valued C^∞ functions of z. Using Cauchy's integral formula we can find a local solution $c(z)$ to $\partial c/\partial\bar{z} = a$. Writing $u = e^{-c}\cdot v$, $f = e^{-c}\cdot g$ the equation $Lu = f$ can be written in the form $\partial v/\partial\bar{z}+b\cdot e^{c-\bar{c}}\cdot\bar{v} = g$, which proves part a) of Theorem 1.

For part b) we remark that the equation

$$(1.4) \qquad f = \tfrac{1}{2}(\partial u/\partial x_1+i\cdot\partial u/\partial x_2)+b\cdot\bar{u}$$

in other local coordinatizations, resp. trivializations as in Theorem 1, a) has the form

$$(1.5) \qquad g = \tfrac{1}{2}(\partial v/\partial y_1+i\cdot\partial v/\partial y_2)+c\cdot\bar{v}.$$

Here $y = y(x)$, $u(x) = S(x)\cdot v(y(x))$, $f(x) = T(x)\cdot g(y(x))$. This leads to $T^{-1}\cdot(\partial/\partial x_1+i\cdot\partial/\partial x_2)y_1\cdot S = I$, $T^{-1}\cdot(\partial/\partial x_1+i\cdot\partial/\partial x_2)y_2\cdot S = i$, so both $T\circ S^{-1} = (\partial/\partial x_1+i\partial/\partial x_2)y_1$ and $T\circ S^{-1}SiS^{-1} = (\partial/\partial x_1+ i\partial/\partial x_2)y_2$ are multiplications with complex numbers. Therefore SiS^{-1} is a multiplication with a complex number which only can be $+i$ or $-i$.

If $SiS^{-1} = i$ then we obtain the Cauchy-Riemann equations for y_1, y_2. Moreover S and therefore also T can only be a multiplication with a complex number. Looking at the zero order terms we obtain that $T^{-1}\partial S/\partial\bar{z}\cdot v+T^{-1}\cdot b\cdot\bar{S}\cdot\bar{v} = c\cdot\bar{v}$ for all v, so $\partial S/\partial\bar{z} = 0$, $c = T^{-1}\cdot b\cdot\bar{S}$. If finally $SiS^{-1} = -i$ then $x\mapsto y(x)$ is anti-holomorphic and S, T are multiplications by complex numbers followed by complex conjugation. This proves b).

For the statement c) in Theorem 1 we observe that A_1, A_2, B in (1.1) are multiplications by complex numbers if L is complex linear and we choose τ^E, τ^F complex linear. The formula $S^{-1}\,A_1^{-1}\,A_2\,S = i$ then implies that $A_1^{-1}\,A_2 = \pm i$. If $A_1^{-1}\,A_2 = +i$ it follows that S and T are multiplications by complex numbers. If $A_1^{-1}A_2 = -i$ then the change of coordinates $(x_1, x_2) \mapsto (x_1, -x_2)$ leads to the above case.

We conclude this section by a discussion of the case that $L = \bar{\partial}+a$ acting on a fixed holomorphic line bundle ξ_0 over the compact Riemann surface M, with varying $a \in \Gamma(M, C^\infty(\bar{\kappa}))$. Let U_α, $\alpha \in A$ be a covering with contractible coordinate neighborhoods in M such that a is given by local sections $a_\alpha \in \Gamma(U_\alpha, C^\infty)$. Let $c_\alpha \in \Gamma(U_\alpha, C^\infty)$ be solutions of

$$2\pi i \cdot \partial c_\alpha/\partial \bar{z} + a_\alpha = 0. \tag{1.6}$$

Then $c_\beta - c_\alpha$ is holomorphic in $U_\alpha \cap U_\beta$, so they define an element $\vartheta(a) \in H^1(M, \mathcal{O})$, which in fact is the element in $H^1(M, \mathcal{O})$ corresponding to $-(2\pi i)^{-1} \cdot a$ under the canonical isomorphism

$$\Gamma(M, C^\infty(\bar{\kappa}))/\bar{\partial}\Gamma(M, C^\infty) \to H^1(M, \mathcal{O}) \tag{1.7}$$

given by the fine resolution $0 \to \mathcal{O} \to C^\infty \xrightarrow{\bar{\partial}} C^\infty(\bar{\kappa}) \to 0$ of the sheaf $\mathcal{O}$.

Writing $u_\alpha = e^{2\pi i \cdot c_\alpha} \cdot v_\alpha$, $f_\alpha = e^{2\pi i \cdot c_\alpha} \cdot g_\alpha$ the equation $\partial u_\alpha/\partial \bar{z} + a_\alpha \cdot u_\alpha = f_\alpha$ is equivalent to $\partial v_\alpha/\partial \bar{z} = g_\alpha$. The transition formula for the v_α is given by $v_\alpha = e^{2\pi i(c_\beta - c_\alpha)} \cdot \xi_{\alpha\beta}^{(0)} \cdot v_\beta$, if $u_\alpha = \xi_{\alpha\beta}^{(0)} \cdot u_\beta$, the

$$\xi_{\alpha\beta}^{(0)} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O})$$

defining ξ_0. In other words, $(\bar{\partial}+a)u = f$ for sections u of ξ_0 is equivalent to $\bar{\partial}v = g$ for sections v of $\xi = \xi(a) = e^{2\pi i \vartheta(a)} \cdot \xi_0$.

In view of the exact sequence

$$0 \to H^1(M, \mathbf{Z}) \to H^1(M, \mathcal{O}) \xrightarrow{e^{2\pi i}} H^1(M, \mathcal{O}^*) \xrightarrow{c} H^2(M, \mathbf{Z}) \to 0 \tag{1.8}$$
$$H^2(M, \mathbf{Z}) \cong \mathbf{Z}$$

the Chern classes $c(\xi)$ and $c(\xi_0)$ of ξ and ξ_0 are equal. Conversely every $\xi \in H^1(M, \mathcal{O}^*)$ with $c(\xi) = c(\xi_0)$ is equal to $\xi(a)$ for some $a \in \Gamma(M, C^\infty(\bar{\kappa}))$. So the solvability properties of the operator $\bar{\partial}+a$ are completely determined by the element $\xi(a) \cdot \xi_0^{-1}$ in $J(M) = H^1(M, \mathcal{O})/H^1(M, \mathbf{Z})$. The complex g-dimensional torus $J(M)$ is called the *Jacobi variety* of the compact Riemann surface M. Here g is the genus of M.

2. The surjectivity of $\bar{\partial} : \Gamma(M, C^\infty(\xi)) \to \Gamma(M, C^\infty(\bar{\kappa}\xi))$

Because $0 \to \mathcal{O}(\xi) \to C^\infty(\xi) \xrightarrow{\bar{\partial}} C^\infty(\bar{\kappa}\xi) \to 0$ is a fine resolution of the sheaf $\mathcal{O}(\xi)$, the surjectivity of $\bar{\partial}$ is equivalent to $H^1(M, \mathcal{O}(\xi)) = 0$, which in turn is equivalent to $\Gamma(M, \mathcal{O}(\kappa\xi^{-1})) = 0$ by Serre duality. Now for any $\zeta \in H^1(M, \mathcal{O}^*)$, $\Gamma(M, \mathcal{O}(\zeta)) \neq 0$ if and only if ζ is trivial or a product of point bundles. Indeed, $\zeta = \zeta_p$ if and only if there exists a non-zero holomorphic section of ζ with precisely one zero at p. Because two holomorphic line bundles ζ, ζ' are equal if there exist non-zero meromorphic sections of ζ, resp. ζ' with equal zeros and poles, the result follows immediately. Defining W^n as in (4), $n = c(\zeta)$ = the number of zeros minus the number of poles of meromorphic sections of ζ, we obtain that $\Gamma(M, \mathcal{O}(\zeta)) = 0$ if and only if $\zeta \cdot \zeta_q^{-n} \notin W^n$.

If γ is a curve from q to p then $h \mapsto \int_\gamma h$, $h \in \Gamma(M, \mathcal{O}(\kappa))$, is an element of $\Gamma(M, \mathcal{O}(\kappa))^* \cong H^1(M, \mathcal{O})$ (Serre duality), which according to Abel's theorem corresponds to $\zeta_p \zeta_q^{-1} \in J(M)$. Therefore $\Phi : p \mapsto \zeta_p \zeta_q^{-1}$ is an analytic mapping: $M \to J(M)$ with image W^1. Φ is injective, hence an analytic embedding of M into $J(M)$ if $g \geqq 1$ (the case $g = 0$ is trivial). Because of Chow's lemma the image W^1 is even an algebraic subvariety (without singularities) of the algebraic variety $J(M)$. So $W^n = W^1 + \cdots + W^1$ (n times) is also an algebraic subvariety of $J(M)$. Because of Riemann-Roch, $\dim \Gamma(M, \mathcal{O}(\zeta)) > 0$ if $c(\zeta) \geqq g$, hence $W^n = J(M)$ for $n \geqq g$. Since $\dim_{\mathbf{C}} W^{n+1} \leqq \dim_{\mathbf{C}} W^n + 1$ it follows that $\dim_{\mathbf{C}} W^n = n$ for $1 \leqq n \leqq g$.

The possible singularities of W^n for $2 \leqq n \leqq g-1$ are studied quite extensively in algebraic geometry. Define

$$(2.1) \quad G_n^r = \{\zeta_{p_1} \cdots \zeta_{p_n} \cdot \zeta_q^{-n} \in W^n; \dim \Gamma(M, \mathcal{O}(\zeta_{p_1} \cdots \zeta_{p_n})) \geqq r+1\}.$$

Alternative description: the mapping $(p_1, \cdots, p_n) \mapsto \zeta_{p_1} \cdots \zeta_{p_n} \cdot \zeta_q^{-n}$: $M^n \to J(M)$ factors through the symmetric product of n copies of M, denoted by $M^{(n)}$, thus leading to a mapping $\Phi^{(n)} : M^{(n)} \to J(M)$. The variety $M^{(n)}$ has no singularities (cf. Andreotti [1]) and the mapping $\Phi^{(n)}$ is analytic. Then $\dim \Gamma(M, \mathcal{O}(\zeta_{p_1} \cdots \zeta_{p_n})) = r+1$ if and only if the rank of the differential of $\Phi^{(n)}$ at $(p_1, \cdots, p_n)$ is equal to $n-r$ (see Gunning [6], Lemma 17).

Now Weil [14] showed that G_n^1 is equal to the set of singularities of W^n for all $n \leqq g-1$. In general G_n^{r+1} is contained in the set of singularities of G_n^r (Mayer [12]), but Martens [11] has given examples of singularities of G_{g-1}^1 not coming from G_{g-1}^2. Martens [10] also proved that

$$(2.2) \quad d = (r+1)(n-r) - rg \leqq \dim G_n^r \leqq n-2r \text{ if } 2 \leqq n \leqq g-1.$$

Kleiman and Laksov [8] proved that $G_n^r \neq \emptyset$ if the number d in the left hand side of (2.2) is non-negative. Finally we mention the work of Kempf [7] containing an infinitesimal study of the singularities of the W^n.

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