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Sylow theory in locally finite groups

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1. Introduction

We make the convention that all groups occurring in this paper are assumed to be locally finite. Let \( \pi \) be a set of primes and \( G \) a group. We shall say that \( G \) is Sylow \( \pi \)-sparse if \( |\text{Syl}_\pi H| < 2^{\aleph_0} \) for every countable subgroup \( H \) of \( G \), where \( \text{Syl}_\pi H \) denotes the set of Sylow, that is maximal, \( \pi \)-subgroups of \( H \). In order for a group \( G \) to be Sylow \( \pi \)-sparse it obviously suffices that every countable subgroup \( H \) of \( G \) be contained in some subgroup \( K \) of \( G \) with \( |\text{Syl}_\pi K| < 2^{\aleph_0} \) (cf. [4] Lemma 2.2) and so a countable group \( G \) is Sylow \( \pi \)-sparse if and only if \( |\text{Syl}_\pi G| < 2^{\aleph_0} \).

We shall say that \( G \) is Sylow \( \pi \)-connected if the Sylow \( \pi \)-subgroups of \( G \) are conjugate in \( G \), and Sylow \( \pi \)-integrated if every subgroup of \( G \) is Sylow \( \pi \)-connected. Thus the class of Sylow \( \pi \)-integrated groups is just the class \( \mathcal{D}_\pi^S \) in the notation of [4].

In this paper one of our concerns is to investigate the relationship between \( \mathcal{D}_\pi^S \) and the class of Sylow \( \pi \)-sparse groups. This relationship seems to be a very intimate one. Obviously every Sylow \( \pi \)-integrated group is Sylow \( \pi \)-sparse, and we know of no example to show that these two properties are distinct.

We prove

**Theorem (A).** If a group \( G \) contains a Sylow \( p \)-sparse subgroup of finite index then \( G \) is Sylow \( p \)-integrated.

Here \( p \), as always, denotes a prime.

**Theorem (B).** Let \( G \) be a \( \pi \)-separable group containing a Sylow \( \pi \)-sparse subgroup of finite index. Then \( G \) is Sylow \( \pi \)-integrated.
Thus for $\pi$-separable groups (groups having a series of finite length in which each factor is either a $\pi$-group or a $\pi'$-group) the concepts 'Sylow $\pi$-sparse' and 'Sylow $\pi$-integrated' coincide, and 'Sylow $p$-sparse' and 'Sylow $p$-integrated' are always equivalent. Similar theorems were proved for countable groups in [4]. The above results may be established by modifying the arguments of [4] (I am indebted to Dr. M. J. Tomkinson for pointing this out) or may be deduced from [4] by considering the closure properties of the class of Sylow $\pi$-integrated groups. In the usual closure operation notation, as used for example in [4], we have

**Lemma (2.1).** The class $\mathcal{D}_\pi$ of Sylow $\pi$-integrated groups is $\{Q, S, D_0, L_\alpha\}$-closed,

where, if $\mathcal{X}$ is some class of groups,

- $G \in D_0 \mathcal{X} \iff G$ is a direct product of finitely many $\mathcal{X}$-groups,
- $G \in L_\alpha \mathcal{X} \iff$ every countable set of elements of $G$ lies in some $\mathcal{X}$-subgroup of $G$.

We also show that the set of Sylow $\pi$-subgroups of a Sylow $\pi$-integrated group is homomorphism invariant (Lemma 2.2). It does not seem to be known whether Sylow $\pi$-connectedness is sufficient to ensure this, and in fact not a great deal is known about even Sylow $p$-connected groups in general.

We are also interested in obtaining information about the structure of Sylow $\pi$-sparse groups. For any group $G$ let $\sigma(G)$ denote the set of prime divisors of the orders of the elements of $G$. Then we have

**Theorem (D).** Let $G$ be locally soluble and let $\sigma(G) = \pi_1 \cup \cdots \cup \pi_n$ be a partition of $\sigma(G)$ into finitely many pairwise disjoint subsets. Suppose that $G$ is Sylow $\pi_i$-sparse for $1 \leq i \leq n$. Then

1. $G/\prod_{i=1}^n O_{\pi_i}(G) \in \mathcal{R}_1$.
2. If $\pi$ is any union of sets $\pi_i$, then $G$ is Sylow $\pi$-integrated.

Subgroups such as $O_{\pi}(G)$, $O_{\pi, \pi'}(G)$, $\cdots$ are defined by: $O_{\pi}(G)$ is the largest normal $\pi$-subgroup of $G$, $O_{\pi, \pi'}(G)/O_{\pi}(G) = O_{\pi'}(G/O_{\pi}/G)$, $\cdots$.

The class $\mathcal{R}_1$ was introduced in [4]; a group $G$ belongs to $\mathcal{R}_1$ if and only if every locally nilpotent subgroup of $G$ is almost abelian and of finite (Mal'cev special) rank. By ([4] Lemma 4.8), locally soluble $\mathcal{R}_1$-groups are countable, metabelian-by-finite, and of finite rank. Consequently the conclusions of Theorem D imply that $G$ is $\pi_i$-separable for $1 \leq i \leq n$.

As corollaries of Theorem D we have

**Corollary (D1).** Let $G$ be locally soluble. Suppose that $\sigma(G)$ is finite and $G$ is Sylow $p$-sparse for all $p \in \sigma(G)$. Then $G/\rho(G)$ satisfies Min and $G$ is Sylow $\pi$-integrated for any subset $\pi$ of $\sigma(G)$.
We write $\rho(G)$ for the Hirsch-Plotkin radical of $G$, $\text{Min}$ for the minimal condition on subgroups and $\text{Min}-p$ for the minimal condition on $p$-subgroups.

**Corollary (D2).** Suppose that $G$ is locally soluble and is Sylow $p$- and $p'$-sparse. Then $G/\text{O}_{p,p',p}(G)$ is finite and $G/\text{O}_p(G)$ satisfies $\text{Min}-p$.

This result is only of interest when $p = 2$; for odd $p$ a much stronger result, in which $G$ is only assumed to be locally $p$-soluble and Sylow $p$-sparse, is proved in [5].

Finally we consider groups which are completely Sylow sparse, that is, are Sylow $\pi$-sparse for every set $\pi$ of primes. In [4] and other places, a certain class $\mathcal{U}$ is considered, whose definition may be given as follows: $G \in \mathcal{U}$ if and only if

1. $G \in \text{PLR}$, that is, $G$ has a finite series with locally nilpotent factors.
2. $G$ is completely Sylow integrated (in the obvious sense).

We prove that the second of these conditions implies the first by proving

**Theorem (E).** Suppose that $G$ is locally soluble. Then $G \in \mathcal{U}$ if and only if $G$ is completely Sylow sparse.

For by a well known theorem of P. Hall, a finite completely Sylow integrated group is soluble and so an arbitrary completely Sylow integrated group is locally soluble. Theorem E can easily be deduced from Corollary D2 and ([4] Theorem D); however we give an independent approach which avoids the use of Thompson’s results on groups of automorphisms of soluble groups.

Theorems A and B may be found in Section 2, Theorem D and other structural results about Sylow $\pi$-sparse groups are established in Section 3, and Section 4 contains Theorem E and some other observations about the class $\mathcal{U}$. It will be useful to write $\mathfrak{S}_\pi$ for the class of $\pi$-groups, and $X^G$ denotes the normal closure of a subset $X$ of a group $G$. The rest of our notation is either standard, to be found in [4], or will be explained when introduced.

### 2. Closure properties

**Lemma (2.1).** The class $\mathfrak{S}_\pi^S$ of Sylow $\pi$-integrated groups is $\{Q, S, D_0, L_0\}$-closed.

**Proof.** The fact that $\mathfrak{S}_\pi^S$ is $s$-closed is immediate from the definition, and the $D_0$-closure follows by the argument of ([6] Lemma 6.7).
Now since $\mathfrak{S}_\pi^S$ is s-closed, a group belongs to $L_\omega \mathfrak{S}_\pi^S$ if and only if every countable subgroup of that group belongs to $\mathfrak{S}_\pi^S$. Therefore, if $\mathfrak{S}_\pi^S$ is not $L_\omega$-closed, there exists a group $G$ such that every countable subgroup of $G$ is Sylow $\pi$-integrated, while $G$ contains two non-conjugate Sylow $\pi$-subgroups $S$ and $T$. Suppose then that this situation holds. We construct a sequence $x_0, x_1, \ldots$ of elements of $G$ and integers $0 \leq n_0 \leq n_1 \leq \ldots$ such that

(i) the set $\{x_0, \ldots, x_{n_i}\}$ is a subgroup $F_i$ of $G$,

(ii) $\langle S^{x_i} \cap F_{i+1}, T \cap F_{i+1} \rangle \notin \mathfrak{S}_\pi^S$,

(iii) $i \leq n_i$.

for each $i \geq 0$.

Suppose that for some $j \geq 0$ we have obtained $x_0, \ldots, x_{n_j}$ and $n_0, \ldots, n_j$ such that (i) and (iii) hold for $i \leq j$ and (ii) for $i < j$. (We may begin the construction by putting $x_0 = 1$, $n_0 = 0$). Then by (iii) the element $x_j$ has already been defined, and as $S$ and $T$ are not conjugate in $G$, we have $S^{x_j} \neq T$ and so $\langle S^{x_j}, T \rangle \notin \mathfrak{S}_\pi^S$. Therefore there exist finite subsets $S_0 \subseteq S$, $T_0 \subseteq T$ such that $\langle S_{0}^{x_j}, T_0 \rangle \notin \mathfrak{S}_\pi^S$. Let $F_{j+1} = \langle F_j, S_0, T_0, y \rangle$ where $y$ is an arbitrary element of $G$ not lying in $F_j$; clearly such an element exists since $G$ is infinite and even uncountable. Then the elements of $F_{j+1}$ not lying in $F_j$ may be indexed as $x_{n_j+1}, \ldots, x_{n_{j+1}}$, where $n_{j+1} > n_j$, and (i) and (iii) hold with $i = j + 1$. Furthermore $\langle S^{x_j} \cap F_{j+1}, T \cap F_{j+1} \rangle$ contains $\langle S_{0}^{x_j}, T_0 \rangle$, which is not a $\pi$-group. Thus (ii) holds and the construction can be carried out.

Let $H = \{x_0, x_1, \ldots\} = \bigcup_{i=0}^\infty F_i$. Then $H$ is countable and so $H$ is Sylow $\pi$-connected. Therefore there is an element $x \in H$ such that

$$\langle (S \cap H)^x, T \cap H \rangle = \langle S^x \cap F_{i+1}, T \cap F_{i+1} \rangle \in \mathfrak{S}_\pi^S.$$ 

But $x = x_i$ for some $i$ and so we obtain $\langle S^{x_i} \cap F_{i+1}, T \cap F_{i+1} \rangle \in \mathfrak{S}_\pi^S$, contradicting (ii). This establishes that $\mathfrak{S}_\pi^S$ is $L_\omega$-closed.

To establish the q-closure it now suffices to show that every homomorphic image of a countable $\mathfrak{S}_\pi^S$-group lies in $\mathfrak{S}_\pi^S$. This follows from ([4] Lemma 2.2). Alternatively we may use the following lemma, which yields more information. It has already been established under further assumptions elsewhere (for example [1] Lemma 2.1, [10] Lemma 9.13).

**Lemma (2.2).** Let $G$ be Sylow $\pi$-integrated and $K \triangleleft G$. Then the Sylow $\pi$-subgroups of $G/K$ are precisely the groups $SK/K$ with $S \in \text{Syl}_\pi G$.

This result will itself be deduced from the following special case:

**Lemma (2.3).** Let $G$ be Sylow $\pi$-integrated, $K \triangleleft G$ and suppose that $K \in \mathfrak{S}_\pi^S$, $G/K \in \mathfrak{S}_\pi^S$. Then $G$ splits over $K$.

**Proof of 2.3.** Suppose that the result is false. Then since any set of
cardinal numbers is well ordered by the usual ordering of cardinals according to magnitude there is a counterexample $K \triangleleft G$ such that the index $|G : K| = \kappa$ is as small as possible. We notice first that $\kappa$ must be infinite. For otherwise $G = KF$ for some finite subgroup $F$ and a well known theorem of Schur shows that $F$ splits over $F \cap K$, giving a contradiction.

Let $\lambda$ be the least ordinal of cardinal $\kappa$. Since $\kappa$ is infinite, $\lambda$ is a limit ordinal. The elements of $G/K$ may be indexed as $\{K_{x_\alpha} : \alpha < \lambda\}$, and if we write $G_\beta = \langle K, x_\beta : \alpha < \beta \rangle$ for $\beta \leq \lambda$, we have

(i) $G_0 = K$,

(ii) $G_\beta \leq G_{\beta+1}$ if $\beta < \lambda$,

(iii) $G_\mu = \bigcup_{\alpha < \mu} G_\alpha$ if $\mu \leq \lambda$ is a limit ordinal,

(iv) $G_\lambda = G$,

(v) $|G_\beta : K| < \kappa$ if $\beta < \lambda$.

In fact, (i)–(iv) are immediate from the definition. To see (v), let $\beta < \lambda$. Then by the choice of $\lambda$, $G_\beta/K$ is generated by a set whose cardinal $\kappa$ is smaller than $\kappa$. Therefore $|G_\beta/K|$ is either finite or equal to $\kappa$, according as $\kappa$ is finite or infinite, and is in any case smaller than $\kappa$.

We now construct inductively subgroups $P_\beta$ of $G_\beta$ ($\beta \leq \lambda$) such that

(vi) $G_\beta = K P_\beta$, $K \cap P_\beta = 1 (\beta \leq \lambda)$,

(vii) $P_\beta \leq P_{\beta+1} (\beta < \lambda)$,

(viii) $P_\mu = \bigcup_{\alpha < \mu} P_\alpha$ if $\mu \leq \lambda$ is a limit ordinal.

Then $P_\lambda$ will complement $K$ in $G$, and this contradiction will establish Lemma 2.3. We begin by putting $P_0 = 1$. Suppose $0 < \gamma \leq \lambda$ and that the subgroup $P_\beta$ have been obtained for $\beta < \gamma$. If $\gamma$ is a limit ordinal we put $P_\gamma = \bigcup_{\alpha < \gamma} P_\alpha$. Otherwise $\gamma$ has the form $\beta + 1$. Then as $\lambda$ is a limit, we have $\gamma < \lambda$. Therefore by (v) and the choice of $\kappa$, there exists a subgroup $P_\gamma^*$ of $G_\gamma$ such that $KP_\gamma^* = G_\gamma$, $K \cap P_\gamma^* = 1$. Then $G_\beta = K(G_\beta \cap P_\gamma^*)$, and $P_\beta$ and $G_\beta \cap P_\gamma^*$ are Sylow $\pi$-subgroups of $G_\beta$. Therefore, as $G_\beta$ is Sylow $\pi$-connected, we have $P_\beta = (G_\beta \cap P_\gamma^*)^x$ for some $x \in K$. Let $P_\gamma = P_\gamma^x$. Then $P_\beta \leq P_\gamma$, and so the construction proceeds and can be completed.

Proof of Lemma 2.2. We have $G \in \mathfrak{D}_\pi^S$ and $K \triangleleft G$. Let $U/K \in \text{Syl}_\pi G/K$ and $T \in \text{Syl}_\pi K$. Then by the Frattini argument

(1) \[ U = KN, \]

where $N = N_G(T)$. Now $T$ is a normal Sylow $\pi$-subgroup of $N \cap K$ and so $N \cap K/T \in \mathfrak{D}_\pi^S$. Also $N/N \cap K \cong NK/K = U/K \in \mathfrak{D}_\pi^S$. We show that $N/T \in \mathfrak{D}_\pi^S$. In fact if $X/T \leq N/T$ and $R_1/T, R_2/T$ are Sylow $\pi$-subgroups of $X/T$, then $R_1$ and $R_2$ are Sylow $\pi$-subgroups of $X$ (since $T \in \mathfrak{D}_\pi^S$)
and so are conjugate in \( X \) as \( G \in \mathcal{D}_p^k \). Therefore \( R_1/T \) and \( R_2/T \) are conjugate in \( X/T \), whence \( N/T \) is Sylow \( \pi \)-integrated as claimed. Lemma 2.3 now shows that \( N/T = (N \cap K/T) \cdot (W/T) \) for some \( \pi \)-subgroup \( W/T \) of \( N/T \). Then \( W \) is a \( \pi \)-subgroup of \( N \) and as \( N = (N \cap K) \cdot W \), we have from (1) that \( U = KW \). It is now easy to see that \( W \in \text{Syl}_x G \), and in any case we may always ensure that this is the case by replacing \( W \) by a Sylow \( \pi \)-subgroup of \( G \) containing it.

We have now shown that every Sylow \( \pi \)-subgroup of \( G/K \) is the natural image of some Sylow \( \pi \)-subgroup of \( G \). In particular \( G \) possesses Sylow \( \pi \)-subgroups whose images modulo \( K \) belong to \( \text{Syl}_x G/K \); by conjugacy, it follows that every Sylow \( \pi \)-subgroup of \( G \) has that property.

The proof of Lemma 2.2 becomes much simpler if one assumes in addition that the Sylow \( \pi \)-subgroups of \( G/O_{p^\infty}(G) \) are countable. For it suffices, as above, to show that every Sylow \( \pi \)-subgroup \( U/K \) of \( G/K \) is the image of some Sylow \( \pi \)-subgroup of \( G \), and in doing this we may assume, by passing to \( G/O_{p^\infty}(G) \), that the Sylow \( \pi \)-subgroups of \( G \) are countable. Let \( S \in \text{Syl}_x U \) and suppose if possible that \( SK < U \). Then as \( S \) is countable there is a countable subgroup \( U^*/K \) of \( U/K \) with \( SK \subseteq U^* \subseteq U \). By well known results ([4] Lemma 2.1) there exists a \( \pi \)-subgroup \( T^* \) of \( U^* \) such that \( U^* = KT^* \), and if \( T \) is a Sylow \( \pi \)-subgroup of \( U \) containing \( T^* \) then we have \( SK < TK \leq U \). Since \( G \) is Sylow \( \pi \)-integrated we have that \( T = S_x \) for some \( x \in U \) and so \( SK/K \) and \( TK/K \) are conjugate in \( U/K \), contradicting the fact that a subgroup of a periodic group cannot be conjugate to a proper subgroup of itself.

We know of no situation in which, in a Sylow \( \pi \)-integrated group \( G \), the Sylow \( \pi \)-subgroups of \( G/O_{p^\infty}(G) \) are uncountable, and it is tempting to conjecture that this cannot occur. Some results in that direction are given in Corollary C1.

**Proofs of Theorems A and B.** In proving Theorem A, we may suppose that \( G \) contains a normal Sylow \( p \)-sparse subgroup \( N \) of finite index. Let \( H \) be a countable subgroup of \( G \). Then \( H \cap N \) is a normal Sylow \( p \)-sparse subgroup of finite index of \( H \). Thus \( \left| \text{Syl}_p H \cap N \right| < 2^{26} \). Now it can be seen from the argument of ([4] Theorem A1) that this implies that \( H \cap N \) is Sylow \( p \)-connected – it is not necessary to invoke the continuum hypothesis for this purpose as the statement of ([4] Theorem A1) might lead us to believe. Therefore by ([4] Theorem A3), \( H \) is Sylow \( p \)-connected. It follows that \( G \in \mathcal{L}_p \mathcal{D}^k_p \) and so by Lemma 2.1 \( G \in \mathcal{D}^k_p \), that is, \( G \) is Sylow \( p \)-integrated.

Theorem B follows from ([4] Theorem B) in the same way; the same remarks about the continuum hypothesis apply.

As an immediate corollary of Theorem B we have:
LEMMA (2.4). Suppose that $G$ contains a normal $\mathfrak{U}$-subgroup $N$ such that $G/N$ is finite and soluble. Then $G \in \mathfrak{U}$.

This follows since $\mathfrak{U}$-groups are $\pi$-separable for all sets $\pi$. Lemma 2.4 was established as ([4] Lemma 6.6), but the present approach is rather more direct.

3. Sylow $\pi$-sparse groups and $\pi$-separability

We begin with a slight extension of Theorem C of [4]. A group $G$ will be called upper $\pi$-separable if its upper $\pi$-series $\{P_\alpha\}$, defined by

$$P_0 = 1, P_{\alpha+1}/P_\alpha = O_{\pi^*}(G/P_\alpha), P_\mu = \bigcup_{\alpha < \mu} P_\alpha$$

for ordinals $\alpha$ and limit ordinals $\mu$, ultimately reaches $G$. The class of upper $\pi$-separable groups is $\mathfrak{S}$-closed, and if $G$ is any non-trivial upper $\pi$-separable group then either $O_{\pi}(G) \neq 1$ or $O_{\pi'}(G) \neq 1$. The following lemma can then be established by the argument of ([4] Lemma 5.4).

LEMMA (3.1). If $G$ is upper $\pi$-separable and $O_{\pi}(G) = 1$ then $O_{\pi^*}(G) \cong C_G(O_{\pi}(G))$.

Our extension of ([4] Theorem C) is the following:

THEOREM (C). Let $G$ be upper $\pi$-separable, locally $\pi$-soluble, and Sylow $\pi$-sparse. Then $|G : O_{\pi}(G)| < \infty$ and $O_{\pi^*}(G)/O_{\pi}(G) \in \mathfrak{S}_1$.

If furthermore $\pi$ is finite, then $|G : O_{\pi}(G)| < \infty$ and $O_{\pi^*}(G)/O_{\pi}(G)$ satisfies $\text{Min}$.

A finite group $X$ is called $\pi$-soluble, if every composition factor of $X$ is either a $\pi'$-group or a cyclic $\pi$-group.

Here $O_{\pi^*}(G) = O_{\pi,\pi}(G), O_{\pi}(G) = O_{\pi,\pi',\pi}(G)$, and so on. The increase in generality comes in the removal of the countability assumption and the replacement of ‘$\pi$-separable’ by ‘upper $\pi$-separable’. The argument given in [4] requires only a few modifications to establish the more general result, and since we require the more general theorem in the sequel we now briefly describe these modifications.

The following elementary fact will be useful:

LEMMA (3.2). Suppose that $G \in \mathfrak{L}_{\pi^*} \mathfrak{S}_\pi$, $G$ is Sylow $\pi$-sparse, and $K \triangleleft G$. Then $G/K$ is Sylow $\pi$-sparse.

PROOF. Let $H/K$ be a countable subgroup of $G/K$. Then $H = KL$ for some countable subgroup $L$ of $G$ and so $H/K \cong L/L \cap K$. By ([4] Lemma 2.1) every Sylow $\pi$-subgroup of $L/L \cap K$ is the natural image of
some Sylow $\pi$-subgroup of $L$ and so, since $|\text{Syl}_\pi L| < 2^{2n}$, we have $|\text{Syl}_\pi H/K| = |\text{Syl}_\pi L/L \cap K| < 2^{2n}$. Hence $G/K$ is Sylow $\pi$-sparse.

Next we obtain some information about locally $\pi$-soluble groups in which the abelian $\pi$-subgroups have finite rank.

**Lemma (3.3).** There exists a function $f$ from the set of non-negative integers into itself with the following property: if $\pi$ is a set of primes and $G$ is a locally $\pi$-soluble group in which, for $p \in \pi$, every abelian $p$-subgroup of $G$ has rank $\leq n$, then $|G : O_{\pi',\pi}(G)| \leq f(n)$.

**Proof.** We begin by considering a finite $\pi$-soluble group $G$ in which, for $p \in \pi$, every abelian $p$-subgroup of $G$ has rank not exceeding $n$. Let $p \in \pi$, let $P$ be any $p$-subgroup of $G$ and let $A$ be any maximal abelian normal subgroup of $P$. Then $A$ can be generated by $n$ elements. It is well known that $A = C_p(A)$; hence $P/A$ is isomorphic to a subgroup of $\text{Aut} A$ and so, by ([8] Lemma 5), $P/A$ can be generated by $\frac{1}{2}n(5n - 1)$ elements. Hence $P$ can be generated by $f_1(n) = n + \frac{1}{2}n(5n - 1)$ elements, and so, for $p \in \pi$, every $p$-subgroup of $G$ can be generated by $f_1(n)$ elements.

Let $U/V$ be any chief factor of $G$. Then $U/V$ is either a $\pi'$-group or an elementary abelian $p$-group with $p \in \pi$. In the latter case $U/V$ has rank at most $f_1(n)$ and so $G/C_G(U/V)$ can be viewed as an irreducible $\pi$-soluble group of linear transformations of a vector space of dimension at most $f_1(n)$ over $\mathbb{Z}_p$; also $p \in \pi$. It follows from the argument of ([4] Lemma 5.5) that $G/C_G(U/V)$ has an abelian normal subgroup of index bounded by a function of $f_1(n)$ alone and so there is a function $f_2(n)$ such that

$$(G^{f_2(n)})' \leq C_G(U/V), \text{ where } G^m = \{x^m : x \in G\}.$$  

This holds for every $\pi$-chief factor $U/V$ of $G$. Therefore, as $G$ is $\pi$-soluble, it follows that $(G^{f_2(n)})' \leq O_{\pi',\pi}(G)$ and hence that $G^{f_2(n)} \leq O_{\pi',\pi}(G)$.

Let $L = G/O_{\pi',\pi}(G)$. Then $L$ is $\pi$-soluble and of exponent dividing $f_2(n)$; also if $p \in \pi$, every $p$-subgroup of $L$ can be generated by $f_1(n)$ elements. Let $p$ be a prime divisor of $|L|$ belonging to $\pi$, let $S_p$ be a Sylow $p$-subgroup of $L$ and $A_p$ a maximal abelian normal subgroup of $S_p$. Then $|A_p| \leq f_2(n)^{f_3(n)}$ and since $A_p$ is self-centralizing in $S_p$ we have that $S_p/A_p$ is isomorphic to a subgroup of $\text{Aut} A_p$ and so $|S_p/A_p|$ is bounded in terms of $|A_p|$. Hence $|S_p| \leq f_3(n)$ for some suitable function $f_3$, and since this holds for every prime divisor of $|L|$ belonging to $\pi$, and every such prime divisor divides $f_2(n)$, it follows that there is a function $f_4$ such that every $\pi$-subgroup of $L$ has order at most $f_4(n)$. In particular $|O_{\pi}(L)| \leq f_4(n)$. Since $O_{\pi}(L) = 1$ we have that $O_{\pi}(L) \geq C_L(O_{\pi}(L))$ (Lemma 3.1) whence we obtain a function $f$ such that $|L| \leq f(n)$. Then $f$ has the property required by Lemma 3.3 for finite $\pi$-soluble groups $G$.

A standard inverse limit argument now gives the general result. Let
Let $G$ be an arbitrary locally $\pi$-soluble group in which, for $p \in \pi$, every abelian $p$-subgroup of $G$ has rank $\leq n$. Let $F$ be the set of all finite subgroups of $G$ and for each $F \in F$ let $\Sigma(F)$ denote the set of all normal subgroups $N$ of $F$ such that $|F : N| \leq f(n)$ and $N = O_{\pi', \pi}(N)$. By the first part of the discussion $\Sigma(F) \neq \emptyset$. If $F_1, F_2, \in F, F_1 \supseteq F_2$ and $N_1 \in \Sigma(F_1)$, then clearly $N_1 \cap F_2 \subseteq \Sigma(F_2)$; thus the sets $\Sigma(F)$ form an inverse system of finite non-empty sets indexed by the set $F$. The inverse limit of such a system is well known to be non-empty and so we may choose a family $\{N_F : F \in F\}$ such that $N_F \in \Sigma(F)$ and $N_{F_2} = N_{F_1} \cap F_2$ whenever $F_1 \supseteq F_2$. Then $N = \bigcup_{F \in F} N_F$ is a normal subgroup of $G$ of index at most $f(n)$ and $N = O_{\pi', \pi}(N)$ (cf. [1] Corollary 3.10). Hence $f$ has the properties required of it.

**Corollary (3.4).** Let $G$ be a locally $\pi$-soluble group in which every abelian $\pi$-subgroup has finite rank. Then $|G : O_{\pi', \pi}(G)| < \infty$.

**Proof.** Since every $\pi$-subgroup of $G$ is locally soluble, a theorem of Gor'kov [2] shows that every $\pi$-subgroup of $G$ has finite rank. By Lemma 3.3 it suffices to deduce from this that the ranks of the $\pi$-subgroups of $G$ are bounded. If this is not the case then there exists, for each integer $n > 0$, a finite $\pi$-subgroup $Q_n$ of $G$ of rank at least $n$. Let $F_n = \langle Q_1, \ldots, Q_n \rangle$ $(n \geq 1)$. Then $F_n$ is finite and $\pi$-soluble, and $F_1 \leq F_2 \leq \cdots$.

Let $P_1 \leq P_2 \leq \cdots$ be a tower with $P_i \in \text{Syl}_\pi F_i$. Then $P = \bigcup_{i=1}^\infty P_i$ is a $\pi$-subgroup of $G$. However, since finite $\pi$-soluble groups are Sylow $\pi$-connected ([3] Theorem 6.3.6), $Q_n$ is conjugate to a subgroup of $P_n$ and so $P_n$ has rank at least $n$. Therefore $P$ has infinite rank, a contradiction.

We now deal with a special case of Theorem C.

**Lemma 3.5.** Suppose that $G = HK$, $H \lhd G$, $H \cap K = 1$, where $H \in \mathcal{G}_\pi'$ and $K \in \mathcal{G}_\pi$. Suppose further that $C_K(H) = 1$, $K$ is locally soluble, and $G$ is Sylow $\pi$-sparse. Then $K \in \mathcal{R}_1$. If in addition $\pi$ is finite, then $K$ satisfies Min.

**Proof.** Let $K_1$ be any countable subgroup of $K$. For each $1 \neq k \in K$ there exists an element $h_k \in H$ such that $[h_k, k] \neq 1$. Then $H_1 = \langle h_k : 1 \neq k \in K_1 \rangle^{K_1}$ is a countable subgroup of $H$ normalized by $K_1$ and such that $C_{K_1}(H_1) = 1$. Since $G$ is Sylow $\pi$-sparse, we have that $|\text{Syl}_\pi H_1 K_1| < 2^{\aleph_0}$. The arguments used to prove ([4] Lemma 4.7) then yield without modification that $K_1 \in \mathcal{R}_1$. Therefore by ([4] Lemma 4.8) every countable subgroup of $K$ has finite rank. Hence $K$ itself has finite rank, and so is countable by a theorem of Kargapolov [7]. Hence $K \in \mathcal{R}_1$.

If $\pi$ is finite, then ([4] Lemma 4.8) shows that $K$ satisfies Min.

Now Theorem C can be established without difficulty.
Proof of Theorem C. Let $\bar{G} = G/\Omega_{\pi}(G)$. We show that every $\pi$-subgroup of $\bar{G}$ lies in $R_1$. In fact let $K$ be any $\pi$-subgroup of $\bar{G}$ and $H = O_{\pi}(\bar{G})$. Then by Lemma 3.1 $C_K(H) = 1$. Since finite $\pi$-soluble groups are Sylow $\pi$-integrated we have from Lemma 3.2 that $\bar{G}$, and hence $HK$, is Sylow $\pi$-sparse. Therefore by Lemma 3.5, $K \in R_1$. It follows from ([4] Lemma 4.8) that every $\pi$-subgroup of $\bar{G}$ has finite rank and hence Corollary 3.4 gives that $|\bar{G} : O_{\pi',\pi}(\bar{G})| < \infty$, that is $|G : O_{\pi}(G)| < \infty$. Furthermore we have by ([4] Lemma 2.1) that every countable subgroup of $O_{\pi',\pi}^{\prime}(\bar{G})/O_{\pi}(\bar{G})$ is the natural image of some $\pi$-subgroup of $\bar{G}$ and so has finite rank. Therefore $O_{\pi',\pi}^{\prime}(\bar{G})/O_{\pi}(\bar{G})$ has finite rank, and so is countable by Kargapolov's theorem [7]. Hence $O_{\pi',\pi}(\bar{G})$ splits over $O_{\pi}(G)$ and it follows that $O_{\pi',\pi}(\bar{G})/O_{\pi}(\bar{G}) \in R_1$. Thus $O_{\pi}(G)/O_{\pi}(G) = R_1$, and the rest of the theorem follows from [4] Lemma 4.8.

Corollary (C1). Let $G$ be upper $\pi$-separable, locally $\pi$-soluble and Sylow $\pi$-sparse. Then the Sylow $\pi$-subgroups of $G/O_{\pi}(G)$ are countable. If $\pi$ is finite then $G/O_{\pi',\pi}(G)$ is countable.

This observation is relevant to the remarks made after the proof of Lemma 2.2. We notice that the hypotheses of Corollary C1 do not imply that $G/O_{\pi,\pi}(G)$ is countable if $\pi$ is infinite. For let $p$ be a prime and let $q_1, q_2, \ldots$ be an infinite sequence of distinct primes satisfying $p|q_i - 1$ for all $i$. Such a sequence exists by Dirichlet’s theorem on the primes in an arithmetic progression. Let $Q = Q_1 \times Q_2 \times \ldots$ where $|Q_i| = q_i$ for each $i$. Then the automorphism group of $Q$ contains a subgroup $P$ which is isomorphic to the Cartesian or complete direct product of a countable infinity of cyclic groups of order $p$. Let $H$ be the semidirect product $QP$. Now $Q$ isomorphic to a subgroup $L$ of the multiplicative group of the algebraic closure $K$ of $\mathbb{Z}_p$. We can view the additive group of $K$ as a $\mathbb{Z}_p$-module by allowing $L$ to act by multiplication. Then every non-trivial element of $L$ acts fixed point freely. Therefore there is $\mathbb{Z}_p Q$-module $W$ such that $C^\pi_W(x) = 0$ for every $1 \neq x \in Q$, and if $V$ is the induced module $W^H$, then we also have $C^\pi_V(x) = 0$ for every $1 \neq x \in Q$, as $Q < H$.

Now let $G$ be the semidirect product $G = VH = VQP$, $V$ being written multiplicatively, and let $\pi = \{q_1, q_2, \ldots\}$. Any countable subgroup of $VQ$ lies in one of the form $V_0 Q$, where $V_0$ is a countable subgroup of $V$ normalized by $Q$. The fact that $C^\pi_{V_0}(x) = 1$ for all $1 \neq x \in Q$, combined with ([4] Lemma 4.3) shows that $V_0 Q$ is Sylow $\pi$-sparse. Hence $VQ$ is Sylow $\pi$-sparse, and since $G/VQ$ is a $\pi'$-group, $G$ is Sylow $\pi$-soluble (and so Sylow $\pi$-integrated by Theorem B). Obviously $G$ is $\pi$-separable and locally $\pi$-soluble; in fact $G$ is soluble of derived length three.

Now $C^\pi_H(Q) = Q$ and so every non-trivial normal subgroup of $H$ meets $Q$ non-trivially. Hence $O_{\pi}(H) = 1$ and also, since $C^\pi_Q(V) = 1$, we
have $C_H(V) = 1$. Therefore $C_G(V) = V$. It follows that $O_\pi(G) = 1$, $O^2_\pi(G) = V$ and $O^3_\pi(G) = VQ$. Thus we even have that $G/O^2_\pi(G)$ is uncountable.

We now consider Theorem D, which will be deduced from the following two results; these may be of some independent interest.

**Lemma (3.6).** Let $G$ be a countable and non-trivial locally $\pi$-soluble group. Suppose that $G$ is Sylow $\pi$-sparse and $O^\pi(G) = 1$. Then $G$ contains a non-trivial normal subgroup $H$ with the following property: if $\tau$ is a set of primes such that $\tau \cap \pi = \emptyset$ and $H$ is Sylow $\tau$-sparse, then every abelian $\tau$-subgroup of $H$ has finite rank.

Notice that by ([5] Theorem A) the hypotheses of Lemma 3.6 cannot be satisfied if $\pi = \{p\}$ and $p$ is odd, and it may be that they cannot be satisfied at all.

**Corollary (3.7).** With the hypotheses of Lemma 3.6, suppose further that $\tau$ is a set of primes such that $\tau \cap \pi = \emptyset$ and $G$ is locally $\tau$-soluble and Sylow $\tau$-sparse. Then $O^\tau(G) \neq 1$.

**Proof of Corollary 3.7.** Let $H$ be a normal subgroup of $G$ with the properties given by Lemma 3.6. Then since $G$ is Sylow $\tau$-sparse so is $H$. Therefore every abelian $\tau$-subgroup of $H$ has finite rank and so by Corollary 3.4 $H$ is $\tau$-separable. Since $H \neq 1$ this gives $O^\tau(H) \neq 1$, and since $H \triangleleft G$ this gives $O^\tau(G) \neq 1$. Thus Corollary 3.7 follows from Lemma 3.6.

**Proof of Theorem D.** We now deduce Theorem D from the foregoing results. We have that $G$ is locally soluble, $\sigma(G) = \pi_1 \cup \cdots \cup \pi_n$ is a partition of $\sigma(G)$ and $G$ is Sylow $\pi_i$-sparse for $1 \leq i \leq n$. With these hypotheses, we first prove

\[ (*) \quad G \text{ is } \pi_i\text{-separable for } 1 \leq i \leq n. \]

If this result does not hold then there exists a group $G$ which satisfies our hypotheses but is not $\pi_1$-separable, and we may suppose that $G$ is chosen to make $n$ as small as possible subject to these conditions. Then $n > 1$. Since $G$ is not $\pi_1$-separable the $\pi_1$-lengths of the finite subgroups of $G$ must be unbounded ([11] Corollary 3.10) and so $G$ contains a countable subgroup which is not $\pi_1$-separable. In obtaining a contradiction we may therefore suppose without loss of generality that $G$ is countable.

Let $X$ be the terminus of the upper $\pi_1$-series of $G$, as defined at the beginning of this section. Then $X$ is upper $\pi_1$-separable, locally soluble and Sylow $\pi_1$-sparse and so is $\pi_1$-separable by Theorem C. Hence $X < G$. By considering $G/X$ instead of $G$ and using Lemma 3.2, we may therefore suppose that $G \neq 1$ but $O^\pi_1(G) = 1$. It follows from Corol-
lary 3.7 that $O_{\pi_2 \pi_3}(G) = 1$. Consequently, since $O_{\pi_2}(G) \leq O_{\pi_1}(G) = 1$, we must have $R = O_{\pi_2}(G) \neq 1$. Now $R$ is Sylow $\pi$-sparse for $1 \leq i \leq n$, and $\pi_2 = \sigma_2 = \pi_1 \cup \pi_3 \cup \cdots \cup \pi_n$. Therefore by the choice of $n$, $R$ is $\pi$-separable. In particular $O_{\pi_1 \pi_1}(R) \neq 1$. But $R \triangleleft G$ and so we have $O_{\pi_1 \pi_1}(G) \neq 1$, a contradiction. This establishes (*).

Now the class $\mathcal{R}_1$ is evidently closed under finite extensions and so it now follows from Theorem C that, if $1 \leq i \leq n$, then the $\pi$-subgroups of $G/O_{\pi_i}(G)$ all belong to $\mathcal{R}_1$. Let $L = \prod_{i=1}^n O_{\pi_i}(G)$ and let $M/L$ be any locally nilpotent subgroup of $G/L$. Then $M/L = M_1/L \times \cdots \times M_n/L$, where $M_i/L$ is the Sylow $\pi_i$-subgroup of $M/L$. Let $X_i = O_{\pi_i \pi_i}(G)$. Then $O_{\pi_i}(G) \leq L \leq X_i$ and so $X_i/L$ is a $\pi_i$-group. Hence $M_i \cap X_i = L$ and so $M_i/L \cong M_i X_i/X_i$, a locally nilpotent $\pi_i$-subgroup of $G/X_i$. By Theorem C, $M_i X_i/X_i$ is abelian-by-finite and of finite rank. It follows that $M/L$ has these properties and so $G/L \in \mathcal{R}_1$.

In establishing (ii), we may suppose without loss of generality that $\pi = \pi_1 \cup \cdots \cup \pi_j (1 \leq j \leq n)$. We have to prove that $G \in \mathfrak{D}_\pi$ and by Lemma 2.1 we may assume that $G$ is countable; also since the hypotheses on $G$ are inherited by subgroups it suffices to prove that $G \in \mathfrak{D}_\pi$. It follows from (*) that $G$ is $\pi$-separable and so $G$ has a finite series of normal subgroups in which each factor is either a $\pi$-group or a $\pi'$-group. By using Lemma 3.2 and induction on the length of such a series we may assume that $G$ contains a normal subgroup $N$ which is either a $\pi$-group or a $\pi'$-group and is such that $G/N \in \mathfrak{D}_\pi$. If $N$ is a $\pi$-group it is then immediate that $G \in \mathfrak{D}_\pi$. Otherwise let $S, T \in \text{Syl}_\pi(G)$. Then as $G/N$ is Sylow $\pi$-connected there is an element $x \in G$ such that $\langle S^x N/N, TN/N \rangle = U/N$ is a $\pi$-group. Since $U/N$ is countable there is a $\pi$-subgroup $V$ of $U$ such that $U = NV$, $N \cap V = 1$. Let $Y$ be any subgroup of $V$. Then since $Y$ is countable and locally soluble we have $Y = \langle Y_1, \ldots, Y_j \rangle$, where $Y_i$ is a $\pi_i$-group. Now $NY_i$ is Sylow $\pi_i$-sparse by hypothesis, and so by the argument of ([4] Lemma 4.2) there is a finite subgroup $F_i$ of $Y_i$ such that $C_N(Y_i) = C_N(F_i)$. Then $C_N(Y) = \bigcap_{i=1}^n C_N(Y_i) = \bigcap_{i=1}^n C_N(F_i) = C_N(F)$, where $F = \langle F_1, \cdots, F_n \rangle$. Since $F$ is finite it now follows from ([4] Lemma 4.3) that $U = NV$ is Sylow $\pi$-connected. Therefore $S^x$ and $T$, which are Sylow $\pi$-subgroups of $U$, are conjugate in $U$. This completes the deduction of Theorem D from Lemma 3.6.

Proofs of Corollaries D1 and D2. To obtain Corollary D1 we apply Theorem D to the partition of $\sigma(G)$ into one-element sets. If $\sigma(G) = \{ p_1, \cdots, p_n \}$ then evidently $\rho(G) = \prod_{i=1}^n O_{p_i}(G)$; also by ([4] Lemma 4.8) an $\mathcal{R}_1$-group $X$ such that $\sigma(X)$ is finite satisfies Min.

If $G$ is Sylow $p$- and $p'$-sparse then Theorem D gives that $G$ is $p$-separable. The desired conclusion then follows from Theorem C.
It remains to establish Lemma 3.6. In order to do this we require one or two technical results, the first of which is similar to ([4] Lemma 3.2).

**LEMMA (3.8).** Let $G$ be Sylow $\pi$-sparse and let $S$ be a $\pi$-subgroup of $G$. Then there exists a finite subgroup $F$ of $S$ such that if $x \in G$ and $F^x \leq S$, then $\langle S, S^x \rangle \in \mathcal{D}_\pi$.

**PROOF.** Suppose that $G$ is a group containing a $\pi$-subgroup $S$ to which no such $F$ corresponds. We shall exhibit a countable subgroup $H$ of $G$ such that $|\text{Syl}_\pi H| = 2^{\aleph_0}$. To do this we recursively construct finite subgroups $F_0, F_1, \ldots$ of $G$ and sets $\Sigma_n (n \geq 0)$ of $2^n \pi$-subgroups of $F_n$ such that:

(i) Each member of $\Sigma_n$ has the form $S^x \cap F_n$ with $x \in F_n$.

(ii) No two distinct members of $\Sigma_n$ generate a $\pi$-group.

(iii) Each member of $\Sigma_{n-1}$ is contained in two distinct members of $\Sigma_n (n > 0)$.

To begin the construction put $F_0 = 1$, $\Sigma_0 = \{1\}$. Assume that we have obtained $\Sigma_m$, and let $F_{m+1}$ consist of the subgroups $S^{x_i} \cap F_{m+1}$ and $S^{x_j} \cap F_{m+1}$ ($1 \leq i \leq 2^m$). Now the subgroup $S \cap F_m$ does not have the property of $F$ in the statement of the lemma, and so there is an element $y \in G$ such that $S \cap F_m y^{-1} \leq S$ but $\langle S, S^y \rangle$ is not a $\pi$-group. Therefore $\langle S, S^y \rangle$ is not a $\pi$-group, and so there is a finite subgroup $F_{m+1} \leq \langle F_m, y \rangle$ such that

$$\langle S \cap F_{m+1}, S^y \cap F_{m+1} \rangle \notin \mathcal{D}_\pi.$$ 

Let $\Sigma_{m+1}$ consist of the subgroups $S^{x_i} \cap F_{m+1}$ and $S^{x_j} \cap F_{m+1}$ ($1 \leq i \leq 2^m$). Then (i) holds. Now since $(S \cap F_m y^{-1}) \leq S$ and $y \in F_{m+1}$ we have $(S \cap F_m y^{-1}) \leq S \cap F_{m+1}$, whence $S \cap F_m \leq (S \cap F_{m+1}) y = S^y \cap F_{m+1}$. Therefore, as $x_i, y \in F_m$, $S^{x_i} \cap F_m = (S \cap F_m)^{x_i}$ is contained in both $S^{x_i} \cap F_{m+1}$ and $S^{x_j} \cap F_{m+1}$. Therefore by (ii) with $n = m$, the group generated by $S^{x_i} \cap F_{m+1}$ and either of $S^{x_j} \cap F_{m+1}$ and $S^{x_j} \cap F_{m+1}$ ($j \neq i$) is not a $\pi$-group. Nor is $\langle S^{x_i} \cap F_{m+1}, S^{x_j} \cap F_{m+1} \rangle = \langle S \cap F_{m+1}, S^{y} \cap F_{m+1} \rangle$, by (1). Therefore (ii) holds with $n = m+1$. In particular the subgroups $S^{x_i} \cap F_{m+1}$ and $S^{x_j} \cap F_{m+1}$ are distinct, and so (iii) holds by the remarks above.

Now let $H = \bigcup_{i=0}^{\aleph_0} F_i$, a countable subgroup of $G$. If $S_0 \leq S_1 \leq \cdots$ is a tower with $S_i \in \Sigma_i$ then $\bigcup_{i=0}^{\aleph_0} S_i$ is a $\pi$-subgroup of $G$. By (iii) there are $2^{\aleph_0}$ such towers, and by (ii), two subgroups obtained as the unions of distinct towers cannot generate a $\pi$-group. Hence $H$ has a set $\Sigma$ of $2^{\aleph_0}$ $\pi$-subgroups, no two of which generate a $\pi$-group. Each member of $\Sigma$ lies
in some Sylow $\pi$-subgroup of $H$ and the Sylow $\pi$-subgroups obtained thus are all distinct. Hence $|\text{Syl}_\pi H| = 2^{\kappa_0}$, as required.

The next lemma is essentially the key step in establishing Lemma 3.6 and Theorem D.

**Lemma (3.9).** Let $1 = F_0 \leq F_1 \leq \cdots$ be a tower of finite subgroups of a group $G = \bigcup_{i=0}^{\infty} F_i$. Let $A$ be an abelian $\pi$-subgroup of $G$ and for $i = 0, 1, \ldots$ let $R_i$ be a $\pi'$-subgroup of $G$ normalized by $A_i = A \cap F_i$. Assume that the ranks of the groups $A_i/C_A(R_i)$ are unbounded. Then $|\text{Syl}_\pi G| = 2^{\kappa_0}$.

**Proof.** We begin with some preliminary remarks. Suppose that $l \geq 0$ and $D \leq A_l$. For each $i \geq 0$ let $S_i$ be a subgroup of $R_i$ normalized by $A_i$. Then since $A_i$ centralizes $D$, $A_i$ normalizes $C_{S_i}(D) = S_i \cap C_G(D)$.

Suppose further that $A_i$ contains subgroups $E$ such that the groups $A_i/C_A(C_{S_i}(E))$ ($i \geq 0$) have unbounded ranks and that $D$ is chosen maximal among these subgroups. We then claim that

$$A_i/D \text{ is cyclic.}$$

In fact, suppose that $A_i = A_i/D$ is not cyclic. Then $A_i$ acts naturally on $C_{S_i}(D)$ if $i \geq l$ and we have from ([4] Lemma 4.4) that $C_{S_i}(D)$ is generated by the centralizers in it of the non-trivial elements of $A_i$. Therefore

$$C_{S_i}(D) = \langle C_{S_i}(D^*) \rangle; \ D < D^* \leq A_i \ (i \geq l).$$

Now the subgroups $C_{S_i}(D^*)$ are $A_i$-invariant if $i \geq l$ and it follows from (3) that $C_{A_i}(C_{S_i}(D)) = \cap C_{A_i}(C_{S_i}(D^*))$, the intersection running over subgroups $D^*$ with $D < D^* \leq A_i$. Therefore $A_i/C_{A_i}(C_{S_i}(D))$ is a subdirect product of the groups $A_i/C_{A_i}(C_{S_i}(D*))$ with $D < D^* \leq A_i$.

Since there are only finitely many such subgroups $D^*$, and since the groups $A_i/C_{A_i}(C_{S_i}(D))$ ($i \geq l$) have unbounded ranks, there must be such a $D^*$ for which the groups $A_i/C_{A_i}(C_{S_i}(D*))$ ($i \geq l$) have unbounded ranks, contradicting the choice of $D$.

We now recursively construct elements $1 = x_0, x_1, \ldots$ of $A$ and integers $n_0, n_1, \ldots$ such that, if $B_j = \langle x_0, \ldots, x_j \rangle$ and $R_{ij} = C_{R_j}(B_j)$, then

(i) $B_j \leq A_{n_j} \ (j \geq 0)$.
(ii) $R_{nj, j-1} > R_{nj, j} \ (j > 0)$.
(iii) For each $j \geq 0$, the groups $A_i/C_A(R_{ij}) \ (i > 0)$ have unbounded ranks.

We begin by putting $x_0 = 1, n_0 = 0$. Then (i) holds. We have $R_{i, 0} = R_i$, and so (iii) holds with $j = 0$ by hypothesis.

Assume now that we have $x_0, \ldots, x_k$ and $n_0, \ldots, n_k$ such that (i)–
(iii) hold for $j \leq k$. Then in particular we have from (iii) that there exists an integer $n_{k+1} \geq n_k$ such that

$$A_{n_{k+1}}/C_{A_{n_{k+1}}}(R_{n_{k+1},k}) \text{ has rank } \geq 2.$$ 

Now by (iii) with $j = k$, we can choose a subgroup $D$ of $A_{n_k+1}$ maximal subject to the condition that the groups $A_j/C_{A_j}(R_{i,j,k}(D))$ have unbounded ranks. It follows from (2) and (4) that $D$ does not centralize $R_{n_k+1,k}$. Let $x_{k+1}$ be an element of $D$ which does not centralize $R_{n_k+1,k}$ and let $B_{k+1} = \langle B_k, x_{k+1} \rangle$. Then (i) holds with $j = k+1$, and since $R_{n_k+1,k+1}$ is just the centralizer in $R_{n_k+1,k}$ of $x_{k+1}$, (ii) holds by the choice of $x_{k+1}$. Finally $R_{i,k+1} = C_{R_{i,k}}(x_{k+1}) \geq C_{R_{i,k}}(D)$ and so (iii) holds with $j = k+1$, by the choice of $D$. Thus the construction can be carried out.

Let $S = \langle x_0, x_1, \ldots \rangle$. We apply Lemma 3.8 to $S$. Let $F$ be any finite subgroup of $S$. Then $F \leq B_{j-1}$ for suitable $j$, and by (ii) there exists an element $x \in R = R_{ij}$ centralized by $B_{j-1}$ but not by $B_j$. In particular $F^x \leq S$. On the other hand $\langle S, S^x \rangle$ contains $\langle B_j, B_j^x \rangle$, which contains $[x, B_j]$. This is a non-trivial subgroup of the $\pi'$-group $R$, since $B_j$ normalizes $R$ by (i). Therefore $\langle S, S^x \rangle$ is not a $\pi$-group, and so no finite subgroup of $S$ has the property of $F$ in Lemma 3.7. Therefore $|S_{\pi}H| = 2^{\aleph_0}$ for some subgroup $H$ of $G$, and so $|S_{\pi}G| = 2^{\aleph_0}$.

**Proof of Lemma 3.6.** We have a countable and nontrivial locally $\pi$-soluble group $G$ with $|\text{Syl}_\pi G| < 2^{\aleph_0}$ and $O_{\pi'}(G) = 1$. We begin by constructing finite subgroups $1 = F_0 \leq F_1 \leq \cdots$ of $G$ such that $G = \bigcup_{i=0}^{\infty} F_i$ and $F_i \cap O_{\pi'}(F_{i+1}) = 1$. Let $1 = x_0, x_1, \ldots$ be the elements of $G$ and suppose that $F_i$ is a finite subgroup of $G$ containing $\langle x_0, \ldots, x_i \rangle$. Since $O_{\pi'}(G) = 1$ we have that, for each $1 \neq x \in O_{\pi'}(F_i)$, $x^G \not\in \Sigma_x \cap \Sigma_x$. Therefore there is a finite subgroup $G_x$ of $G$ such that $x^{G_x} \not\in \Sigma_x \cap \Sigma_x$. Put $F_{n+1} = \langle x_{n+1}, F_n, G_x; 1 \neq x \in O_{\pi'}(F_n) \rangle$. Then $F_{n+1} \supseteq \langle x_0, \ldots, x_{n+1} \rangle$ and $F_n \cap O_{\pi'}(F_{n+1}) = 1$.

Since $G$ is locally $\pi$-soluble but not $\pi$-separable it follows from Corollary 3.4 that $G$ contains an abelian $\pi$-subgroup $A$ of infinite rank. Let $R_i = O_{\pi'}(F_i)$ ($i \geq 0$). Then $A_i = A \cap F_i$ normalizes $R_i$ for each $i$, and it follows from Lemma 3.9 that the ranks of the groups $A_i/C_{A_i}(R_i)$ are bounded by some integer $n$. Let $X$ be a finite subgroup of rank $n+1$ of $A$. Then for each sufficiently large $i$ there is a non-trivial element of $X$ which centralizes $R_i$. Since $X$ is finite it follows that there is an element $1 \neq x \in X$ which centralizes infinitely many of the subgroups $R_i$. Since $O_{\pi'}(F_j) \cap F_i = 1$ whenever $j > i$ we may assume, by omitting some of the $F_i$ and renumbering the rest, that $x \in F_1$ and $x$ centralizes every $R_i$.

Now let $H = x^G$. Then $1 \neq H \lhd G$ and $H = \bigcup_{i=0}^{\infty} H_i$, where $H_i = x^{F_i}$. 

Now since $R_i \preceq F_i$ we have that $x^{F_i} = H_i$ centralizes $R_i$, and so $S_i = H_i \cap R_i$ is central in $H_i$. Since $H_i \preceq F_i$ we have $S_i = O_{\pi}(H_i)$, and it follows that $O_{\pi}(H_i) = S_1 \times T_1$ where $T_1 = O_{\pi}(H_i)$. Since $H_i$ is $\pi$-separable it now follows from Lemma 3.1 that $C_{H_i}(T_i) \leq S_i \times T_i$. Now let $\tau$ be a set of primes with $\tau \cap \pi = \emptyset$ and suppose that $H$ contains an abelian $\tau$-subgroup $B$ of infinite rank. Now $H_i \cap O_{\pi}(H_{i+1}) \leq F_i \cap O_{\pi}(F_{i+1}) = 1$ and so if $B_i = B \cap H_i$ then $B_i \cap (S_{i+1} \times T_{i+1}) = 1$. Therefore $B_i$ normalizes the $\tau'$-subgroup $T_{i+1}$ and acts faithfully on it. Since the groups $B_i$ have unbounded ranks, Lemma 3.9 shows that $|Syl_H| = 2^{\aleph_0}$, so that $H$ is not Sylow $\tau$-sparse. This establishes Lemma 3.6, and completes the proof of Theorem D.

4. The class $I1$

The following result is immediate from Corollary D1:

**Lemma (4.1).** Suppose that $G$ is locally soluble, that $\sigma(G)$ is finite, and that $G$ is Sylow $p$-sparse for all $p \in \sigma(G)$. Then $G \in I1$.

It is easy to see, by considering suitable direct products of finite groups, that in general a locally soluble group $G$ may be Sylow $p$-sparse for all $p \in \sigma(G)$ without belonging to $I1$. However, we now prove Theorem E, according to which if such a $G$ is completely Sylow sparse, then $G \in I1$.

**Proof of Theorem E.** Let $G$ be a locally soluble group. If $G \in I1$ then $G$ is completely Sylow integrated and so completely Sylow sparse, and so we assume $G$ completely Sylow sparse in order to prove the converse.

We recall that a group $H$ is called a radical group (in the sense of Plotkin [9]) if the upper $LR$-series of $H$, defined by $R_0 = 1$, $R_{\alpha+1}/R_{\alpha} = \rho(G/R_{\alpha})$, $R_{\alpha} = \bigcup_{\mu < \alpha} R_{\mu}$ for ordinals $\alpha$ and limit ordinals $\mu$, reaches $H$. It is well known and easy to prove that if $H$ is a radical group then $R_1 \geq C_H(R_1)$ ([9] Theorem 7); hence if $R_1$ is finite, so is $H$. Suppose now that $H$ is a completely Sylow sparse radical group. Then $R_4$ is obviously $\pi$-separable for any set $\pi$, and so Theorem B gives that $R_4$ is completely Sylow integrated. Hence $R_4 \in I1$ and so, by ([4] Theorem E), $R_4/R_3$ is finite. It follows from the above remarks that $H/R_3$ is finite, and so $R_n = H$ for some finite $n$. This gives $H \in I1$. After that brief digression we now return to the proof of Theorem E and take $H$ to be the terminus of the upper $LR$-series of our completely Sylow sparse group $G$. Then by our digression it suffices to prove $H = G$. We therefore assume $H < G$; then $L = G/H \neq 1$ but $\rho(L) = 1$. Also $L$ is completely Sylow sparse and locally soluble. Therefore by Corollary D2 and the fact that $O_{\rho}(L) \leq \rho(L)$, we have that $H$ satisfies Min-$p$ for all primes $p$. Let $M$ be the subgroup of
L generated by the radicable abelian subgroups of L. Then M has no proper subgroup of finite index and by local solubility and Min-p, every chief factor of M is a finite elementary abelian group. Therefore every chief factor of M is central and M is locally nilpotent. Since M ≤ G we must have M = 1, and since every p-subgroup of L contains a radicable abelian subgroup of finite index it follows that the Sylow p-subgroups of L are finite for each prime p.

Now L contains an abelian subgroup A of infinite rank, otherwise Gorčakov's theorem [2] shows that L has finite rank and a theorem of Kargapolov [7] gives that L ∈ (ⅨR)Ω defenses, contradicting the fact that L ≠ 1 but ρ(L) = 1. We may suppose that A is countably infinite, and by arguing as in the proof of Lemma 3.6 we can then construct a tower 1 = F₀ ≤ F₁ ≤ ⋅⋅⋅ of finite subgroups of L such that A ≤ ∪₀≤i∞ Fᵢ = K and Fᵢ ∩ ρ(Fᵢ₊₁) = 1. Here, of course, ρ(Fᵢ) is the Fitting subgroup of Fᵢ.

Let i > 0 be a given integer and σᵢ, πᵢ be given finite sets of primes with σᵢ ∩ πᵢ = ∅. Then since the Sylow p-subgroups of A are finite for each prime p but A has infinite rank there exists a prime q not belonging to σᵢ ∪ πᵢ such that the Sylow q-subgroup Aᵢ of A has rank at least i. Furthermore Aᵢ ≤ Fₙᵢ for some nᵢ, which we may choose sufficiently large to ensure that Fₙᵢ contains a Sylow σᵢ ∪ πᵢ ∪ {q}-subgroup of K. Then Aᵢ normalizes Rᵢ = ρ(Fₙᵢ₊₁), and since Fᵢ ∩ Rᵢ = 1 we have that Cₐᵢₐ(Rᵢ) = 1 and that no prime in σᵢ ∪ πᵢ ∪ {q} divides |Rᵢ|. We put πᵢ₊₁ = πᵢ ∪ {q}, σᵢ₊₁ = σᵢ ∪ σ(Rᵢ) and continue in this way. Finally let π = ∪₀≤i∞ πᵢ, σ = ∪₀≤i∞ σᵢ, and let B be the Sylow π-subgroup of A. Then we have that Bᵢ = B ∩ Fₙᵢ normalizes Rᵢ and that the rank of Bᵢ/Cᵢ(Bᵢ₋₁(Rᵢ)) is at least i. Also σ(Rᵢ) ≤ σ for all i, and σ ∩ π = ∅. Therefore by Lemma 3.9, |Sylₓ K| = 2⁸₀. However, K is a countable subgroup of L, which is completely Sylow sparse, and so we have a contradiction. Thus Theorem E is established.

We conclude with the following remark, which is independent of what goes before.

Lemma (4.2). Π ≤ (ⅨR)Ω₁.

Proof. Let R = ρ(G) and let A/R be a locally nilpotent subgroup of G/R. By ([4] Theorem E), A/R has finite rank, and so we only need show that A/R is abelian-by-finite. Since G/R ∈ Ω₂ defense (by ([4] Theorem E)), we may evidently suppose that G/R ∈ Ω₂. Then A' ≤ R₂, the second term of the upper Ω₂-series of G. For each prime p let Sₚ be a Sylow p-subgroup of A. Now A ∈ (ⅨR)² ∩ Π and so by ([4] Lemma 6.4), A/ρ(A) ∈ Ω₁. Hence the Sylow p-subgroup of A/ρ(A) is abelian for all but finitely many
primes $p$, and so there is a finite set $\sigma$ of primes such that $S'_p \leq \rho(A)$ if $p \notin \sigma$. Since $\rho(A) \supseteq R$, we have $[S'_p, R_p] = 1$ if $p \notin \sigma$, where $R_p'$ is the Sylow $p'$-subgroup of $R$. However $S'_p \leq R_2$, and by ([4] Lemma 6.3) we have that $C_{S_p \cap R_2}(R_p') \leq \rho(R_2) = R$. Therefore $S'_p \leq R$ if $p \notin \sigma$, and hence the Sylow $p$-subgroup $S_p R/R$ of $A/R$ is abelian if $p \notin \sigma$. Therefore $A/R$ is abelian-by-finite, and the lemma is proved.

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