COMPOSITIO MATHEMATICA

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The zeros of exponential polynomials (I)

Compositio Mathematica, tome 26, nº 1 (1973), p. 69-78
<http://www.numdam.org/item?id=CM_1973__26_1_69_0>

COMPOSITIO MATHEMATICA, Vol. 26, Fasc. 1, 1973, pag. 69-78
Noordhoff International Publishing
Printed in the Netherlands

THE ZEROS OF EXPONENTIAL POLYNOMIALS (I)

by

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1. Introduction

The function sin (s) of the complex variable s possesses the remarkable property that all its zeros are located on the real axis. The reason for this phenomenon becomes apparent when the function sin (s) is defined as an exponential polynomial by the formula

$$\sin(s) = \frac{e^{is}-e^{-is}}{2i}, \quad s = \sigma+it.$$

From this identity we easily observe that if $s_0 = \sigma_0+it_0$ is a zero of $\sin(s) = 0$, then

$$|e^{is_0}| = |e^{-is_0}|$$

which implies that $t_0 = 0$. An equally remarkable phenomenon occurs for the exponential function e^s which does not vanish anywhere in the finite part of the plane.

Questions concerning the existence and location of zeros of functions like sin (s) and e^s which are defined by linear combinations of exponential functions have received a considerable attention in the past fifty years. More generally, the emphasis has been on the investigation of the geometric position and asymptotic distribution of the zeros of functions defined by exponential polynomials of the type

$$\varphi(s) = \sum_{k=1}^{m} A_k(s)e^{\alpha_k s}, \quad s = \sigma+it, \tag{1}$$

where the frequencies α_k are complex numbers and the $A_k(s)$ are polynomials in s. Through the efforts of several mathematicians we now possess a fairly complete knowledge of the global position of the zeros of the exponential polynomial $\varphi(s)$. The most interesting results in this direction were first obtained by Pólya [8]. His results were subsequently improved by Schwengler and others. See the survey article of Langer [6] where Pólya's work and other related topics are discussed. The question of the asymptotic distribution of the zeros of the exponential polynomial $\varphi(s)$ is closely related to the geometric position of the zeros and has been extensively treated since the work of Pólya. In recent years there has been a resurgence of interest on these questions. First in the hands of Turán who studied exponential polynomials of the type (1) in connection with his work on the Riemann zeta function. Secondly in the work of Dickson [4] which gives a refinemenent and simplification of the ideas of Pólya. In a recent paper Tijdeman [10] improved further the results of (Dancs and) Turán and also simplified their proof.

The many results that have been established concerning the position and distribution of the zeros of the exponential polynomial $\varphi(s)$ do not yet explain the phenomenon we described earlier to the effect that all the zeros of the function sin (s) are real. For a long time it had been known (see Wilder [12] and Tamarkin [9]), and follows easily from the work of Pólya that the zeros of the exponential polynomial $\varphi(s)$ are located in strips parallel to the imaginary axis if it is assumed that the coefficients $A_k(s)$ are complex constants and the frequencies α_k are all real. The aim of this paper is to give a precise description in this case as to where are the strips of zeros of the exponential polynomial $\varphi(s)$ located, and what is the position of the zeros within the strips. See statement of Main Theorem in § 2. Throughout our work we assume that the exponential polynomial is of the form

$$\varphi(s) = \sum_{k=1}^{m} A_k e^{\alpha_k s}, \quad s = \sigma + it, \tag{2}$$

where the A_k are non-zero complex constants and the frequencies α_k are m real numbers for which $1, \alpha_1, \cdots, \alpha_m$ are linearly independent over the field of rational numbers. One of the main results we prove is that the real parts of the zeros of $\varphi(s)$ are dense in those intervals of the real line which lie entirely inside a strip of zeros. The extension of our results to exponential polynomials of the type (1) with the $A_k(s)$ not necessarily constant seems difficult to obtain.

The theorem we prove can also be looked at as giving a sufficient condition for the location of the zeros of an exponential polynomial in a given strip of the s-plane. In this sense our result resembles the Fundamental Theorem of algebra to the effect that any non-constant polynomial has roots. In this respect we should also mention that the fact that an exponential polynomial in more than one term has zeros follows readily from Hadamard's Factorization Theorem for entire functions, or from Pólya's Theorem. The latter furthermore gives a description of where the zeros are located, i.e. on strips perpendicular to the sides that bound the convex hull generated by the frequencies of the exponential polynomial. Our main result is somehow weaker than Pólya's since it deals with exponential polynomials whose frequencies satisfy rather restrictive conditions. Nevertheless, for the exponential polynomials under consideration our main result gives finer information than that which can be extracted from Pólya's result. In fact our main result suggests the following conjecture.

Conjecture. Let $\varphi(s)$ be an exponential polynomial with constant coefficients and with complex frequencies which are linearly independent over the rationals. Let S be one of the strips where $\varphi(s)$ has zeros (the existence of S follows from Pólya's Theorem). Let L be any line inside S and parallel to the sides of S. The claim is that any ε-neighborhood of L (i.e. a strip of constant width $\varepsilon > 0$ covering L) contains an infinite number of zeros of $\varphi(s)$.

Broadly speaking we can say that our result on the existence and location of the zeros of the exponential polynomial $\varphi(s)$ is a direct consequence of the fact that $\varphi(s)$ defines an almost periodic function of the complex variable s.

In the last part of this article we give an application of our results to showing that a certain number theoretic function has zeros in the critical strip ot the Riemann zeta function.

The results in this paper arose in connection with the author's work [7] on a conjecture of Professor Leon Ehrenpreis (see [5] page 320) concerning the mutual distance between distinct zeros of exponential polynomials. The author wishes to express his thanks to Professor Ehrenpreis for much valuable advice. The author is also very grateful to the referee, without whose many useful suggestions the present paper would have contained many obscurities.

2. Preliminaries

We start by proving a simple generalization of Euclid's Theorem on the inequality of the triangle. We shall later use it to show that under suitable conditions the exponential polynomial $\varphi(s)$ takes small values.

Geometric Principle. Let $A_1, A_2, \cdots, A_m$ be m positive real numbers satisfying the inequalities

$$A_j \leqq \sum_{\substack{k=1 \\ k \neq j}}^{m} A_k, \; (j = 1, 2, \cdots, m). \tag{3}$$

then there is at least one m-sided polygon $P(A_1, \cdots, A_m)$ whose sides have lengths $A_1, A_2, \cdots, A_m$.

PROOF. Let us index the A_k in decreasing order so that $A_1 \geqq A_2 \geqq \cdots \geqq A_m$. The result is trivial if $A_1 = \sum_{k=2}^{m} A_k$. For we might just take as our m-sided polygon the geometric figure consisting of just two sides, one of length A_1, the other of length $\sum_{k=2}^{m} A_k$, the one superimposed on the other. Let us then assume that $A_1 < \sum_{k=2}^{m} A_k$, and let j be that unique integer for which

$$A_1 \geqq \sum_{k=2}^{j-1} A_k \text{ and } A_1 < \sum_{k=1}^{j} A_k.$$

Clearly $3 \leqq j \leqq m$. Now consider the three new positive real numbers

$$B_1 = A_1, \; B_2 = \sum_{k=2}^{j-1} A_k, \; B_3 = A_j.$$

It is easy to verify that B_1, B_2, B_3 satisfy the inequalities (3). Consider then the triangle $P(B_1, B_2, B_3)$. If $j = m$, this triangle can be viewed as an m-side polygon. Suppose that $j < m$, and adjoin to $P(B_1, B_2, B_3)$ an isosceles triangle with base equal to A_{j+1} and one of its two equal sides coinciding with A_j. In the resulting configuration drop the old segment of length A_j. The new figure can now be viewed as a closed $(j+1)$-sided polygon. It is clear that after a finite number of steps a closed m-sided polygon will be reached with sides A_1, A_2, $\cdots$, A_m. This completes the proof of the Geometric Principle.

In the proof of the main result we shall use the following version of the Kronecker-Weyl theorem on the uniform distribution of the fractional parts of the integral multiples of irrational numbers.

THEOREM (*Kronecker-Weyl*). *If $1, \alpha_1, \alpha_2, \cdots, \alpha_m$ are real numbers which are linearly independent over the rational number field, $\gamma_1, \gamma_2, \cdots, \gamma_m$ are arbitrary real numbers, and T and ε are positive real numbers, then there exist a real number t and integers $p_1, \cdots, p_m$ such that $t > T$ and*

$$\left| t\alpha_k - p_k - \frac{\gamma_k}{2\pi} \right| < \varepsilon$$

for $k = 1, 2, \cdots, m$.

A proof of this result can be found in most standard works on diophantine analysis; see for example Cassels [3].

MAIN THEOREM. *Assume that* $1, \alpha_1, \cdots, \alpha_m$ *are real numbers linearly independent over the rationals. Consider the exponential polynomial*

$$\varphi(s) = \sum_{k=1}^{m} A_k e^{\alpha_k s}, \; s = \sigma + it,$$

where the A_k *are complex numbers. Then a necessary and sufficient condition for* $\varphi(s)$ *to have zeros near any line parallel to the imaginary axis inside the strip* $I = \{\sigma+it|\sigma_0 < \sigma < \sigma_1, \; -\infty \leqq t \leqq \infty\}$ *is that*

$$|A_j e^{\sigma\alpha_j}| \leqq \sum_{\substack{k=1 \\ k \neq j}}^{m} |A_k e^{\sigma\alpha_k}|, \; (j = 1, 2, \cdots, m) \tag{4}$$

for any σ *with* $\sigma+it \in I$.

In the above theorem, the statement 'zeros near any line inside the strip I' is taken to mean that given any σ_2 with $\sigma_0 < \sigma_2 < \sigma_1$ and $\varepsilon > 0$, an $s^* = \sigma^* + it^*$ can be found such that $\sigma_2 - \varepsilon < \sigma^* < \sigma_2 + \varepsilon$ and $\varphi(s^*) = 0$.

To prove the Main Theorem we shall use a familiar argument from the theory of almost periodic functions.

3. Proof of Main Theorem

We first prove that the conditions are necessary. Assume that $\varphi(s)$ has zeros near any line in the strip I with fixed Re $(s) = \sigma$. If the inequalities (4) hold for this σ, we are done. Suppose then that for some j

$$|A_j e^{\sigma\alpha_j}| > \sum_{\substack{k=1 \\ k \neq j}}^{m} |A_k e^{\alpha_k \sigma}| \tag{5}$$

and consider the function defined by

$$f(x) = |A_j| e^{\alpha_j x} - \sum_{\substack{k=1 \\ k \neq j}}^{m} |A_k| e^{\alpha_k x}.$$

Clearly this is a continuous function. Therefore the inequality (5) implies that a $\delta > 0$ can be found for which $f(x) > 0$ for all x satisfying $\sigma - \delta < x < \sigma + \delta$. But this would contradict the fact that the equation $\varphi(s) = 0$ had roots near the line Re $(s) = \sigma$.

To establish the sufficiency part, we now assume that the inequalities (4) hold and proceed to show that $\varphi(s) = 0$ has roots near any line in I with $\sigma_0 \leqq$ Re $(s) \leqq \sigma_1$. We first prove the following lemma.

LEMMA. *If the function* $\varphi(s)$ *has the properties*

(a) *It is bounded and analytic for all* s *with* $\sigma_0 \leqq$ Re $(s) \leqq \sigma_1$,

(b) *For some* σ_3 *with* $\sigma_0 \leqq \sigma_3 \leqq \sigma_2$, *the line* $\{\sigma_3 + it | -\infty \leqq t \leqq \infty\}$ *contains a sequence of points* $\{\sigma_3 + it_n\}$ *such that* $\varphi(\sigma_3 + it_n) \to 0$ *as* $n \to \infty$,

(c) *There exist positive numbers d and l such that on any segment of the line* $\{\sigma_3 + it | -\infty \leqq t \leqq \infty\}$ *of length l there can be found a point* $\sigma_3 + it^*$ *such that* $|\varphi(\sigma_3 + it^*)| \geqq d$, *then* $\varphi(s)$ *has zeros in any strip*

$$\{\sigma + it | \sigma_3 - \delta < \sigma < \sigma_3 + \delta, \ -\infty \leqq t \leqq \infty\},$$

where δ *is any positive number.*

PROOF OF LEMMA. The idea of the proof consists in constructing a sequence of functions which converge to a function that does have a zero at the point σ_3 and which does not differ greatly from the original function. Then with the help of Rouché's Theorem we deduce that $\varphi(s)$ also has a zero in a small circle about σ_3. The proof we give is probably due to H. Bohr and is included here only for the sake of completeness. Let us then consider the sequence of functions $\{\varphi_n(s)\}$ defined by $\varphi_n(s) = \varphi(s + it_n)$ which by (a) is bounded in the rectangle $\sigma_0 \leqq \mathrm{Re}\,(s) \leqq \sigma_1$, $-l \leqq \mathrm{Im}\,(s) \leqq l$. By Montel's Theorem a subsequence $\{\varphi_{n_k}(s)\}$ can be found which is uniformly convergent in a subrectangle of the form

$$R(\delta, l) = \left\{ s = \sigma + it \,|\sigma_3 - \delta \leqq \sigma \leqq \sigma_3 + \delta, \ -\frac{l}{2} \leqq t \leqq \frac{l}{2} \right\},$$

where $\delta < \min\,(\sigma_3 - \sigma_0, \sigma_1 - \sigma_3)$. Let $\psi(s)$ be the limit of this subsequence. Clearly assumption (b) implies that $\psi(\sigma_3) = 0$. On the other hand $\psi(s)$ cannot vanish identically, for otherwise this would contradict the fact that the subsequence $\varphi_{n_k}(s)$ has a maximum which is bounded away from zero on any segment of length l on the line $\{\sigma_3 + it | -\infty \leqq t \leqq \infty\}$. By regularity, $s = \sigma_3$ is an isolated zero and hence, given any $\varepsilon > 0$, a positive $r < \min\,(\delta, \varepsilon)$ can be found such that $\psi(s)$ does not vanish on the circumference of the circle $|s - \sigma_3| = r$. Now, a $k_0 = k_0(\varepsilon)$ can be found such that

$$|\psi(s) - \varphi_{n_k}(s)| < |\psi(s)|, \ k \geqq k_0, \ |s - \sigma_3| \leqq r.$$

A simple application of Rouché's Theorem then shows that each function

$$\varphi_{n_k}(s) = \psi(s) + \{\varphi_{n_k}(s) - \psi(s)\}, \ k \geqq k_0$$

has in the disc $|s - \sigma_3| \leqq r$ as many zeros as $\psi(s)$ does, hence at least one. But this is equivalent to saying that the function $\varphi(s)$ has at least one zero inside the disc $|s - (\sigma_3 + it_{n_k})| \leqq r$ for each $k \geqq k_0$. This completes the proof of the lemma.

The above result is a very useful tool in investigations concerning the values taken by analytic almost periodic functions. Here what this Lemma does for us is to show that under the conditions of the Main Theorem, the exponential polynomial $\varphi(s)$ has zeros near any line inside the strip I. To prove this we fix a σ and observe that the inequalities (4) and our Geometric Principle guarantee the existence of some m-sided polygon P with sides

$$|A_k|e^{\alpha_k \sigma},\ k = 1, 2, \cdots, m.$$

Let this polygon be drawn on the complex z-plane, and denote its vertices by $a_0, a_1, \cdots, a_m$, where we assume that the vertex a_0 coincides with the origin. Assume furthermore that the vertices bounding the side of length $|A_k|e^{\alpha_k \sigma}$ are a_{k-1}, a_k. Suppose that the angle which the side $|A_k|e^{\alpha_k \sigma}$ makes at the vertex a_{k-1} with the real axis, measured in a counter clockwise manner, is η_k. Also put $A_k = |A_k|e^{i\vartheta_k}$, $0 \leqq \vartheta_k < 2\pi$ ($k = 1, \cdots, m$). Consider the sequence of real numbers $\gamma_k = \eta_k - \vartheta_k$, ($k = 1, 2, \cdots, m$). Let there be given a decreasing sequence of positive real numbers $\varepsilon_n > 0$ and to each of these apply the Kronecker-Weyl theorem to obtain an infinite sequence of $m+1$ tuples of integers $(t_n, p_1^{(n)}, \cdots, p_m^{(n)})$ such that

$$\left| t_n \alpha_k - p_k^{(n)} - \frac{\gamma_k}{2\pi} \right| < \varepsilon_n, \quad \begin{matrix} (k = 1, 2, \cdots, m) \\ (n = 1, 2, 3, \cdots). \end{matrix} \tag{6}$$

Using now the polar representation $z = re^{i\vartheta}$ in the z-plane we have on the one hand

$$\sum_{k=1}^{m} |A_k| e^{\alpha_k \sigma + i\eta_k} = 0$$

or equivalently

$$\sum_{k=1}^{m} A_k e^{\alpha_k \sigma + i\gamma_k} = 0.$$

The inequalities (6) then show that

$$\sum_{k=1}^{m} A_k e^{\alpha_k \sigma + i\alpha_k(2\pi t_n)} = 0(\varepsilon_n m \sum_{k=1}^{m} |A_k| e^{\alpha_k \sigma}).$$

We have thus shown that on the line Re $(s) = \sigma$, there exist an infinite number of points $2\pi t_n$ for which the exponential polynomial $\varphi(\sigma + 2\pi i t_n) \to 0$ as $n \to \infty$. We now show that positive numbers l and d can be found such that on any interval of length l on the line Re $(s) = \sigma$ there is some point s^* at which $|\varphi(s^*)| \geqq d$. Clearly if such a pair can be found, any smaller, but positive, d would also do. On the contrary, let us assume that whatever positive value of l is taken and that no matter how small d is, there is always a t_0 such that $|\varphi(\sigma + it)| < d$ for all t in the interval $(t_0, t_0 + l)$. Let us in particular consider the m values taken by $\varphi(s)$ at the points

$$s_n = \sigma + i\left(t_0 + \frac{hl}{m}\right), \ (h = 0, 1, \cdots, m-1).$$

This gives rise to m equations

$$\sum_{k=1}^{m} A_k e^{\alpha_k(\sigma+it_0)} e^{i\alpha_k kl/m} = \varphi(s_h), \ (h = 0, 1, \cdots, m-1).$$

Solving these equations for $A_k e^{\alpha_k(\sigma+it_0)}$ we obtain

$$A_k e^{\alpha_k(\sigma+it_0)} = \sum_{h=1}^{m} \varphi(s_h) \frac{\Delta_{hk}}{\Delta}, \tag{7}$$

where Δ_{hk} is the (h, k) cofactor of the determinant

$$\Delta = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{i\alpha_1 l/m} & e^{i\alpha_2 l/m} & \cdots & e^{i\alpha_m l/m} \\ \vdots & \vdots & & \vdots \\ e^{i\alpha_1 l(m-1)/m} & e^{i\alpha_2 l(m-1)/m} & \cdots & e^{i\alpha_m l(m-1)/m} \end{bmatrix}.$$

But this is a Vandermonde determinant and can easily be evaluated as

$$\Delta = \prod_{1 \leqq j \leqq k \leqq m} (e^{i\alpha_k l/m} - e^{i\alpha_j l/m}).$$

We may assume without restriction that l is such that any $\alpha_k l$ does not differ from any $\alpha_j l$, $(k \neq j)$ by an integral multiple of π, then we obtain

$$|\Delta| \geqq \operatorname*{Min}_{k \neq j} |e^{i(\alpha_k - \alpha_j) l/m} - 1|^{m(m-1)/2} = D > 0.$$

Similarly

$$|\Delta_{kj}| \leqq (m-1)! \ (k, j = 1, 2, \cdots, m).$$

Taking absolute values on both sides of (7) we obtain

$$|A_k| e^{\alpha_k \sigma} \leqq \frac{m!}{D} \operatorname*{Max}_{h=1,\cdots,m} |\varphi(s_h)| \leqq \frac{m!}{D} d, \ (k = 1, \cdots, m).$$

But if

$$d < \frac{D}{m!} \operatorname*{Min}_{k=1,\cdots,m} |A_k| e^{\alpha_k \sigma},$$

then we obtain a contradiction. This then shows that condition (c) of the Lemma is satisfied. We therefore conclude that the exponential polynomial $\varphi(s)$ has zeros near any line in the strip I. This completes the proof of the Main Theorem.

In a subsequent paper we shall show that even when the inequalities (4) are satisfied on the line $\operatorname{Re}(s) = \sigma$, the exponential polynomial may fail to have any zeros on the line Re $(s) = \sigma$. A simple example of this phenomenon is the exponential polynomial

$$\varphi(t) = 1+e^{i\beta_1 t}+e^{i\beta_2 t}+e^{i\beta_3 t}$$

which cannot have real zeros. Here we assume the β_1, β_2, β_3 are real numbers linearly independent over the rational number field.

An interesting consequence of the Main Theorem is that the function defined by the polynomial

$$\varphi_M(s) = \sum_{p \leqq M} \frac{1}{p^s},$$

where the summation is taken over all primes $p \leqq M$, $(M \geqq M_0)$, has zeros near any line on the strip $0 \leqq \text{Re}\,(s) \leqq 1$.

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(Oblatum 22–X–1971)

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