

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 26, n° 2 (1973), p. 119-128

[http://www.numdam.org/item?id=CM\\_1973\\_\\_26\\_2\\_119\\_0](http://www.numdam.org/item?id=CM_1973__26_2_119_0)

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## COVERINGS OF FIBRATIONS

by

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### 1. Introduction

Let  $E \xrightarrow{p} B$  be a fibration. Then a covering map  $\tilde{E} \xrightarrow{\pi} E$  gives rise to a fibration  $\tilde{E} \xrightarrow{p\pi} B$ . If the fibre of this fibration is connected, then the fibre is a covering space of the fibre of  $E \rightarrow B$ . In this case, we shall call  $\tilde{E}$  a *covering fibration of the fibration  $F \rightarrow E \xrightarrow{p} B$  extending the covering  $\pi : \tilde{F} \rightarrow F$* .

This note takes up the following questions.

A. Given a fibration  $F \rightarrow E \rightarrow B$  and a covering  $\tilde{F} \xrightarrow{\pi} F$ , when can we find a covering  $\tilde{E}$  extending  $\tilde{F}$ ?

B. Given a space  $F$  and a covering space  $\tilde{F}$ , under what conditions can we always find a covering fibration extending  $\tilde{F}$  for any fibration with fibre  $F$ ?

For  $\tilde{F}$  a universal covering space, we can answer question A (see Theorem 1) in terms of conditions on the fundamental groups of the fibration  $F \rightarrow E \rightarrow B$ . For oriented fibrations and universal coverings  $\tilde{F}$ , we find the answer to B depends upon  $G_1(F)$ , (see Theorem 2). Some results on covering fibrations extending non-universal coverings  $\tilde{F}$  are found in § 3.

The existence of covering fibrations leads to various applications. The most striking of them is theorem 15 which generalizes a theorem of Borel's [1], lemma 3.2.

By fibration, we shall mean Hurewicz fibration (i.e.  $F \rightarrow E \xrightarrow{p} B$  has the homotopy covering property). We assume that  $E$  and  $F$  are path connected, locally path connected, and semi-locally 1-connected. Most of the results proved will obviously be true for other types of fibrations. In § 3, we consider fibre bundles (locally homeomorphic to a product of a neighborhood in the base and the fibre). By an oriented fibration, we shall mean the strongest possible interpretation. That is,  $F \rightarrow E \rightarrow B$  is *oriented* if  $\pi_1(B)$  operates trivially on  $\zeta(F)$ , where  $\zeta(F)$  is the group of homotopy classes of self homotopy equivalences. By  $\xi_0(F)$  we shall mean the group of based homotopy classes of based self homotopy equivalences.

\* Partially supported by a National Science Foundation grant.

Also  $\tilde{X}$  will always denote a covering space of  $X$ . Covering spaces are always assumed to be path connected.

### 2. Covering fibrations extending universal coverings

First we answer question A for universal coverings.

**THEOREM 1:** *Let  $F \xrightarrow{i} E \xrightarrow{p} B$  be a fibration with connected fibre  $F$  and let  $\tilde{F}$  be the universal cover of  $F$ . Then there exists a covering fibration  $\tilde{E}$  of  $E \xrightarrow{p} B$  extending  $\tilde{F}$  if and only if*

a)  $i_* : \pi_1(F) \rightarrow \pi_1(E)$  is injective

and

b)  $p_* : \pi_1(E) \rightarrow \pi_1(B)$  has a right inverse (which is a homomorphism).

**PROOF:** Suppose that  $\tilde{E}$  is a covering fibration. Then we have

$$\begin{array}{ccc}
 \tilde{F} & \xrightarrow{\pi} & F \\
 i \downarrow & & \downarrow i \\
 \tilde{E} & \xrightarrow{\pi} & E \\
 p \downarrow & & \downarrow p \\
 B & \xrightarrow{1} & B
 \end{array}$$

where  $\pi$  denotes covering maps. From the exact ladder associated with this diagram, we have

$$\begin{array}{ccc}
 \pi_2(B) & \xrightarrow{\cong} & \pi_2(B) \\
 \downarrow & & \downarrow d \\
 0 & \longrightarrow & \pi_1(F).
 \end{array}$$

Thus  $d$  is trivial, hence  $i_* : \pi_1(F) \rightarrow \pi_1(E)$  is injective. On the other hand  $(p\pi)_* : \pi_1(\tilde{E}) \xrightarrow{\cong} \pi_1(B)$ . So  $(p\pi)^*$  has an inverse  $j : \pi_1(B) \rightarrow \pi_1(\tilde{E})$ . Then  $\pi_* j : \pi_1(B) \rightarrow \pi_1(E)$  is the required inverse to  $p_* : \pi_1(E) \rightarrow \pi_1(B)$ .

Conversely, suppose a) and b) hold, Let  $\tilde{E}$  be the covering space of  $E$  corresponding to a subgroup  $H$  of  $\pi_1(E)$  such that  $p_* : H \rightarrow \pi_1(B)$  is an isomorphism. Then  $p' = p\pi : \tilde{E} \rightarrow B$  is a fibration with fibre  $F'$  and we

have the commutative diagram

$$\begin{array}{ccc}
 F' & \xrightarrow{\bar{\pi}} & F \\
 i' \downarrow & & \downarrow i \\
 \tilde{E} & \xrightarrow{\pi} & E \\
 p' \downarrow & & \downarrow p \\
 B & \xrightarrow{1} & B
 \end{array}$$

where  $\bar{\pi}$  is the restriction of  $\pi$ .

Now we will show that  $F' = \tilde{F}$ . Notice that

$$p'_* : \pi_1(\tilde{E}) \rightarrow \pi_1(B)$$

is the composition of isomorphisms

$$\pi_1(\tilde{E}) \xrightarrow{\pi_*} H \xrightarrow{p_*} \pi_1(B)$$

and is therefore an isomorphism. Hence  $F'$  is path connected. Since  $\bar{\pi} : F' \rightarrow F$  is a fibration with discrete fiber, it is a covering space [9, Theorem 10, p. 78].

We will show now that  $\pi_1(F') = 0$ . If  $v \in \pi_1(F')$  then  $i'_*(v) = 0$  by exactness. Hence  $0 = \pi_* i'_*(v) = i_* \bar{\pi}_*(v)$ . But  $\bar{\pi}_*$  is injective since  $\bar{\pi}$  is a covering and  $i_*$  is injective by hypothesis a). Therefore  $v = 0$  as was to be shown.

Next we turn to question B in the case of universal coverings and oriented fibrations. We need a few technical remarks.

First, let us recall the definition of the first evaluation subgroup  $G_1(F) \subset \pi_1(F)$ . A homotopy  $h_t : F \rightarrow F$  is called a *cyclic homotopy* if  $h_0 = h_1 = 1_F$ . The loop  $\tau$  given by  $\tau(t) = h_t(*)$  (where  $*$   $\in F$  is the base point) is called the *trace* of the homotopy. Then  $G_1(F)$  consists of the subgroups of  $\pi_1(F, *)$  whose elements are represented by loops which are traces of cyclic homotopies. For a list of properties of  $G_1(F)$ , see [5].

We also need the fact (see theorem 16.9, [3]) that for any fibre  $F$ , there exists a universal fibration  $F \rightarrow E_\infty \rightarrow B_\infty$ . Now the transgression  $d_\infty : \pi_2(B_\infty) \rightarrow \pi_1(F)$  is related to  $G_1(F)$  by the fact that  $d_\infty(\pi_2(B_\infty)) = G_1(F)$ . Now, as always, we assume that  $F$  is connected. Then let  $\tilde{B}_\infty$  be the universal covering space of  $B_\infty$ . Let  $F \rightarrow D \rightarrow \tilde{B}_\infty$  denote the fibration classified by the covering map  $\pi : \tilde{B}_\infty \rightarrow B_\infty$ . It is easy to see that  $d(\pi_2(\tilde{B}_\infty)) = G_1(F)$ , where  $d$  is the transgression  $d : \pi_2(\tilde{B}_\infty) \rightarrow \pi_1(F)$ . It also is true that every oriented fibration may be regarded as a pullback of  $F \rightarrow D \rightarrow \tilde{B}_\infty$ . This follows since a classifying map  $k : B \rightarrow B_\infty$  of an oriented fibration  $F \rightarrow E \rightarrow B$  maps  $\pi_1(B)$  to the identity of  $\pi_1(B_\infty)$ .

Hence, there is a lifting of  $k$  to  $\tilde{k} : B \rightarrow \tilde{B}_\infty$  such that  $k = \pi\tilde{k}$ .

Now we can state and prove

**THEOREM 2:** *There exists a covering fibration of any oriented fibration with connected fibre  $F$  extending the universal covering space  $\tilde{F}$  if and only if  $G_1(F)$  is trivial.*

**PROOF:** Let  $F \xrightarrow{i} D \xrightarrow{p} \tilde{B}_\infty$  be the fibration mentioned above. We shall show that there exists a covering fibration  $\tilde{D} \rightarrow \tilde{B}_\infty$  which extends  $\tilde{F}$  if and only if  $G_1(F) = 0$ . The pull back of the classifying map for  $F \rightarrow E \rightarrow B$ , denoted by  $k : B \rightarrow B_\infty$ , of  $\tilde{F} \rightarrow D \rightarrow B$  will be the required covering fibration. That is,  $\tilde{D} = k^*(D)$ .

Now  $\pi_1(\tilde{B}_\infty)$  is trivial. Thus (b) of theorem 1 is satisfied. On the other hand,  $G_1(F) = 0$  implies that  $d : \pi_2(\tilde{B}_\infty) \rightarrow \pi_1(F)$  is trivial. Hence  $i_* : \pi_1(F) \rightarrow \pi_1(D)$  is injective. So condition (a) is satisfied and so we may apply theorem 1.

If we assume that  $G_1(F) \neq 0$ , then by theorem 1, we cannot find a covering fibration of  $F \rightarrow D \rightarrow \tilde{B}_\infty$  extending  $\tilde{F}$ .

**COROLLARY 3:** *Let  $F$  be a compact polyhedron and  $\chi(F) \neq 0$ . Then there exists a covering fibration of any oriented fibration extending  $F$ , the universal covering  $F$ .*

**PROOF:** We know that  $\chi(F) \neq 0$  implies that  $G_1(F) = 0$ .

**COROLLARY 4:** *Let  $F$  be a compact polyhedron with  $\chi(F) \neq 0$ . Then there exists a cross-section over the two-skeleton of the base space of any oriented fibration with fibre  $F$ .*

**PROOF:** By the corollary above, we have

$$\begin{array}{ccc}
 \tilde{F} & \xrightarrow{\pi} & F \\
 \downarrow & & \downarrow \\
 \tilde{E} & \xrightarrow{\pi} & E \\
 \downarrow p & & \downarrow \\
 B & \longrightarrow & B
 \end{array}$$

where  $B$  is a 2-dimensional CW complex.

The groups in which the obstructions to a cross-section of  $\tilde{E} \xrightarrow{p} B$  lie must vanish. Thus there is a cross-section  $c : B \rightarrow \tilde{E}$ . Now  $\pi c$  is the required cross-section.

Now we shall study those fibrations with fibres  $RP^n$ , real projective space. We begin by giving alternative proofs to some results of Olum.

LEMMA 5 (Olum [8]) :

$$\xi_0(RP^n) \cong Z_2.$$

PROOF: Let  $\hat{\xi}_0(S^n)$  denote the group of homotopy classes of base point preserving equivariant homotopy equivalences of  $S^n$ . We have an isomorphism

$$\xi_0(RP^n) = \hat{\xi}_0(S^n)$$

defined by sending  $f$  to the unique base point preserving map  $\tilde{f}$  that covers  $f$ . By the method of proof of [2, TH. 2.5], the operation of suspension defines an isomorphism

$$\hat{\xi}_0(S^n) = \hat{\xi}_0(S^{n+1}), \quad n \geq 1.$$

Therefore, we have

$$\xi_0(RP^n) = \xi_0(RP^{n+1}), \quad n \geq 1.$$

Finally,

$$\xi_0(RP^1) = \xi_0(S^1) \cong Z_2.$$

COROLLARY 6.

$$\xi(RP^{2n+1}) \cong Z_2; \xi(RP^{2n}) \cong 0.$$

PROOF: Let  $L$  denote the space of self homotopy equivalences of  $RP^n$ . Let  $L_0$  be the subspace of maps of  $L$  which preserve the base point. Now  $G_1(RP^{2n}) = 0$  and

$$G_1(RP^{2n+1}) = \pi_1(RP^{2n+1});$$

see [4], corollary I.6 and theorem II.5. From the exact sequence arising from the fibration  $L_0 \rightarrow L \xrightarrow{\omega} RP^n$ , and noting that the image of  $\omega_*$  is  $G_1(RP^n)$ , we see that  $d : \pi_1(RP^n) \rightarrow \pi_0(L_0)$  is zero if  $n$  is odd and injective if  $n$  is even. Thus  $i_* : \pi_0(L_0) \rightarrow \pi_0(L)$  is injective if  $n$  is odd and has kernel  $Z_2$  if  $n$  is even. But  $\pi_0(L_0) = \xi_0(RP^n) \cong Z_2$  by lemma 5. Also  $i_* : \pi_0(L_0) \rightarrow \pi_0(L)$  must be onto since  $RP^n$  is connected. Thus  $\xi(RP^n) = \pi_0(L)$  is  $Z_2$  when  $n$  is odd and 0 when  $n$  is even.

COROLLARY 7. *Every fibration with fibre  $RP^{2n}$  is orientable.*

PROOF: Since  $\xi(RP^{2n})$  is trivial, the fundamental group of the base must act trivially on  $\xi(RP^{2n})$ .

THEOREM 8. *Every fibration with fibre  $RP^{2n}$  is covered by an  $S^{2n}$  fibration with an involution.*

PROOF: The conclusion means that we can always find a covering fibration which extends the universal cover  $S^{2n}$  of  $RP^{2n}$ . But this follows immediately from theorem 2 and corollary 7. The extended total space must be a 2-fold covering of the original total space, and the deck transformation is the involution.

REMARK: Theorem 8 is not true for  $RP^{2n+1}$  even when the fibration is orientable. Consider the “universal oriented fibration”,

$$RP^{2n+1} \xrightarrow{i} D \rightarrow \tilde{B}_\infty.$$

Now

$$i_* : \pi_1(RP^{2n+1}) \rightarrow \pi_1(D)$$

is trivial since

$$G_1(R^{2n+1}) = Z_2.$$

Thus in theorem 1 condition a) fails.

The next corollary is an application of theorem 8.

COROLLARY 9. *Let  $RP^{2n} \rightarrow E \rightarrow B$  be a fibration. Then*

$$H^*(E; Z_2) = H^*(RP^{2n}; Z_2) \otimes H^*(B; Z_2)$$

*as  $Z_2$ -vector spaces.*

PROOF: Let  $S^{2n} \rightarrow \tilde{E} \rightarrow B$  be an  $S^{2n}$ -fibration with an involution which covers  $RP^{2n} \rightarrow E \rightarrow B$ . Let  $\lambda : \tilde{E} \rightarrow S_\infty$  be an equivariant map and let  $\lambda : E \rightarrow RP^\infty$  denote the quotient map. Then the composition  $RP^{2n} \xrightarrow{i} E \xrightarrow{\lambda} RP^\infty$  sends the generator  $c \in H^1(RP^\infty; Z_2)$  to the generator  $\hat{c} \in H^1(RP^{2n}; Z_2)$ . A cohomology extension of the fiber

$$\theta : H^*(RP^{2n}; Z_2) \rightarrow H^*(E; Z_2)$$

is defined by  $\theta(\hat{c}^t) = \lambda^*(c^t)$ ,  $t \geq 0$ . The corollary now follows from the Leray-Hirsch theorem [9, p. 257].

### 3. Arbitrary coverings

We shall restrict our attention to coverings of fibre *bundles*. Our technique can probably be extended to Hurewicz fibrations, or Dold fibrations, but the needed propositions have not been written down yet. First we answer question B for fibre bundles. Then follow applications.

Let  $G$  be a group of homeomorphisms of  $F$  onto itself. Let  $G^*$  be the self homeomorphisms of a covering of  $F$ , denoted  $\tilde{F}$ , which are liftings of homeomorphisms in  $G$ . Let  $\Phi : G^* \rightarrow G$  be the map which takes a map  $f \in G^*$  to the induced map  $\Phi(f) \in G$ . Then  $\Phi$  is continuous and also is a homomorphism.

THEOREM 10: *If there exists a cross-section  $c : G \rightarrow G^*$  to  $\Phi$  which is also a homomorphism, then there is a covering bundle of any  $G$ -bundle with fibre  $F$  which extends  $\tilde{F}$ .*

PROOF: Let  $F \rightarrow E \rightarrow B$  be a  $G$ -bundle and let  $G \rightarrow E^* \rightarrow B$  be the associated principal  $G$ -bundle. Then consider the  $G$ -bundle  $\tilde{F} \rightarrow E \times_G \tilde{F} \rightarrow$

$B$  where  $G$  acts on  $\tilde{F}$  by means of the cross-section  $c$ . Then  $\tilde{E} = E \times_G \tilde{F}$  is the required covering. The projection  $\pi : \tilde{E} \rightarrow E$  is given by

$$\pi(\langle e, \tilde{x} \rangle) = \langle e, \pi(\tilde{x}) \rangle \in E^* \times_G F = E.$$

The next theorem gives conditions for Theorem 10 to hold. We shall let  $G_e^*$  be the identity component of  $G^*$  and  $G_e$  the identity component of  $G$ .

**THEOREM 11:** *A covering bundle extending  $\tilde{F}$  always exists if a)  $G_1(F) \subset \pi_1(\tilde{F})$ , and b) there is a homomorphism*

$$\bar{c} : G/G_e \rightarrow G^*/G_e^*$$

*which is a cross-section to the homomorphism*

$$\Phi : G^*/G_e^* \rightarrow G/G_e$$

*induced by  $\Phi$ .*

**PROOF:** Note that  $\Phi : G^* \rightarrow G$  is a fibration with a discrete fibre. The fibre over  $1_F$  consists of the liftings of  $1_F$ . Assume that  $\tilde{f} \in G_e^*$  is a lifting of  $1_F$ . Then the path from  $\tilde{f}$  to  $1_{\tilde{F}}$  induces a cyclic homotopy  $h_t : F \rightarrow F$ . The trace of  $h_t$  must represent an element in  $\pi_1(\tilde{F})$  by a). Thus the trace lifts to a closed path in  $\tilde{F}$ . Thus  $\tilde{f}$  has a fixed point and hence  $\tilde{f} = 1_{\tilde{F}}$ .

This fact allows us to conclude that  $G_e$  and  $G_e^*$  are isomorphic. Thus we may construct the cross-section required by theorem 10 over  $G_e$ , and condition b) allows us to extend the cross-section over the other components.

**REMARK:** If  $G$  is connected and  $G_1(F) = 0$ , we may extend any covering  $\tilde{F}$  to a covering  $G$ -bundle of any arbitrary  $G$ -bundle.

**COROLLARY 12:** *If  $F$  is a compact polyhedron and  $\chi(F) \neq 0$ , then  $\tilde{F}$  can be extended to a covering bundle for any fibre bundle with connected structural group.*

Let  $M$  be a closed topological manifold which is unorientable and let  $\tilde{M}$  denote its oriented double covering. Let  $0(M)$  be the subgroup of  $\pi_1(M)$  of elements represented by orientation preserving loops. Then  $0(\tilde{M})$  is the subgroup of  $\pi_1(\tilde{M})$  corresponding to the oriented double covering  $\tilde{M}$ .

From this point on, we shall consider fibre bundles with fibre  $M$  and study covering bundles which extend  $\tilde{M}$ . The end result will yield the main application of these techniques, Theorem 15.

**THEOREM 13:** *For any fibre bundle with fibre  $M$ , a closed unorientable topological manifold, the oriented double covering  $\tilde{M}$  extends to a covering*



bundle. In addition, the structural group of this covering bundle preserves the orientation of  $\tilde{M}$ .

PROOF: Every homeomorphism  $h : M \rightarrow M$  has two liftings  $\tilde{M} \rightarrow \tilde{M}$ , one of which preserves and the other reverses the orientation of  $\tilde{M}$ . Consider the correspondence  $i$  which sends every  $h \in G$  to its orientation preserving lifting in  $G^*$ . It is easy to see that  $i : G \rightarrow G^*$  is continuous, a homomorphism of groups and a cross-section of  $\Phi : G^* \rightarrow G$ . Thus theorem 10 is satisfied. The group of the covering fibre bundle is  $i(G)$ , so the second statement of the theorem is clearly true.

If we compare theorem 13 with condition a) of theorem 11, we are lead to conjecture that  $G_1(M) \subset O(M)$ . This is in fact true, as the following theorem shows.

THEOREM 14:

$$G_1(M) \subset O(M).$$

PROOF: Let  $\alpha \in G_1(M)$ . Then there is a cyclic homotopy  $h_t : M \rightarrow M$  whose trace represents  $\alpha$ . Now  $h_t$  lifts to a homotopy  $\tilde{h}_t : \tilde{M} \rightarrow \tilde{M}$  and  $\tilde{h}_1$  is a lifting of the identity. Since  $\tilde{h}_1$  is homotopic to  $1_{\tilde{M}}$ ,  $\tilde{h}_1$  preserves the orientation on  $\tilde{M}$ . There are only two liftings of the identity and one of them reverses orientation. So  $\tilde{h}_1 = 1_{\tilde{M}}$ . Thus the trace of  $\tilde{h}_t$  is a loop which covers the trace of  $h_t$ . Hence  $\alpha$  must be in  $O(M)$ .

THEOREM 15: Let  $M \rightarrow E \xrightarrow{\pi} B$  be a fibre bundle with fibre a closed topological  $n$ -manifold and with structural group  $G$ . If  $\chi(M) \neq 0 \pmod p$ , where  $p$  is a prime, then

$$\pi^* : H^*(B; Z_p) \rightarrow H^*(E; Z_p)$$

is injective.

PROOF: First consider the case where  $\pi_1(B)$  operates trivially on  $H^n(M; Z_p)$ . Then the theorem is true if  $M$  is orientable or if  $p = 2$  [6, Theorem 12]. Suppose now that  $M$  is unorientable and  $p \neq 2$ . Let  $\tilde{M}$  denote the orientable double covering of  $M$ . By theorem 13, we have

$$\begin{array}{ccc} \tilde{M} & & M \\ \downarrow & & \downarrow \\ \tilde{E} & \longrightarrow & E \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ B & \xrightarrow{1} & B \end{array}$$

and  $\pi_1(B)$  operates trivially on  $H^n(\tilde{M}; Z_p)$ . Now  $\chi(\tilde{M}) = 2\chi(M) \neq 0 \pmod p$ . Since  $\tilde{M}$  is orientable it follows that  $\tilde{\pi}^*$  is injective. Then by commutativity  $\pi^*$  is injective.

Now consider the case where  $\pi_1(B)$  operates non-trivially on  $H^n(M; Z_p)$   $p$  odd. Let  $K \subset \pi_1(B)$  denote the normal subgroup of index 2 consisting of elements which operate trivially on  $H^n(M; Z)$ . Let  $\tilde{B} \xrightarrow{q} B$  denote the 2-fold covering which corresponds to  $K$ . We have

$$\begin{array}{ccc}
 M & & M \\
 \downarrow & & \downarrow \\
 \tilde{E} & \xrightarrow{\tilde{q}} & E \\
 \tilde{\pi} \downarrow & & \downarrow \pi \\
 \tilde{B} & \xrightarrow{q} & B
 \end{array}$$

where  $\tilde{E} \xrightarrow{\tilde{q}} \tilde{B}$  is the bundle induced by  $q$ . Since  $\pi_1(B)$  operates trivially on  $H^n(M; Z_p)$  we have from the preceding paragraph that  $\tilde{\pi}^*$  is injective. To show that  $\pi^*$  is injective it is now sufficient to show that

$$q^* : H^*(B; Z_p) \rightarrow H^*(\tilde{B}; Z_p)$$

is injective. There is the transfer map [10, Chapter 5]

$$\tau : H^*(\tilde{B}; Z_p) \rightarrow H^*(B; Z_p),$$

and  $\tau q^*$ , being multiplication by 2, is an isomorphism. Therefore  $q^*$  is injective. This completes the proof of the theorem.

This theorem was first noted by A. Borel in [1] with the extra hypotheses that the bundle is differentiable and oriented in some sense and  $M$  is oriented. In [6], the second author removed the differentiability hypothesis and the above theorem removes the orientability hypotheses.

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(Oblatum 8-V-72)