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EXTENSION PATHOLOGY IN REGRESSIVE ISOLS *

by

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1. Introduction

Let E, Λ and Λ_R denote the collections of all non-negative integers, isols and regressive isols respectively. Let f be a recursive function from E^n to E and Q a recursive n-ary relation. We denote the extensions of f and Q to Λ by f_Λ and Q_Λ, and write Q_R for $Q_\Lambda \cap \Lambda_R^n$. A. Nerode has shown in [16] that if $f(x)$ is a recursive combinatorial function of one variable $f(E)_\Lambda = f_\Lambda(\Lambda)$ if and only if $f(E)$ differs from an arithmetic progression by at most finitely many elements. J. Barback has shown in [4] that $f(E)_R = f_\Lambda(\Lambda_R)$ for any eventually increasing recursive function f of one variable. The latter result would seem to imply that the arithmetical structure of Λ_R is considerably nicer than that of Λ. The main theorems of this paper show that this is not the case for regressive arithmetic involving functions of more than one variable. Techniques are developed which enable one to produce isols in $f(E^n)_R - (f_\Lambda(\Lambda_R^n) \cap \Lambda_R)$ for a large class of recursive functions f. In the case of relations with simple arithmetical definitions, these techniques can be used to produce regressive isols belonging to the canonical extension of a relation but not satisfying the usual arithmetical definition – e.g., a prime regressive isol belonging to the extension of $\{x \in E | x \text{ is composite}\}$.

A precise statement of the main theorem requires the following definitions: Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ be elements of E^n. $x \leqq y$ if $x_i \leqq y_i$ for $1 \leqq i \leqq n$, and $x < y$ if $x \leqq y$ and $x \neq y$. A function f from E^n to E is called *strictly increasing* if $x < y \Rightarrow f(x) < f(y)$. f is called *almost strictly increasing* (a.s.i.) if there is a number M such that $f(x_1 + M, \cdots, x_n + M)$ is strictly increasing and for each fixed $k < M$, each of the functions $f(k, x_2, \cdots, x_n), f(x_1, k, x_2, \cdots, x_n), \cdots, f(x_1, \cdots, x_{n-1}, k)$ is a.s.i. (This inductive definition reduces in the case $n = 1$ to the usual definition of an eventually strictly increasing function.) Let α and β be subsets of E. We write $\alpha =_e \beta$ to indicate that $(\alpha - \beta) \cup (\beta - \alpha)$ is finite. The main result is the following.

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THEOREM. *Let $f(x_1, \cdots, x_n)$ be recursive and a.s.i. Then $f_A(\Lambda_R^n) \cap \Lambda_R = f(E^n)_R$ if and only if there are numbers j and $k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_n$ such that $f(E^n) =_e g(E)$, where $g(x_j) = f(k_1, \cdots, k_{j-1}, x_j, k_{j+1}, \ldots, k_n)$.*

For the sake of simplicity, we shall prove this theorem only for the case $n = 2$; the general proof is completely similar. The following sections also include various applications of the main theorem and a discussion of why the class of a.s.i. functions is appropriate for study here. Again for the sake of simplicity, this material will deal almost entirely with functions of two variables.

2. Preliminaries

We shall assume in the following that the reader is familiar with the concepts and main results of the papers listed as references. Of particular importance are [4], [11] and [12]. Let $g(x)$ and $f(x, y)$ be recursive functions. We use the notations Dg and Df of [12] for the difference functions of g and f (which have the property that for $T, U \in \Lambda_R$, $g_A(T) = \sum_{T+1}^{*} Dg$ and $f_A(T, U) = \sum_{T+1, U+1}^{*} Df$.) We will use $j(x, y)$ and $j(x, y, z)$ to denote the standard recursive one-to-one maps of E^2 and E^3 onto E used in [1]–[12]. We will denote the associated projection functions of $j(x, y)$ by $k(x)$ and $l(x)$ and the associated projection functions of $i(x, y, z)$ by $k_1(x)$, $k_2(x)$ and $k_3(x)$. For any set $\alpha \subseteq E$, we will denote the function which enumerates the elements of α in increasing order by $h_\alpha(x)$. If f is a function, we denote the domain and range of f by δf and ρf respectively. We denote the finite set $\{0, 1, \cdots, n-1\}$ by $v(n)$ or v_n.

We shall make use of some results which have appeared in print only recently (or not at all). In [7], the authors use the notation δ_Σ to denote the collection $\{\Sigma_A Dh_\delta, A \in \Lambda_R, A \geqq 1, A \leqq_* Dh_\delta\}$. The theorem of [7] which is most important to us here is Theorem 3, which states that if α is an r.e. set, $\alpha_R = \bigcup_{\delta \subseteq \alpha} \delta_\Sigma$. The counter-examples needed to obtain our main theorem are obtained using T-regressive isols, which were first defined and studied in [11]. The following lemma of J. Gersting does not appear in [11]; it states a basic property of sums over T-regressive isols in a form which is quite useful to us.

LEMMA T. *Let U be a T-regressive isol. Suppose that $\Sigma_V g(n) = \Sigma_U a(n)$, where $V \leqq_* g(n)$ and $U \leqq_* a(n)$. Then there exist a number k and a strictly increasing function $h(n)$ such that $a(0)+\cdots+a(k) = g(0)+\cdots+g(h(0))$ and $a(n+1) = g(h(n)+1)+\cdots+g(h(n+1))$.*

3. Almost strictly increasing functions of two variables

The question "*Is $f(E, E)_R = f_A(\Lambda_R, \Lambda_R)$?*" is non-trivial only for functions $f(x, y)$ which map Λ_R^2 into Λ_R, since $f(E, E)_R \subseteq \Lambda_R$. However, the class of recursive functions mapping Λ_R^2 into Λ_R is quite limited (cf. [14]). Thus it is of more interest to answer the question "Is $f(E, E)_R = f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$?" for some wider class of recursive functions f. The results of [12] indicate that the latter question might be appropriate for the class of all almost increasing recursive functions. However, the technique used here to construct isols belonging to $f(E, E)_R - (f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R)$ involves consideration of isols of the form $\Sigma_T\, Dh_{\rho f}$, where $T \in \Lambda_R$ and $T \leqq_* Dh_{\rho f}$. To assure that $T \leqq_* Dh_{\rho f}$, we require that $f(E, E)$ be recursive. The reader can easily verify that this is the case if f is a.s.i. recursive. The following example shows that $f(E, E)$ need not be recursive when f is almost increasing recursive: Let α be an r.e. set which is not recursive and $a(x)$ a one-to-one recursive function ranging over α. Define $f(n, k)$ by

$$f(n, k) = \begin{cases} 2n, & n \notin \{a(0), \cdots, a(k)\} \\ 2n+1, & n \in \{a(0), \cdots, a(k)\}. \end{cases}$$

f is clearly recursive and increasing. However, ρf cannot be recursive since $n \in \alpha$ iff $2n+1 \in \rho f$.

4. Summation representation of functions of isols

In [5] the notation $T \leqq_* a_n$ is applied to infinite regressive isols T and total functions a_n to indicate that there is a regressive function t_n ranging over a set in T such that $t_n \leqq_* a_n$. In the statement of the following theorem it is convenient to extend the use of this notation as follows: If T is a finite regressive isol and a_n a partial function, we write $T \leqq_* a_n$ when the domain of a is the set $\{0, 1, \cdots, T-1\}$.

THEOREM 1. *Let $f(x, y)$ be a recursive function of two variables. Let $A, B \in \Lambda_R$ and $f_\Lambda(A, B) \in \Lambda_R$. Then there exist functions $c(n)$, $s(n)$ and $w(n)$ with identical domains and a regressive isol T such that*

(1) $T \leqq_* c(n)$, $T \leqq_* s(n)$ *and* $T \leqq_* w(n)$,
(2) $f_\Lambda(A, B) = \Sigma_T c(n)$,
(3) $\sum_{j=0}^{n} c(j) = f(s(n), w(n))$ *for* $n \in \delta c$,
(4) $(s(n), w(n)) < (s(n+1), w(n+1))$ *for* $n+1 \in \delta s$,
(5) $c(n) > 0$ *for all* $n \in \delta c$.

PROOF. Let a_n and b_n be retraceable functions ranging over sets α and β belonging to the regressive isols $A+1$ and $B+1$ respectively. Let

$$\theta^+ = \bigcup_{(i,j) \in E^2} \{j(a_i, b_j, k) | k < Df^+(i, j)\}$$
$$\theta^- = \bigcup_{(i,j) \in E^2} \{j(a_i, b_j, k) | k < Df^-(i, j)\}.$$

Since $\Sigma_{A+1,B+1} Df^{+} - \Sigma_{A+1,B+1} Df^{-} = f_A(A, B)$ and $f_A(A, B) \in \Lambda_R$, there is a one-to-one partial recursive function $q(x)$ such that $\theta^{-} \subset \delta_q$, $q(\theta^{-}) \subset \theta^{+}$ and $q(\theta^{-}) | \theta^{+} - q(\theta^{-})$. We denote $\theta^{+} - q(\theta^{-})$ by γ and note that Req $(\gamma) = f_A(A, B)$. Let g_n be a regressive function ranging over γ. Our proof requires the following definitions and notations. Let η be a finite subset of γ, say $\eta = g_{n(0)}, \cdots, g_{n(k)}$ where $n(0) < n(1) < \cdots < n(k)$. We will use $\bar{\eta}$ to denote the set $\{g_0, g_1, g_2, \cdots, g_{n(k)}\}$. Let ζ be a finite subset of $\theta^{+} \cup \theta^{-}$, and let $\{k_1(x) | x \in \zeta\} = \{a_{i(1)}, a_{i(2)}, \cdots, a_{i(s)}\}$, $i(1) < i(2) < \cdots < i(s)$ and $\{k_2(x) | x \in \zeta\} = \{b_{j(1)}, \cdots, b_{j(t)}\}$, $j(1) < j(2) < \cdots < j(t)$. We define

$$P(\zeta) = \{j(a_i, b_j, k) | i \leqq i(s) \;\&\; j \leqq j(t) \;\&\; k < Df^{+}(i, j)\}$$
$$N(\zeta) = \{j(a_i, b_j, k) | i \leqq i(s) \;\&\; j \leqq j(t) \;\&\; k < Df^{-}(i, j)\}$$
$$W(\zeta) = P(\zeta) \cup N(\zeta) \cup \overline{(P(\zeta) \cap \gamma)} \cup qN(\zeta) \cap q^{-1}(P(\zeta) - \gamma)$$
$$B(\zeta) = \bigcup_{n=1}^{\infty} W^{n}(\zeta).$$

Since α and β are isolated, $B(\zeta)$ is finite and can be obtained effectively if ζ is given. It is readily shown that

(6) $B(\zeta) = PB(\zeta) \cup NB(\zeta)$,

(7) $qNB(\zeta) \subseteq PB(\zeta) - \gamma$,

(8) $q^{-1}(PB(\zeta) - \gamma) \subseteq NB(\zeta)$,

(9) $B(\zeta) \cap \gamma = \overline{B(\zeta) \cap \gamma}$.

Combining (7) and (8) we obtain

(10) $qNB(\zeta) = PB(\zeta) - \gamma$.

It follows readily from (6) that there exist numbers p and q such that $\{k_1(x) | x \in B(\zeta)\} = \{a_i | i \leqq p\}$ and $\{k_2(x) | x \in B(\zeta)\} = \{b_j | j \leqq q\}$. Then by (10),

(11) $$\text{card } (B(\zeta) \cap \gamma) = \text{card } (PB(\zeta)) - \text{card } (NB(\zeta)) = \sum_{(i,j) \leqq (p,q)} (Df^{+}(i, j) - Df^{-}(i, j)) = f(p, q).$$

By (9) and (11), $B(\zeta) \cap \gamma = \{g_i | i < f(p, q)\}$.

We will now give a simultaneous inductive definition of a sequence of numbers and two sequences of finite sets.

$$t_0 = g_0,\; Q(0) = B(\{t_0\}),$$
$$t_{n+1} = g_k, \text{ where } k = \mu y[g_y \notin Q(n)],$$
$$Q(n+1) = B(Q(n) \cup \{t_{n+1}\}),$$
$$R(0) = Q(0) \cap \gamma,$$
$$R(n+1) = (Q(n+1) - Q(n)) \cap \gamma.$$
$$c(n) = \text{card } (R(n)).$$

Let $\{k_1(x)|x \in Q(n)\} = \{a_0, a_1, \cdots, a_p\}$ and $\{k_2(x)|x \in Q(n)\} = \{b_0, b_1, \cdots, b_q\}$. Define $s(n) = p$ and $w(n) = q$.

Assertions (1), (4) and (5) of the theorem are straightforward consequences of the definitions of c, s and w. Assertion (3) follows from those definitions and (11). Assertion (2) requires a proof that $\gamma \cong \sigma$ where $\sigma = \{j(t_n, k)|k < c(n)\}$. However, it is easily shown that $\gamma \leqq_* \sigma$ and $\sigma \leqq_* \gamma$ under the natural correspondence $j(t_{n+1}, i) \leftrightarrow g_{k+i}$; we leave the details of this to the reader.

COROLLARY. *Let $f(x, y)$ be recursive. Then $f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R \subseteq f(E, E)_R$.*

PROOF. Let $f_\Lambda(A, B) \in f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$. Then there exist T, $c(n)$, $s(n)$ and $w(n)$ satisfying Theorem 1. If T is finite, $f_\Lambda(A, B) = \sum_T c(n) \in f(E, E)$. If T is infinite, let $\delta = \{f(s(n), w(n))|n \in E\}$. Then $\delta \subseteq f(E, E)$ and $\Sigma_T c(n) \in \delta_\Sigma$. By Theorem 3 of [7], $\Sigma_T c(n) \in f(E, E)_R$.

5. Extensions to *T*-regressive isols and the main theorem

PROPOSITION 5.1. *Let $f(x, y)$ be a.s.i. and recursive. If there is no increasing sequence $(s(0), w(0)), (s(1), w(1)), \cdots$ of ordered pairs such that $\{f(s(n), w(n))|n \in E\} =_e f(E, E)$, then for any T-regressive isol W, $\Sigma_w Dh_{\rho f} \in f(E, E)_R - (f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R)$.*

PROOF. By Theorem 3 of [4], $\Sigma_w Dh_{\rho f} \in f(E, E)_R$. Suppose that $\Sigma_w Dh_{\rho f} \in f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$ for some T-regressive W. By Theorem 1, $\Sigma_w Dh_{\rho f} = \Sigma_U a(n)$, where $U \leqq_* a(n)$ and there is a strictly increasing sequence $(s(n), w(n))$ such that $\sum_{i=0}^{n} a(i) = f(s(n), w(n))$. Then lemma T yields the contradiction $f(E, E) =_e \{f(s(n), w(n))|n \in E\}$.

NOTATION. Let $k \in E$. We denote $\{f(k, x)|x \in E\}$ by Row (k, f) and $\{f(x, k)|k \in E\}$ by Col (k, f).

PROPOSITION 5.2. *Let $g(x, y)$ be strictly increasing with domain E^2. If there is a strictly increasing sequence of pairs $(s(n), t(n))$ such that $\rho g = \{g(s(n), t(n)|n \in E\}$, then $\rho g =$ Row $(0, g)$ or $\rho g =$ Col $(0, g)$.*

PROOF. Let g, s and t be as above. It is easily seen that $(s(n+1), t(n+1)) \in \{(s(n)+1, t(n)), (s(n), t(n)+1)\}$ for all n. Making use of this fact, one can prove by induction on n the statement:

(*) For all n, $s(n)+t(n) = n$, $\{g(i, 0)|i \leqq s(n)\} = \{h_{\rho g}(i)|i \leqq s(n)\}$ and $\{g(0, i)|i \leqq t(n)\} = \{h_{\rho g}(i)|i \leqq t(n)\}$.

The desired result follows immediately from (*).

COROLLARY 1. *Let $f(x, y)$ be a.s.i. and recursive. If there is a strictly increasing sequence of pairs $(s(i), t(i))$ such that $f(E, E) =_e \{f(s(i),$*

$t(i))|i \in E\}$, then there is a number k such that $f(E, E) =_e$ Row (k, f) or $f(E, E) =_e$ Col (k, f).

COROLLARY 2. *Let $f(x, y)$ be a.s.i. and recursive. If $f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R = f(E, E)_R$, then there is a number k such that $f(E, E) =_e$ Row (k, f) or $f(E, E) =_e$ Col (k, f).*

PROPOSITION 5.3. *Let $f(x, y)$ be a.s.i. and recursive. If there is a number k such that $f(E, E) =_e$ Row (k, f) or $f(E, E) =_e$ Col (k, f), then $f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R = f(E, E)_R$.*

PROOF. We need only prove that $f(E, E)_R \subseteq f_\Lambda(\Lambda_R, \Lambda_R)$. Suppose that $f(E, E) =_e$ Row (k, f), and let $g(x) = f(k, x)$. By Theorem 3 of [4], $f(E, E)_R = \{\Sigma_T Dh_{\rho f} | T \in \Lambda_R\}$. Since ρf is infinite, it is clear that the only finite isols in $f(E, E)_R$ are the numbers in $f(E, E)$. Since $\rho g =_e \rho f$, Lemma 6.2 of [5] shows that any infinite isol in $f(E, E)_R$ is a member of $g(E)_R$. Hence if T is infinite and a member of $f(E, E)_R$, there is a $U \in \Lambda_R$ such that $T = g_\Lambda(U)$. Since $g_\Lambda(U) = f_\Lambda(k, U)$, $T \in f_\Lambda(\Lambda_R, \Lambda_R)$. The case in which $f(E, E) =_e$ Col (k, f) is completely similar.

We have now completely proved our main result.

THEOREM 2. *Let $f(x, y)$ be a.s.i. and recursive. Then $f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R = f(E, E)_R$ if and only if there is a number k such that $f(E, E) =_e$ Row (k, f) or $f(E, E) =_e$ Col (k, f).*

This theorem can be improved in the special case in which f is strictly increasing. We first must prove the following proposition.

PROPOSITION 5.4. *Let $f(x, y)$ be strictly increasing. If there is a number k such that $f(E, E) =_e$ Row (k, f) or $f(E, E) =_e$ Col (k, f), then $f(E, E) =_e$ Row $(0, f)$ or $f(E, E) =_e$ Col $(0, f)$.*

PROOF. If $k > 0$ and $f(E, E) =_e$ Row (k, f), the omission of infinitely many members of ρf from Row $(0, f)$ will force the n^{th} element of Row $(0, f)$ to be greater than the n^{th} element of Row (k, f) for sufficiently large values of n. A similar contradiction can be obtained if it is assumed that $f(E, E) =_e$ Col (k, f) but $f(E, E) \neq_e$ Col $(0, f)$.

THEOREM 3. *Let $f(x, y)$ be strictly increasing and recursive. Then $f(E, E)_R = f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$ if and only if $f(E, E) =_e$ Row $(0, f)$ or $f(E, E) =_e$ Col $(0, f)$.*

The conditions of Theorem 2 and Theorem 3 can be used readily in certain specific applications (some of which will appear in the following section). However the general problem of determining, given $f(x, y)$, whether or not $f(E, E)_R = f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$ is undecidable if $f(x, y)$ ranges over either the class of all a.s.i. recursive functions or the class of all strictly increasing recursive functions. This follows from the proposition below.

PROPOSITION 5.6. *There is no uniform effective procedure which will determine, given* $f(x, y)$ *strictly increasing and recursive, whether or not* $f(E, E) =_e \mathrm{Row}\,(0, f)$ *or* $f(E, E) =_e \mathrm{Col}\,(0, f)$.

PROOF. Left to the reader.

6. Some applications

A1. Let $f(x, y) = ax+by$, where $\{a, b\} \subseteq E-\{0\}$. Then $f(E, E)_R = f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$ if and only if $a|b$ or $b|a$.

PROOF. f is strictly increasing and satisfies the condition of Theorem 3 if and only if $a|b$ or $b|a$.

Let $f(x, y)$ be a.s.i. and recursive. Suppose that there is a number M such that for each fixed $k \in E$ the single variable functions $Df\,(x, k)$ and $Df\,(k, y)$ are eventually greater than or equal to M, while $Dh_{\rho f}(x) < M$ for infinitely many x. Then the differences between successive elements in Row (k, f) and Col (k, f) are eventually larger than differences which must appear infinitely often between successive elements of $f(E, E)$. Thus we cannot have $f(E, E) =_e \mathrm{Row}\,(k, f)$ or $f(E, E) =_e \mathrm{Col}\,(k, f)$, and $f(E, E)_R - (f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R) \neq \emptyset$. We use this principle in the following two applications.

A2. Let $f(x, y)$ be a polynomial of the form $p(x)+q(y)$, where $p(x)$ and $q(y)$ are single variable polynomials of degree $\geqq 2$. Then there is a regressive isol T such that $T \in f(E, E)_R - (f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R)$.

PROOF. Take $M = q(1)$ and observe that for each fixed k, $Df(x+1, k) = Dp(x+1)$ and $Df(k, y+1) = Dq(y+1)$, where Dp and Dq are eventually strictly increasing.

A3. Let $f(x, y) = (x+2)(y+2)$. There is an isol T such that $T \in f(E, E)_R$ but for all $U, V \in \Lambda_R$, $T \neq (U+2)(V+2)$.

PROOF. Let $M = 2$. For each fixed k, $Df\,(x, k) \equiv k+2 \equiv Df\,(k, y)$.

The isol T of A3 is prime but belongs to the canonical extension of the relation "x is composite". It is of the form $\Sigma_U Dh_{\rho f}$ where U is T-regressive; hence, it is neither multiple free nor of the form $pr_\Lambda(w)$, $W \in \Lambda_R$. Thus there are prime regressive isols which do not lie in either of the two main classes of prime regressive isols studied in [6].

An apparently minor modification of the function $f(x, y)$ of A3 leads us to the function $f(x, y) = xy$ which does have the property that $f(E,E)_R = f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$, since Row $(1, f) =$ Col $(1, f) = E$. The following proposition provides additional examples of functions $f(x, y)$ satisfying the equation $f(E, E)_R = f_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$.

A4. Let $f(u)$ be a strictly increasing recursive function. Let $g(x, y)$ be an a.s.i. recursive function of two variables satisfying the equation $g(E, E)_R = g_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$. Let $h(x, y) = fg(x, y)$. Then $h(x, y)$ is a.s.i. and $h(E, E)_R = h_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$.

PROOF. Left to the reader.

In particular, if $p(u)$ is any polynomial in U with coefficients in E, $p(x+y)$ and $p(xy)$ are polynomials in x and y such that $p(E, E)_R = p_\Lambda(\Lambda_R, \Lambda_R) \cap \Lambda_R$. This provides further interesting contrasts with the previous applications; e.g., $f(x, y) = x^2+y^2$ falls under A2 while $x^2+2xy+y^2$ falls under A4.

We conclude this paper with the observation that many concrete results of interest can be obtained without appealing to our theorems. A glaring example for functions of four variables is the following: Let $f(x, y, z, w) = x^2+y^2+z^2+w^2$. Since every positive integer can be expressed as a sum of no more than four squares, $f(E^4) = E$ and $f(E^4)_R = \Lambda_R$. But no infinite multiple free regressive isol can be written as the sum of four squares. Hence $f(\Lambda_R^4) \cap \Lambda_R$ is a proper subset of $f(E^4)_R$.

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