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DERIVATIONS OF VECTOR FIELDS

by

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1. Statement of the result

Let M be a differentiable, i.e. C^∞, manifold. We denote the Lie-algebra of C^∞ vectorfields on M by $\chi(M)$. A map $D : \chi(M) \to \chi(M)$ is called a *derivation* if D is $\boldsymbol{R}$-linear and if $D([X, Y]) = [D(X), Y] + [X, D(Y)]$ for all $X, Y \in \chi(M)$. It is clear that every $X \in \chi(M)$ defines a derivation $DX : DX(Y) = [X, Y]$. In this note we want to show that every derivation can be obtained in this way.

THEOREM. *For each derivation $D : \chi(M) \to \chi(M)$ there is a vectorfield $Z \in \chi(M)$, such that for each $X \in \chi(M)$, $D(X) = [Z, X]$.*

This theorem has a certain relation with recent work of M. Gel'fand, D. B. Fuks, and others [1] on the cohomology of Lie-algebras of smooth vectorfields, because it implies that $H^1(\chi(M); \chi M)) = 0$; $H^1(\chi(M); \chi(M))$ being the first cohomology group of $\chi(M)$ with coefficient in $\chi(M)$ with the adjoined representation (this was pointed out to me by M. Hazewinkel). There is however one difference in their approach: in defining their cohomology they only use cochains which are continuous mappings (with respect to the C^∞ topology). It is however not difficult to show that the nullety of $H^1(\chi(M); \chi(M))$ follows from our theorem in either case.

The theorem will follow from the following lemmas:

LEMMA 1. *Let $D : \chi(M) \to \chi(M)$ be a derivation and let $X \in \chi(M)$ be zero on some open subset $U \subset M$. Then $D(X)|U \equiv 0$.*

LEMMA 2. *Let $X \in \chi(\boldsymbol{R}^n)$ be a vectorfield on $\boldsymbol{R}^n$ with $j^3(X)(0) = 0$, i.e. the 3-jet of each of the component functions of X is zero in the origin. Then there are vectorfields $Y_1, \cdots, Y_q Z_1, \cdots, Z_q$ and there is a neighbourhood U of the origin in $\boldsymbol{R}^n$ such that:*

$$X|U = \sum_i [Y_i, Z_i]|U \quad \textit{and}$$

$$j^1(Y_i)(0) = 0, \; j^1(Z_i)(0) = 0 \text{ for all } i = 1, \cdots, q.$$

* During the preparation of this paper, the author was a visiting member of the Mathematical Institute of the University of Strasbourg.

LEMMA 3. *Let $D : \chi(M) \to \chi(M)$ be a derivation and let $X \in \chi(M)$ and $p \in M$ be such that $j^3(X)(p) = 0$. Then $D(X)(p) = 0$. In other words, $DX(p)$ is determined by $j^3(X)(p)$, also if $j^3(X)(p) \neq 0$.*

Lemma 3 will be derived from the lemmas 1 and 2. Finally we shall use lemma 3 to derive:

LEMMA 4. *Let $U \subset \boldsymbol{R}^n$ be an open connected and simply connected set and let $D_U : \chi(U) \to \chi(U)$ be a derivation. Then there is a unique vectorfield $Z \in \chi(U)$ such that $D_U(X) = [Z, X]$ for all $X \in \chi(U)$.*

Finally, we shall see that the theorem follows from lemma 1 and lemma 4.

2. The Proofs

PROOF OF LEMMA 1. Suppose $X|U \equiv 0$ and $D(X)(q) \neq 0$ for some point $q \in U$. We take a vectorfield $Y \in \chi(M)$ such that $\sup(Y) \subset U$ and $[D(X), Y](q) \neq 0$. By definition we have $D[X, Y] = [DX, Y] + [X, DY]$; evaluating this in q we get $0 = [DX, Y](q) \neq 0$, which contracts our assumption. Hence the lemma is proved.

PROOF OF LEMMA 2. It is clearly enough to show the lemma for the case

$$X = X(x_1, \cdots, x_n) \frac{\partial}{\partial x_1},$$

with $j^3(X)(0) = 0$. Such vectorfields can be written as a finite sum of vectorfields of the following two types:
type I:

$$\tilde{X} = x_1^{m_1} \bullet \cdots \bullet x_n^{m_n} \cdot \alpha(x_2, \cdots, x_n) \frac{\partial}{\partial x_1},$$

with $\sum m_i \geqq 4$ and α a C^∞ function;
type II:

$$\tilde{X} = x_1^4 \cdot g(x_1, \cdots, x_n) \frac{\partial}{\partial x_1}$$

with g a C^∞ function.

To prove the lemma we show that each vectorfield, which is either of type I or of type II, can be written as the Lie-product of two vedtorfields with zero 1-jet in the origin. For $\tilde{X}$ of type I as above, we observe that

$$\tilde{X} = \left[\frac{1}{k_1 - h_1} \cdot x_1^{h_2} \bullet \cdots \bullet x_n^{h_n} \frac{\partial}{\partial x_1},\ x_1^{k_2} \bullet \cdots \bullet x_n^{k_n}, \alpha(x_2, \cdots, x_n) \frac{\partial}{\partial x_1}\right],$$

$$\begin{aligned} h_1 + k_1 &= m_1 + 1 \text{ and } h_1 \neq k_1 \\ h_2 + k_2 &= m_2 \\ &\vdots \\ h_n + k_n &= m_n. \end{aligned}$$

Using the fact that $\sum m_i \geqq 4$, we see that we can choose $h_1, \cdots, h_n$, $k_1, \cdots, k_n$ so that $\sum h_i \geqq 2$ and $\sum k_i \geqq 2$; hence for type one we have the required Lie-product.

Suppose that that

$$\tilde{X} = x_1^4 \cdot g(x_1, \cdots, x_n) \frac{\partial}{\partial x_1}$$

is of type II. We want to show that there is a function H, defined on a neighbourhood of the origin in $\boldsymbol{R}^n$ such that

$$\tilde{X} = \left[x_1^2 H(x_1, \ldots, x_n) \frac{\partial}{\partial x_1}, \ x_1^2 \frac{\partial}{\partial x_1} \right]$$

in a neighbourhood of the origin. The existence of such H follows from

SUB-LEMMA (2.1) *Let Z, X be vectorfields on $\boldsymbol{R}^1$, which depend on real variables $\mu_1, \cdots, \mu_r$, and which can be written in the form*

$$Z = x^k \cdot f(x, \mu_1, \cdots, \mu_r) \frac{\partial}{\partial x}, \quad X = x^l \cdot g(x, \mu_1, \cdots, \mu_r) \frac{\partial}{\partial x}$$

where f, g are C^∞ functions on $\boldsymbol{R}^{r+1}$ (at least on a neighbourhood of the origin), $l \geqq 2k$ and $f(0,)0, \cdots, 0) \neq 0$.

Then there is a vectorfield Y, also depending on $\mu_1, \cdots, \mu_r$, of the form

$$Y = x^k \cdot H(x, \mu_1, \cdots, \mu_r) \frac{\partial}{\partial x},$$

such that

$$[Y, Z] = X$$

for all $(x, \mu_1, \cdots, \mu_r)$ in a small neighbourhood of the origin in $\boldsymbol{R}^{r+1}$.

PROOF of (2,1).

$$[Y, Z] = X \text{ or } \left[x^k \cdot H(x, \mu) \frac{\partial}{\partial x}, \ x^k \cdot f(x, \mu) \frac{\partial}{\partial x} \right] = x^l \cdot g(x, \mu) \frac{\partial}{\partial x}$$

is equivalent with

$$x^k \cdot H(x, \mu) \cdot \left[k \cdot x^{k-1} \cdot f(x, x) + x^k \frac{\partial f}{\partial x}(x \ u) \right]$$
$$- x^k \cdot f(x, \mu) \left[k \cdot x^{k-1} \cdot H(x, \mu) + x^k \frac{\partial H}{\partial x}(x, \mu) \right] = x^l \cdot g(x, \mu).$$

The terms with x^{2k-1} cancel and $l \geqq 2k$, so we can devide by x^{2k} and obtain:

$$H(x, \mu) \cdot \frac{\partial f}{\partial x}(x, \mu) - f(x, \mu) \frac{\partial H}{\partial x}(x, \mu) = x^{l-2k} \cdot g(x, \mu)$$

Restricting ourselfs to a small neighbourhood of the origin in the (x, μ) space, we may devide by f and obtain:

$$\frac{\partial H}{\partial x}(x, \mu) = \frac{\frac{\partial f}{\partial x}(x, \mu)}{f(x, \mu)} \cdot H(x, \mu) - x^{l-2k} \cdot \frac{g(x, \mu)}{f(x, \mu)}$$

This is an ordinary differential equation depending on the parameters $\mu = (\mu_1, \cdots, \mu_r)$. Hence, by the existence and smoothness of solutions of differential equations depending on parameters, it follows that there is a function H which has the required properties.

Proof of Lemma 3. For X and p as in the statement of the lemma (i.e. $j^3(X)(p) = 0$) we can find, using local coordinates and lemma 2, a neighbourhood U of p in M and vectorfields on M $Y_1, \cdots Y_q$ and $Z_1, \cdots, Z_q$ such that

$$X|U = \sum_i [Y_i, Z_i]|U$$

and

$$j^1(Y_i)(p) = 0, \quad j^1(Z_i)(p) = 0 \text{ for all } i = 1, \cdots, q.$$

Let $D : \chi(M) \to \chi(M)$ be any derivation. It follows from Lemma 1 that

$$D(X)(p) = D(\sum_i [Y_i, Z_i])(p).$$

By the definition of derivation, this last expression equals

$$\sum_i [D(Y_i), Z_i](p) + \sum_i [Y_i, D(Z_i)](p)$$

which is zero because the 1-jets of Y_i and Z_i are zero in p. This proves lemma 3.

Proof of Lemma 4. For D_U and $U \subset \mathbf{R}^n$ as in the statement of Lemma 4 and $x_1, \cdots, x_n$ coordinates on $\mathbf{R}^n$, we define the functions $D_{ij} : U \to \mathbf{R}$, $i, j = 1, \cdots, n$ by

$$D_U\left(\frac{\partial}{\partial x_i}\right) = \sum D_{ij} \frac{\partial}{\partial x_j}.$$

We know that

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] \equiv 0,$$

for all i, j so

$$0 \equiv D_U \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = \left[D_U\left(\frac{\partial}{\partial x_i}\right), \frac{\partial}{\partial x_j}\right] + \left[\frac{\partial}{\partial x_i}, D_U\left(\frac{\partial}{\partial x_j}\right)\right]$$

$$= -\sum_h \frac{\partial D_{ih}}{\partial x_j} \frac{\partial}{\partial x_h} + \sum_h \frac{\partial D_{jh}}{\partial x_i} \frac{\partial}{\partial x_h}.$$

Hence, for all i, j, h, we have

$$\frac{\partial D_{ih}}{\partial x_j} = \frac{\partial D_{jh}}{\partial x_i};$$

as U is 1-connected, there are functions $\bar{D}_h : Y \to \boldsymbol{R}$, $h = 1, \cdots, n$ such that

$$\frac{\partial \bar{D}_h}{\partial x_i} = -D_{ih}.$$

Now we define $\bar{Z} \in \chi\ (U)$

$$\bar{Z} = \sum \bar{D}_h \frac{\partial}{\partial x_i}.$$

From the above construction it follows that we have for each

$$i = 1, \cdots, n : D_U\left(\frac{\partial}{\partial x_i}\right) = \left[\bar{Z}, \frac{\partial}{\partial x_i}\right].$$

Now we define the derivation

$$D_U^1 : \chi(U) \to \chi(U) \text{ by } D_U^1(X) = D_U(X) - [\bar{Z}, X];$$

clearly

$$D_U^1\left(\frac{\partial}{\partial x_i}\right) \equiv 0$$

for all i.

Next we define the functions $D_{ijk} : U \to \boldsymbol{R}$, $i, j, k = 1, \cdots, n$ by

$$D_U^1\left(x_i \frac{\partial}{\partial x_j}\right) = \sum D_{ijk} \frac{\partial}{\partial x_k}.$$

First we show that all these functions are constant: as

$$\left[\frac{\partial}{\partial x_i}, x_j \frac{\partial}{\partial x_k}\right] = \delta_{ij} \frac{\partial}{\partial x_k}$$

we have

$$0 \equiv D_U^1\left[\frac{\partial}{\partial x_i}, x_j \frac{\partial}{\partial x_k}\right] = \left[D_U^1\left(\frac{\partial}{\partial x_i}\right), x_j \frac{\partial}{\partial x_k}\right] + \left[\frac{\partial}{\partial x_i}, D_U^1\left(x_j \frac{\vartheta}{\partial x_k}\right)\right] = \sum \frac{\partial D_{jkh}}{\partial x_i} \frac{\partial}{\partial x_h}.$$

Hence, for all i, j, k, h,

$$\frac{\partial D_{jkh}}{\partial x_i} \equiv 0;$$

so the functions D_{jkh} must be constant (because U is connected). We denote these constans by c_{jkh}. Next we want to show that

a) $c_{ijk} = 0$ whenever $j \neq k$ and
b) $c_{ijj} = c_{ikk}$ for all i, j, k.

To prove this we observe that

$$\left[x_i \frac{\partial}{\partial x_j}, x_k \frac{\partial}{\partial x_l}\right] = \delta_{jk} x_i \frac{\partial}{\partial x_l} - \delta_{li} x_k \frac{\partial}{\partial x_j}.$$

Applying D_U^1 to this, we obtain

$$\delta_{jk} \sum_h c_{ilh} \frac{\partial}{\partial x_h} - \delta_{li} \sum_h c_{kjh} \frac{\partial}{\partial x_h} = c_{ijk} \frac{\partial}{\partial x_l} - c_{kli} \frac{\partial}{\partial x_j} \quad \ldots *$$

If we take in * $k \neq l = j = i$ (this assumes that the dimension $n \neq 1$ because for $n = 1$ we cannot take $k \neq l$; if $n = 1$ however a) and b) above are trivially true) we obtain:

$$-\sum_h c_{klh} \frac{\partial}{\partial x_h} = c_{llk} \frac{\partial}{\partial x_l} - c_{kll} \frac{\partial}{\partial x_l}$$

from which it follows that $c_{klh} = 0$ if $l \neq h$ which proves a) above.

Next we take in * $k \neq j$ and $l = i$ and obtain (using the above result):

$$-c_{kjj} \frac{\partial}{\partial x_j} = -c_{kll} \frac{\partial}{\partial x_j} \text{ and hence:}$$

$$c_{kjj} = c_{kll} \text{ if } k \neq j, \text{ which implies } b).$$

From the above calculations it follows that for all i, j,

$$D_U^1\left(x_i \frac{\partial}{\partial x_j}\right) = \left[\sum_h c_{hhh} \frac{\partial}{\partial x_h}, x_i \frac{\partial}{\partial x_j}\right].$$

We now define $Z \in \chi(U)$ by

$$Z = \bar{Z} + \sum_h c_{hhh} \frac{\partial}{\partial x_h}$$

and observe that for all i, j,

$$D_U\left(\frac{\partial}{\partial x_i}\right) = \left[Z, \frac{\partial}{\partial x_i}\right] \text{ and } D_U\left(x_i \frac{\partial}{\partial x_j}\right) = \left[Z, x_i \frac{\partial}{\partial x_j}\right];$$

it is not hard to see that Z is uniquely determined by these properties. In order to complete the proof of this lmma we have to show that the derivation D_U^2, defined by $D_U^2(X) = D_U(X) - [Z, X]$ is identically zero:

SUB-LEMMA (4.1). *Let D_U and $U \subset \boldsymbol{R}^n$ be as in Lemma* 4. *If, for all i, j,*

$$D_U\left(\frac{\partial}{\partial x_i}\right) \equiv 0 \text{ and } D_U\left(x_i \frac{\partial}{\partial x_j}\right) \equiv 0$$

then $D_U(X) \equiv 0$ for all $X \in \chi(U)$.

PROOF of (4.1) We define the functions $D_{ijkl} : U \to \boldsymbol{R}$ by

$$D_U\left(x_i x_j \frac{\partial}{\partial x_k}\right) = \sum D_{ijkl} \frac{\partial}{\partial x_l}.$$

To prove that these functions are all constant one can proceed just as in the case with D_{ijk} above, but now we use the fact that

$$D_U\left(\left[\frac{\partial}{\partial x_i}, x_j x_k \frac{\partial}{\partial x_l}\right]\right) \equiv 0;$$

we omit the computation. We denote the corresponding constants again by c_{ijkl}. Next we observe that

$$\left[\sum_h x_h \frac{\partial}{\partial x_h}, x_i x_j \frac{\partial}{\partial x_k}\right] = x_i x_j \frac{\partial}{\partial x_k};$$

applying D_U to this we obtain:

$$\left[\sum_h x_h \frac{\partial}{\partial x_h}, \sum_l c_{ijkl} \frac{\partial}{\partial x_l}\right] = \sum c_{ijkl} \frac{\partial}{\partial x_l}, \text{ or} - \sum_l c_{ijkl} \frac{\partial}{\partial x_l} = \sum c_{ijkl} \frac{\partial}{\partial x_l};$$

hence all the constants c_{ijkl} are zero. In the same way one can show that

$$D_U\left(x_i x_j x_k \frac{\partial}{\partial x_l}\right) \equiv 0$$

for all i, j, k, l. Finally, we apply lemma 3 to obtain the proof: Let $X \in \chi(U)$ and $p \in U$, we want to show $D_U(X)(p) = 0$. There is a vectorfield $\hat{X} \in \chi(U)$ such that the coefficient functions of $\hat{X}$ are polynomials of degree $\leqq 3$ and such that $j^3(X)(p) = j^3(\hat{X})(p)$. By our previous computations we have $D_U(X) \equiv 0$ and by lemma 3 we have $D(X)(p) = D(\hat{X})(p)$; hence $D_U(X)(p) = 0$, this proves (4.1).

PROOF OF THE THEOREM. For a given derivation $D : \chi(M) \to \chi(M)$ and an open $U \subset M$, we get an induced derivation $D_U : \chi(U) \to \chi(U)$. This D_U is constructed as follows:

For $X \in \chi(U)$ an $p \in U$ one defines $D_U(X)(p)$ to be $D(X)(p)$, where $\tilde{X} \in \chi(M)$ is some vectorfield which equals X on some open neighbourhood of p. Clearly $D_U(X)(p)$ is well defined (by Lemma 1) and D_U is a derivation on $\chi(U)$.

Now we take an atlas $\{U_i, \varphi_i(U_i) \to \boldsymbol{R}^n\}$ of M such that each U_i is connected and simply connected. Using the coordinates $x_j\varphi_i$ on each U_i we can apply Lemma 4 to each D_{U_i} and obtain on each U_i a vectorfield $Z_i \in \chi(U_i)$ such that $D_{U_i}(X) = [Z_i, X]$ for each $X \in \chi(U_i)$.

As D_{U_i} and D_{U_j} both restricted to $U_i \cap U_j$ are equal, Z_i and Z_j both restricted to $U_i \cap U_j$ also have to be equal. Hence there is a vectorfleld $Z \in \chi(M)$ such that for each i, $Z_i = Z|U_i$. It follows easily that, for each $X \in \chi(M)$, $D(X) = [Z, X]$.

3. Remark

One can also take, instad of $\chi(M)$, the set of vectorfields which respect a certain given structure. To be more explicit, let ω be a differential form on M defining a symplectic structure or a volume structure, and let $\chi_\omega(M)$ be the Lie-algebra of those vectorfields X for which $L_X\omega \equiv 0$ (L_X means : Lie derivative with respect to X). Now one can ask again whether every derivation $D : \chi_\omega(M) \to \chi_\omega(M)$ is induced by a vectorfield $Z \in \chi_\omega(M)$. This is in general not the case. Take for example $M =$ è^n and $\omega = dx_1 \wedge \cdots \wedge dx_n$ the usual volume form and

$$Z = \sum_{i=1}^{\infty} x_i \frac{\partial}{\partial x_i}.$$

Then $Z \notin \chi_\omega(\boldsymbol{R}^n)$ but for each $X \in \chi_\omega(\boldsymbol{R}^n)$, $[Z, X] \in \chi_\omega(\boldsymbol{R}^n)$; so '$[Z, -]$' is a derivation on $\chi_\omega(\boldsymbol{R}^n)$. This derivation cannot be induced by any $Z' \in \chi_\omega(\boldsymbol{R}^n)$.

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