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CORRECTION TO 'ON THE PURITY OF THE BRANCH LOCUS'

by

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The proof ([2, p. 464]) fails because the algebra of principal parts $P^m(A)$ is not a finitely generated A-module. However, the proof does go through if we replace $P^m(A)$ by the algebra of topological principal parts ${}^tP^m(A)$, defined below. We check this after proving two preliminary results of independent interest.

PROPOSITION. *Let R be a ring, q an ideal of R, and M a finitely generated R-module. Consider the following separated completions:*

$$\hat{R} = \varprojlim (R/q^n); \ \hat{M} = \varprojlim (M/q^nM).$$

Assume $\hat{R}$ is noetherian and q is finitely generated. Then there is a canonical isomorphism,

$$\hat{R} \otimes_R M = \hat{M}.$$

PROOF. With no finiteness assumptions on $\hat{R}$ and q, the canonical map,

$$\hat{R} \otimes_R M \to \hat{M},$$

is surjective (the proof is straightforward, see [3, p. 108]). Therefore, since q is finitely generated, we have an equality,

$$q^n\hat{R} = (q^n)\hat{}\,,$$

for each positive integer n. So, since $q^n\hat{R}$ is obviously equal to $(q\hat{R})^n$, we obtain an equality,

$$(\hat{q})^n = (q^n)\hat{}\,.$$

Consequently, (GD II, 1.10), there is a canonical isomorphism,

$$(1) \qquad \hat{R}/(\hat{q})^n = R/q^n.$$

Hence, $\hat{R}$ is equal to $\varprojlim (\hat{R}/(\hat{q})^n)$; in other words, $\hat{R}$ is separated and complete with respect to the $\hat{q}$-adic topology.

Since $\hat{R}$ is noetherian and $\hat{R} \otimes_R M$ is, obviously, finitely generated over $\hat{R}$, the $\hat{q}$-adic separated completion $(\hat{R} \otimes_R M)\hat{}$ is equal, by (GD

II, 1.18), to $\hat{R} \otimes_{\hat{R}} (\hat{R} \otimes_R M)$, so to $\hat{R} \otimes_R M$; in other words, we have a canonical isomorphism,

$$\hat{R} \otimes_R M = \varprojlim ((\hat{R} \otimes_R M)/(\hat{q})^n(\hat{R} \otimes_R M)).$$

Now, by basic properties of tensor product and by (1), for each n we have

$$\begin{aligned}(\hat{R} \otimes_R M)/(\hat{q})^n(\hat{R} \otimes_R M) &= (\hat{R}/(\hat{q})^n) \otimes_{\hat{R}} (\hat{R} \otimes_R M) \\ &= (\hat{R}/(\hat{q})^n) \otimes_R M \\ &= (R/q^n) \otimes_R M \\ &= M/q^n M.\end{aligned}$$

Passing to the projective limit over n, we obtain the proposition.

In the next two results, let k be a noetherian ring, let A be a noetherian k-algebra that is separated and complete with respect to the adic topology of an ideal m such that $K = A/m$ is a finitely generated k-algebra, and let B be an A-algebra that is a finitely generated A-module. The complete tensor products $A \hat{\otimes}_k A$ and $B \hat{\otimes}_k B$ are defined as the separated completions of $A \otimes_k A$ and $B \otimes_k B$ with respect to the adic topology of the ideals,

$$\begin{aligned} M &= (m \otimes_k A + A \otimes_k m) \\ N &= ((mB) \otimes_k B + B \otimes_k (mB)) = M(B \otimes_k B). \end{aligned}$$

The k-algebras of mth order topological principal parts are defined by

$$\begin{aligned} {}^tP^m(A) &= (A \hat{\otimes}_k A)/I^{m+1} \\ {}^tP^m(B) &= (B \hat{\otimes}_k B)/J^{m+1} \end{aligned}$$

where I (resp. J) denotes the kernel of the map $A \hat{\otimes}_k A \to A$ (resp. $B \hat{\otimes}_k B \to B$) that takes $a \hat{\otimes} b$ to ab.

COROLLARY. *Under the above conditions, $A \hat{\otimes}_k A$ is noetherian and there is a canonical $(A \hat{\otimes}_k A)$-algebra isomorphism,*

$$(A \hat{\otimes}_k A) \otimes_{(A \otimes_k A)} (B \otimes_k B) = B \hat{\otimes}_k B.$$

PROOF. The ring $(A \otimes_k A)/M$ is noetherian, for it is equal to $K \otimes_k K$, which is, clearly, a finitely generated algebra over the noetherian ring k. Moreover, M is a finitely generated ideal of $(A \otimes_k A)$, for m is an ideal in the noetherian ring A. Hence, $A \hat{\otimes}_k A$ is noetherian (GD II, 1. 22). Clearly, $B \otimes_k B$ is a finitely generated $(A \otimes_k A)$-module. Therefore, the second assertion follows from the proposition.

LEMMA. *Under the above conditions, assume that the structure morphism,*

$$f : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A),$$

is étale over a nonempty open subset V of Spec (A).

(i) *For each $m \geqq 0$, the $(A \hat{\otimes}_k A)$-algebra homomorphisms,*

$$\begin{array}{ll} {}_m v : {}^tP^m(A) \otimes_A B \to {}^tP^m(B), & v_m : B \otimes_A {}^tP^m(A) \to {}^tP^m(B) \\ (a \hat{\otimes} a') \otimes b \mapsto a \hat{\otimes} (a'b) & b \otimes (a \hat{\otimes} a') \mapsto (ab) \hat{\otimes} a', \end{array}$$

are isomorphisms over V, where ${}^tP^m(A)$ and ${}^tP^m(B)$ are regarded as A-algebras first from the right, then from the left.

(ii) *The canonical map,*

$$v : gr_I^{\bullet}(A \hat{\otimes}_k A) \otimes_A B \to gr_J^{\bullet}(B \hat{\otimes}_k B),$$

is an isomorphism over V, where I (resp. J) denotes the kernel of the map $A \hat{\otimes}_k A \to A$ (resp. $B \hat{\otimes}_k B \to B$) that takes $a \hat{\otimes} b$ to ab.

PROOF. (i) Filtered by the powers of I (resp. of I, resp. of J), the $(A \hat{\otimes}_k A)$-algebra ${}^tP^m(A) \otimes_A B$ (resp. $B \otimes_A {}^tP^m(A)$, resp. ${}^tP^m(B)$) is separated and complete, the filtration being finite; so, by (GD II, 1.5, 1.21), it suffices to prove that $gr^{\bullet}({}_m v)$ and $gr^{\bullet}(v_m)$ are isomorphisms over V.

Consider the composition

$$[gr_I^{\bullet}({}^tP^m(A))] \otimes_A B \to gr_I^{\bullet}({}^tP^m(A) \otimes_A B) \to gr_J^{\bullet}({}^tP^m(B)).$$

The right hand map is obviously equal to both $gr({}_m v)$ and $gr(v_m)$. The left hand map is an isomorphism over V since f is flat over V. Finally, the composition is a truncation of v, so an isomorphism by (ii). Thus, (i) holds.

(ii) Consider the following diagram:

$$\begin{array}{ccccc} \operatorname{Spec}(B \otimes_A B) & \longrightarrow & \operatorname{Spec}(B \hat{\otimes}_k B) & \longrightarrow & \operatorname{Spec}(B \otimes_k B) \\ \downarrow & \square & \downarrow & \square & \downarrow \\ \operatorname{Spec}(A) & \longrightarrow & \operatorname{Spec}(A \hat{\otimes}_k A) & \longrightarrow & \operatorname{Spec}(A \otimes_k A). \end{array}$$

The right hand square is cartesian by the corollary; the left, by the following computation involving the corollary:

$$(2) \quad \begin{aligned} A \otimes_{(A \hat{\otimes}_k A)} (B \hat{\otimes}_k B) &= A \otimes_{(A \hat{\otimes}_k A)} [(A \hat{\otimes}_k A) \otimes_{(A \otimes_k A)} (B \otimes_k B)] \\ &= A \otimes_{(A \otimes_k A)} (B \otimes_k B) = (B \otimes_A B). \end{aligned}$$

The right vertical map is equal to $f \times f$, and $f \times f$ is, by (GD V, 2.7 iv), flat over $V \times V$. Hence, by (GD V, 2.7. iii), the middle vertical map is flat over the inverse image of $V \times V$ in Spec $(A \hat{\otimes}_k A)$. Therefore, by (GD V, 3.2), the canonical map of modules over $A = (A \hat{\otimes}_k A)/I$,

$$v' : [gr_I^{\bullet}(A \hat{\otimes}_k A)] \otimes_A (B \otimes_A B) \to gr_I^{\bullet}(B \hat{\otimes}_k B)$$

is an isomorphism over V, because $(B \hat{\otimes}_k B)/I(B \hat{\otimes}_k B)$ is equal to $B \otimes_A B$ by (2).

Since f is unramified over V, the diagonal map,

$$\operatorname{Spec}(B) \to \operatorname{Spec}(B \otimes_A B),$$

is an open embedding (GD VI, 3.3). However, it is also the closed embedding defined by J. Now, whenever Z, Y are two closed subschemes of a scheme X, and Z is an open subscheme of Y, then the canonical map,

$$gr^{\bullet}_{I(Y)}(\mathcal{O}_X)|Z \to gr^{\bullet}_{I(Z)}(\mathcal{O}_Z),$$

is an isomorphism, where $I(Y)$ is the ideal of Y and $I(Z)$, that of Z; indeed, the assertion is local on X and need only be checked on Z, and the ideals $I(Y)$ and $I(Z)$ coincide in a neighborhood of each point of Z. Therefore, the canonical map,

$$v'' : gr^{\bullet}_I(B \hat{\otimes}_k B) \otimes_{(B \otimes_A B)} B \to gr^{\bullet}_J(B \hat{\otimes}_k B),$$

is an isomorphism over V. So, since $v = v'' \circ (v' \otimes_{(B \otimes_A B)} B)$ holds, v is also an isomorphism over V.

THEOREM. *Let k be a noetherian ring, $A = k[[T_1, \cdots, T_n]]$ a formal power series ring. Let B be a finite A-algebra that is étale over every prime ideal p of A where depth $(B_p) \leqq 1$ holds. Then, there exists a canonical A-algebra isomorphism $u_0 : A \otimes_k B_0 \xrightarrow{\sim} B$ with $B_0 = B/(T_1 B + \cdots + T_n B)$.*

PROOF. Give A the $(T_1 A + \cdots + T_n A)$-adic topology. We are going to construct an isomorphism of $(A \hat{\otimes}_k A)$-algebras,

$$u : (A \hat{\otimes}_k A) \otimes_A B \to B \otimes_A (A \hat{\otimes}_k A),$$

where $A \hat{\otimes}_k A$ is regarded as an A-algebra via the second factor in $(A \hat{\otimes}_k A) \otimes_A B$ and via the first in $B \otimes_A (A \hat{\otimes}_k A)$. Then, u yields u_0 as follows. Consider the diagram,

$$\begin{tikzcd} A & A \hat{\otimes}_k A \arrow[l, "w"'] \\ k \arrow[u, "j"] & A \arrow[l, "e"'] \arrow[u, "j_2"'] \end{tikzcd}$$

where j is the structure map, where $j_2(a) = 1 \hat{\otimes} a$ holds, where $e(a)$ is the constant term of a, and where $w(a_1 \hat{\otimes} a_2) = e(a_2) \cdot a_1$ holds. The diagram is obviously commutative and so we have a canonical isomorphism,

$$A \otimes_{(A \hat{\otimes}_k A)} [(A \hat{\otimes}_k A) \otimes_A B] = A \otimes_k (k \otimes_A B).$$

Since $k \otimes_A B$ is obviously equal to B_0, we obtain

$$A \otimes_{(A\hat{\otimes}_k A)} [(A \hat{\otimes}_k A) \otimes_A B] = A \otimes_k B_0.$$

On the other hand, setting $j_1(a) = a \hat{\otimes} 1$, we have $w \circ j_1 = id_A$, and so we have another canonical isomorphism,

$$A \otimes_{(A\hat{\otimes}_k A)} [B \otimes_A (A \hat{\otimes}_k A)] = B.$$

Therefore, $A \otimes_{(A\hat{\otimes}_k A)} u$ is equal to the desired isomorphism u_0.

We now construct u. Since A is a formal power series ring, ${}^tP^m(A)$, regarded as an A-algebra either on the left or right, clearly has the form

$${}^tP^m(A) = A[[U_1, \cdots, U_n]]/(U_1, \cdots, U_n)^{m+1},$$

where $U_1, \cdots, U_n$ are indeterminates (in fact, $U_i = T_i \hat{\otimes} 1 - 1 \hat{\otimes} T_i$ holds); thus, ${}^tP^m(A)$ is a free A-module of finite rank, say r. Therefore, $B \otimes_A {}^tP^m(A)$ and ${}^tP^m(A) \otimes_A B$ are both isomorphic to $B^{\oplus r}$. Hence, by the hypothesis on B, these A-modules have depth $\leqq 1$ only at points of Spec (A) over which B is étale.

Consider the A-module,

$$M = \mathrm{Hom}_A ({}^tP^m(A) \otimes_A B,\ B \otimes_A {}^tP^m(A)),$$

where both arguments are considered as A-modules on the left (so the second is isomorphic to $B^{\oplus r}$, but not necessarily the first). The lemma implies that both arguments are canonically isomorphic to ${}^tP^m(B)$ as $(A \hat{\otimes}_k A)$-algebras over the open subset V of Spec (A) where B is étale; hence, since by (EGA I, 1.3.12) we have

$$\tilde{M} = \underline{\mathrm{Hom}}\, (({}^tP^m(A) \otimes_k B)^{\sim}, (B \otimes_A {}^tP^m(A))^{\sim}),$$

$\tilde{M}$ has a canonical section over V. By Lemma 2 ([2], p. 463), V contains every point p where depth $(M_p) \leqq 1$ holds. So, by Lemma 3(ii) ([2], p. 463), this section is defined by an element u_m of M; in fact, by Lemma 3(i) ([2], p. 463), u_m is an $(A \hat{\otimes}_k A)$-algebra homomorphism since it is on V. Similarly, we obtain an inverse to u_m (first on V, then globally).

Clearly $A \hat{\otimes}_k A$ is I-adically separated and complete. So, since $(A \hat{\otimes}_k A) \otimes_A B$ is a finitely generated $(A \hat{\otimes}_k A)$-module, it is also I-adically septarad and complete. By right exactness of $\otimes_A B$, we have

$${}^tP^m(A) \otimes_A B = ((A \hat{\otimes}_k A) \otimes_A B)/(I^{m+1}((A \hat{\otimes}_k A) \otimes_A B)).$$

Hence, we have

$$(A \hat{\otimes}_k A) \otimes_A B = \varprojlim\, ({}^tP^m(A) \otimes_A B).$$

Similarly, we have

$$B \otimes_A (A \hat{\otimes}_k A) = \varprojlim (B \otimes_A {}^tP^m(A)).$$

Finally, the various isomorphisms clearly form a compatible system of maps, so they induce the desired $(A \hat{\otimes}_k A)$-algebra isomorphism u.

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