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ON A NEW METHOD IN INTUITIONIST LINEAR ANALYSIS

by

Sahab Lal Shukla

1. Introduction

A prominent feature of intuitionist functional analysis is that many spaces which are classically Banach spaces, i.e., complete normed linear spaces, cannot be constructively equipped with a norm. It is the first impulse of an intuitionist used to attaching greater importance to more stringent constructive conditions to confine himself only to normed spaces, but this attitude is bound to lead to a serious curtailment of his field of activity because even the dual spaces of the simplest infinite-dimensional normed spaces cannot be normed constructively. Thus, e.g., only a very partial result concerning the Riesz Representation theorem for Hilbert spaces is obtained in [6]. The major difficulty in dealihg with non-normable spaces is of course the problem of treating nearness and derived notions such as convergence, continuity etc. An attempt to deal with this type of difficulty has been made in the last chapter of [6] but, a satisfactory mathematical tool namely that of a distance delimitation together with the necessary distinction between cataloguings of the first and the second kind appears finally in [7]. Using this tool it became possible to determine the dual and the second dual of the Hilbert space as well as of the space of continuous functions (c.f. [8]). The same technique is further exploited in [11] for the determination of dual spaces of various sequence spaces. In the same article the dual space of b^1 is determined not by appealing to the cataloguing of its closed unit sphere but by concentrating on the action of the full linear functional on a certain suitably chosen fan extracted from the closed unit sphere. In this article we shall present a considerable extension of that method. This procedure of course has the disadvantage that whereas the use of the cataloguing of the closed unit sphere of the space permits one even to conclude the continuity of non-linear operators, this method is adapted only for use with linear operators. However, for these, it definitely has an advantage, viz., that the exercise of a little inguinity in the extraction of a suitable dressed spread from the closed unit sphere of the space in question renders the determination of its cataloguing completely superfluous. However, the greatest advantage is that this method can even be applied successfully for the determination of duals of non-linear spaces and of spaces whose cataloguings do not permit us to form a definite conclusion about the dual spaces (e.g., b^1, and b_0^p later in this article), or whose cataloguings have been not determined so far and which might tern out to be uncataloguable (e.g., ${}_1b^p$ and q^1). Let us mention that it is by the use of this method that the first examples of dual pairs of spaces neither of which is normable (e.g., b_0^p, ${}_1b^q$) as well as of non-α-reflexive spaces (e.g., l^∞, q^1) were given. Moreover, it has recently been shown that this method is applicable with equal facility to function spaces (e.g., L^∞).

We refer the reader to [11], § 1 for a definition of all relevant notions in intuitionist functional analysis: at that place references are also given to the original articles in which the concerned definitions were introduced. We summarize here the points of departure from the conventions laid down there and establish some further notations which we shall use.

In the first place, the correct definition of quasi-number, given in [10], ought to be substituted for that cited in [11]. Secondly, we should more precisely say that a space B is *α-reflexive* if to every full linear functional Φ on B^* there corresponds an f in B such that $\Phi(f^*) = f^*(f)$ for every f^* in B^*, while also $|||\Phi||| = |||g|||$. The definitions of β-, γ- and δ-reflexivity should also be similarly interpreted. Thirdly, it may be remarked here that there is an apparent discrepancy between the statements of results in this article and in [11], where the notation of [19] has been used, the isomorphic equality in the context of dual spaces signifying a conjugate linear (anti-linear) one-one correspondence. In the present article, we use the term isomorphic equality to signify a linear one-one quasi-norm preserving correspondence (c.f. e.g., [18]). Accordingly, whereas duality in [11] is studied in terms of the sesquilinear form (f, g), here we have rather used the bilinear form $[f, g]$ (which are both defined below).

The elements of our sequence spaces will be denoted by $f, g, h, \cdots$. The n-th component of a vector f will be denoted by $f(n)$, i.e.,

$$f = (f(1), f(2), f(3), \cdots) = (f(n))_{n=1}^{\infty}.$$

If the sequence $f(1), f(2), \cdots$ is of bounded variation then the quasi-number containing it will also be denoted by $f(\infty)$. In accordance with customary usage, $f(0), f(-1), f(-2), \cdots$ will be taken to zero for a sequence $(f(n))_{n=1}^{\infty}$. For any f, ${}^nf$ will denote the vector given by ${}^nf(k) = f(k)$ for $k \leqq n$ and ${}^nf(k) = 0$ for $k > n$.

e_m denotes the vector given by $e_m^{(n)} = \delta_{mn}$ where δ_{mn} is the Kronecker delta. e_0 is given by $e_0(0) = 1$; $e_0(n) = 0$ for $n = 1, 2, \cdots$. e is the vector given by $e(n) = 1$ for each n, i.e., $e = (1, 1, 1, \cdots)$.

For two sequences f and g, we shall use the notation $[f, g]$ to denote the infinite series $\sum f(n)\, g(n)$, and (f, g) to denote $\Sigma f(n)\, \overline{g(n)}$.

A particular distance delimitation, which will be used for the closed unit spheres of several different sequence spaces is ω given by the following specification: n lies in $\omega(f, 0)$ if $|f(k)| < 2^{-n}$ for $1 \leqq k \leqq n$, $\omega(f, g) = \omega(f-g, 0)$. It follows immediately that ${}^{n}f$ ω-converges to f.

We shall use in this article the fan L whose direction consists of two nodes of order one, and each node in it containing only l's has exactly two immediate descendants, while each node in which a 2 occurs somewhere has exactly one immediate descendant.

We introduce the following concise notation. A node p of order $n = m_1 + m_2 + \cdots + m_{2r}$ which consists of m_1 1's, followed by m_2 2's, followed by m_3 1's, $\cdots$ will be denoted by

$$1^{m_1}2^{m_2}1^{m_3} \cdots 2^{m_{2r}} :$$

nodes of other forms (e.g., beginning with a 2, or ending with a 1) are denoted similarly. If p is the node

$$1^{m_1}2^{m_2} \cdots 1^{m_{2r-1}}2^{m_{2r}}$$

and p' its descendant

$$1^{m_1}2^{m_2} \cdots 1^{m_{2r-1}}2^{m_{2r}}1^{m_{2r+1}} \cdots 1^{m_{2s+1}}$$

then we shall also write

$$p' = p\, 1^{m_{2r+1}}2^{m_{2r+2}} \cdots 1^{m_{2s+1}}.$$

Instead of an arrow, we shall always consider the corresponding sequence of last constituents of nodes; and denote it by $1^{m_1}2^{m_2}1^{m_3}, \cdots$ in case a sequence is of the form

$$1^{m_1}2^{m_2}1^{m_3} \cdots 2^{m_{2r}}1, 1, 1, \cdots,$$

we shall briefly write

$$1^{m_1}2^{m_2}1^{m_3} \cdots 2^{m_{2r}}1^{\infty};$$

or even $p1^{\infty}$ where p denotes the node

$$1^{m_1}2^{m_2} \cdots 2^{m_{2r}}.$$

In the course of construction of various dual spaces we shall utilize the spread Λ whose direction consists of countably many nodes of order one, in which each node containing only l's has countably many immediate descendants, while each node in which a natural number $n (n > 1)$ occurs somewhere has exactly one immediate descendant. For technical ease of description later on, we dress the spread Λ as follows. The node

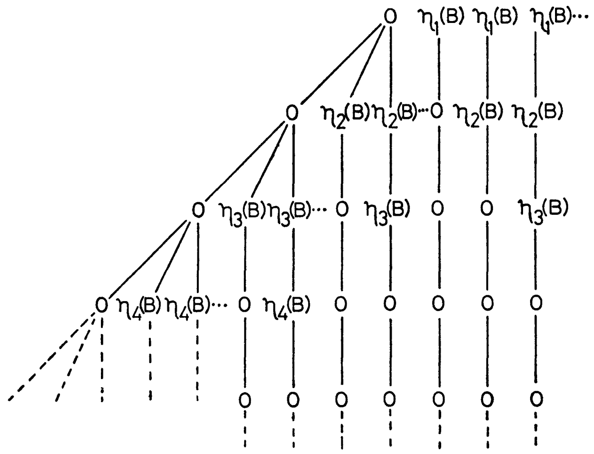

Fig. 1. The Fan L

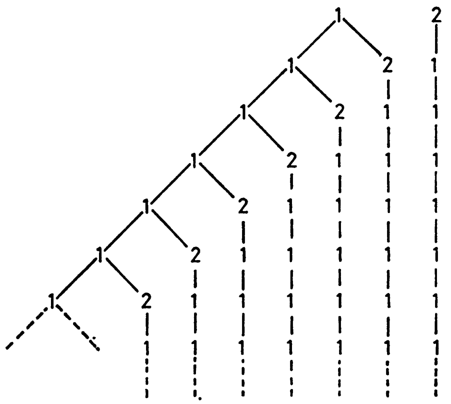

Fig. 2a. The spread Λ

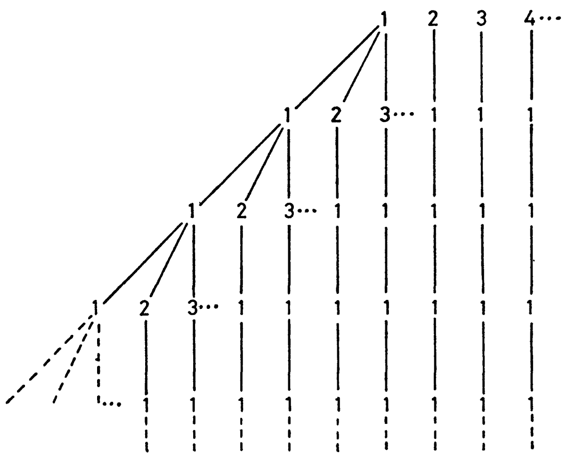

Fig. 2b. The dressed spread $\Lambda(B)$

(1^n) (which consists of n 1's) is dressed by n zeros, while a node $(1^m k 1^n)$ (which consists of m 1's followed by a k followed by n 1's) is dressed by m zeros followed by complex numbers

$$\eta_{m+1}, \eta_{m+2}, \cdots \eta_{m+n+1} \text{ if } n \leqq k-2,$$

while it is dressed by

$$\eta_{m+1}, \cdots, \eta_{m+k-1}$$

followed by $n-k+2$ zeros if $n > k+2$.

The η_k's used will depend upon the space in question, and if necessary, this dependence will be indicated by writting $\eta_k(B)$. The corresponding dressed spread will be denoted by $\Lambda(B)$. As usual, q will denote the index conjugate to

$$p : q = \frac{p}{p-1} \left(\text{i.e., } \frac{1}{p} + \frac{1}{q} = 1\right)$$

for $p > 1$, $q = \infty$ for $p = 1$, $q = 1$ for $p = \infty$. In addition, r and s will also be used for conjugate indices, i.e.,

$$\frac{1}{r} + \frac{1}{s} = 1 \text{ for } r > 1,\ s = \infty \text{ for } r = 1, s = 1 \text{ for } r = \infty.$$

2. Determination of duals of dual spaces of certain normed spaces and of related spaces

2.1. The first space with which we commence is the space $b^p (1 \leqq p < \infty)$ of all complex sequences f for which

$$\sum_{i=1}^{\infty} |f(i)|^p$$

is bounded, the quasi-norm of f in b^p being $|||f|||$ = the quasi-number containing

$$(|f(1)|, (|f(1)|^p + |f(2)|^p)^{1/p}, (|f(1)|^p + f(2)|^p + |f(3)|^p)^{1/p}, \cdots).$$

Let Φ be a full linear functional on b^p, and set $g(i) = \Phi(e_i)$. For $p > 1$, we dress the spread Λ as follows. For each fixed k, for each n, we can choose exactly one assertion from those recognized as valid of the two assertions for a sequence g:

$$\text{(i) } |g(n)| < 2^{-k-n} \text{ and (ii) } |g(n)| > 2^{-k-n-1}.$$

Now take $y_g^k(n) = 0$ if (i) has been chosen and $y_g^k(n) = |g(n)|^{q-1} \operatorname{sgn} \overline{g(n)}$ if (ii) has been chosen. Obviously

$$|y_g^k(n) - y_g^{k+k'}(n)| < (2^{-k-n})^{q-1}$$

for

$$k' \geqq 0 : \lim_{k\to\infty} y_g^k(n)$$

therefore exists; and we denote it by $y_g(n)$, and we have

$$g(n)y_g(n) = \lim_{k\to\infty} g(n)y_g^k(n) = |g(n)|^q.$$

For the space b^p we take $\eta_k = y_g(k)$ in the dressed spread Λ to obtain the dressed spread $\Lambda(b^p)$. Considering the dressed spread $\Lambda(b^p)$ as a catalogued space, we conclude by the continuity theorem that to each k, an $N(k)$ can be assigned such that $|\Phi(0)-\Phi(f)| < 2^{-k}$ for any f in $\Lambda(b^p)$ which passes through the dressed node consisting of N zeros. If we take the vector $(0, 0, 0, \cdots, y_g(n+1), y_g(n+2), \cdots, y_g(n+m), 0, 0, \cdots)$ (n zeros in the beginning), then for $n \geqq N$,

$$|\sum_{i=n+1}^{n+m} g(i)y_g(i)| = \sum_{i=n}^{n+m} |g(i)|^q < 2^{-k}.$$

From this follows that

$$\sum_{i=1}^{\infty} |g(i)|^q$$

is convergent.

We now introduce the normed space $l^p (1 < p < \infty)$, consisting of all f with convergent

$$\sum_{i=1}^{\infty} |f(i)|^p,$$

which is a congruent subspecies of b^p but cannot coincide with it, the norm being the same as in the case of b^p. With this notation we have proved that to each full linear functional Φ on b^p, a vector g of l^q can be assigned such that $\Phi(f) = [f, g]$.

We shall now establish the ω-continuity of Φ. Given k, we first of all indicate an $N = N(k)$ such that

$$\sum_{i=n}^{n+m} |g(i)|^q < 2^{-(k+1)q}$$

and $n \cdot 2^{-n} \cdot ||g|| \not> 2^{k-1}$ for $n \geqq N(k)$ and $m \geqq 0$. If now $n \geqq N$ lies in $\omega(f, 0)$, then

$$\begin{aligned}|\Phi(f| &= |\sum_{i=1}^{\infty} f(i)g(i)| \not> \sum_{i=1}^{n} |f(i)g(i)| + \sum_{i=n+1}^{\infty} |f(i)g(i)| \\ &\not> (\sum_{n=1}^{n} |f(i)|^p)^{1/p} (\sum_{i=1}^{n} |g(i)|^q)^{1/q} + (\sum_{i=n+1}^{\infty} |f(i)|^p) \quad (\sum_{i=n+1}^{\infty} |g(i)|^q)^{1/q} \\ &\not> n2^{-n}||g|| + 1.2^{-k-1} \not> 2^{-k-1} + 2^{-k-1} = 2^{-k}.\end{aligned}$$

We shall prove that every full linear functional Φ has the supremum $||g||$ on the closed unit sphere of b^p. It is a consequence of Hölder's inequality that Φ is bounded by $||g||$ for all vectors f belonging to the closed unit sphere of b^p. Let

$$^{n}f^{k} = (y_g^k(1), y_g^k(2), \cdots, y_g^k(n), 0, 0, 0, \cdots).$$

However,

$$\sum_{i=1}^{n} |g(i)|^q - 2^{-(k+2)q} - 2^{-(k+3)q} - \cdots - 2^{-(k+n+1)q} \ngtr \sum_{i=1}^{n} y_g^k(i) g(i) = \Phi(^{n}f^{k})$$

for every n. Now corresponding to given k, there exists an $N(k)$ such that

$$\sum_{i=N+1}^{\infty} |g(i)|^q < 2^{-(k+1)q}.$$

Hence

$$\Phi(^{n}f^{n}) > \sum_{i=1}^{\infty} |g(i)|^q - 2^{-(k+1)q} - 2^{-(k+1)q} > \sum_{i=1}^{\infty} |g(i)|^q - 2^{-kq-q+1},$$

which proves our assertion, so that Φ is a normed linear functional on b^p and $||g||$ is the supremum of Φ on b^p.

It will be noticed that the cataloguing of the closed unit sphere of b^p has been not utilized in this proof.

Let f belong to the closed unit sphere of b^p and g be an element of l^q, then Φ defined by $\Phi(f) = [f, g]$ is convergent (by Hölder's inequality) and hence Φ is a full linear functional on b^p. We have now proved that corresponding to each full linear functional Φ of b^p a vector g of l^q can be assigned such that $\Phi(f) = [f, g]$ and conversely, for each f belonging to b^p and g belonging to l^q, $[f, g]$ is convergent and defines a full linear functional on b^p. Thus we have proved

THEOREM 1. $(b^p)^* = (b^p)' = l^q$.

It has been proved in [11] Theo. I that $(l^p)^* = b^q$ and $(l^p)' = l^q$. A proof of $(b^1)^* = (b^1)' = c_0$ utilizing a dressing of the fan direction L is given in [11] theo. III.

2.2. b^∞ is the space of all bounded complex sequences f for which the quasi-norm of f is the quasi-number containing the bounded monotone sequence $(|f(1)|, \max(|f(1)|, |f(2)|), \max(|f(1)|, |f(2)|, |f(3)|) \cdots)$.

Let Φ be a full linear functional defined on b^∞ and let $\Phi(e_i) = g(i)$. For each fixed j, for each n, we can choose exactly one assertion from those recognized as valid of the two assertions for a sequence $(g(n))_n$:

$$\text{(i) } |g(n)| < 2^{-j-n} \text{ and (ii) } |g(n)| > 2^{-j-n-1}.$$

Now take $y_g^j(n) = 0$ if (i) has been chosen and $y_g^j(n) = \operatorname{sgn} \overline{g(n)}$ if (ii) has been chosen. We dress the direction of the spread Λ by taking

$\eta_k = y_g^j(k)$ to obtain the dressed spread $\Lambda(b^\infty)$. As Φ is full on $\Lambda(b^\infty)$, we conclude that to each k, an $N(k)$ can be assigned such that $|\Phi(0)-\Phi(f)| < 2^{-k}$ for any f in $\Lambda(b^\infty)$ passing through the dressed node consisting of N zeros. Applying this to the vector

$$(0, 0, \cdots, 0, y_g^j(n+1), y_g^j(n+2), \cdots, y_g^j(n+m), 0, 0, 0, \cdots)$$

(n zeros in the beginning for $n \geqq N(k)$, we have

$$|\sum_{i=n+1}^{n+m} g(i)y_g^j(i)| \not> \sum_{i=n+1}^{n+m} |g(i)| - 2^{-j-n} < 2^{-k}.$$

From this follows that

$$\sum_{i=1}^{\infty} |g(i)|$$

is convergent. We have thus proved that to each full linear functional Φ on b^∞, a vector g of l^1 can be assigned such that $\Phi(f) = [f, g]$.

The ω-continuity of Φ as well as the fact that any full linear functional Φ necessarily possesses a supremum on the closed unit sphere of b^∞ equal to $||g||$ can be proved exactly in the same way as for b^p : we have thus completely established the theorem.

THEOREM 2. $(b^\infty)^* = (b^\infty)' = l^1$.

2.3. We now treat the non-linear space l^∞ of all bounded complex sequences f which possess suprema. The norm of any vector f belonging to l^∞ is

$$||f|| = \sup_i |f(i)|.$$

We shall show that our method has the greater scope of being applicable even to such non-linear spaces.

By mapping a vector f of $\Lambda(b^\infty)$ into a vector f' of l^∞ given by $f'(1) = 1$, $f'(n) = f(n-1)$ for $n \geqq 2$, we obtain a species $\Lambda'(b^\infty)$ contained in the closed unit sphere of l^∞, on which the above considerations are applicable without change and we conclude as above that $(l^\infty)^* = (l^\infty)' = l^1$. It has been mentioned in ([11]) that the dual space of l^1 is b^∞, i.e., $(l^1)^* = b^\infty$ and $(l^1)' = l^\infty$.

2.4. We now introduce the space qc which is the species of all sequences

$$f = (f(n))_{n=1}^{\infty}$$

for which a sequence of natural numbers $N(1), N(2), \cdots$ can be indicated such that

$$\sum_{k=0}^{\infty} |f(n(k+1))-f(n(k)+1)|$$

is bounded whenever $n(0) < n(1) < \cdots$ are chosen in such a way that $0 \leqq n(0) < N(1)$, $N(k) \leqq n(k) < N(k+1)$. The quasi-norm of b^∞ is used here. It is obvious that qc is contained in the space b^∞ and contains c. The cataloguing of the closed unit sphere of qc has not been determined here, but we determine the dual space of qc by our method.

Let Φ be a full linear functional defined on qc and let $\Phi(e_k) = g(k)$. Now the dressed spread $\Lambda(b^\infty)$ constructed in theorem II above is contained in the space qc so by the same reasoning as given there we can prove that to the full linear functional Φ of qc, a vector g of l^1 can be assigned such that $\Phi(f) = [f, g]$.

Conversely, if g is any vector of l^1, then $[f, g]$ converges for every vector f of qc since f even lies in b^∞; and this thus determines a full and hence a normed linear functional of b^∞, and hence, a fortiori of qc. Thus we have proved

THEOREM 3. $(qc)^* = (qc)' = l^1$.

2.5. We now discuss the space bv which is the space of all complex sequences of bounded variation, i.e., for any f in bv

$$\sum_{i=1}^{\infty} |\Delta f(i)|$$

(where $f(0) = 0$) is bounded. $|||f|||$ is defined as the quasi-number core containing the bounded monotone sequence $t_2, t_3, \cdots$ where

$$t_n = \sum_{i=2}^{n} |\Delta f(i)| + |f(n)|.$$

We now proceed to determine the dual space of bv without using the cataloguing of the closed unit sphere of this space.

Let Φ be a full linear functional on bv, and let $\Phi(e_i) = g(i)$. For each fixed j, for each n, we can choose exactly one assertion from those recognized as valid of the two assertions for a sequence $g = (g(n))_n$:

$$\text{(i) } |g(n)| < 2^{-j-n} \text{ and (ii) } |g(n)| > 2^{-j-n-1}.$$

Now take $y_g^j(n) = 0$ if (i) has been chosen and $y_g^j(n) = 1$ if (ii) has been chosen. We take $\eta_k = y_g^j(k)$ to obtain the dressed spread $\Lambda(bv)$. We conclude by the continuity theorem that to each k, an $N(k)$ can be assigned such that $|\Phi(0) - \Phi(f)| < 2^{-k}$ for any f in $\Lambda(bv)$ passing through the dressed node consisting of N zeros. Applying this to the vector

$$(0, 0, 0, \cdots, 0, y_g^j(n+1), y_g^j(n+2), \cdots, y_g^j(n+m), 0, 0, 0, \cdots)$$

(with initial n zeros) for $n \geqq N(k)$, we have

$$|\sum_{i=n}^{n+m} g(i)| - 2^{-j-n} < 2^{-k}.$$

From this follows that

$$\sum_{i=1}^{\infty} g(i)$$

is convergent.

We now introduce the normed space w^1 consisting of all f with convergent

$$\sum_{i=1}^{\infty} f(i).$$

The norm is

$$||f|| = \sup_n |\sum_{i=1}^{n} f(i)|.$$

With this notation we have proved that to each full linear functional Φ on bv, a vector g of w^1 can be assigned such that $\Phi(f) = [f, g]$.

The continuity of Φ follows from the fact that the closed unit sphere of bv is catalogued. We insert here an alternative proof of this continuity, which does not utilize this cataloguing. Given k, we first of all indicate an $N = N(k)$ such that $n \cdot 2^{-2}||g|| \not> 2^{-k}$ for $n \geqq N(k)$. If now f is in the closed unit sphere of bv and $n \geqq N(k)$ lies in $\omega(f, 0)$, then

$$|\Phi(f)| = |\sum_{i=1}^{\infty} f(i)g(i)| \not> 2^{-k}.$$

For, taking $n \geqq N(k)$, we have

$$\begin{aligned} |\sum_{i=1}^{n} f(i)g(i)| &= |\sum_{i=1}^{n} f(i)\Delta G(i)| \quad (\text{where } G(i) = \textstyle\sum_{j=1}^{i} g(j)) \\ &= |\sum_{i=1}^{n-1} G(i)(-\Delta f(i+1)) + G(n)f(n)| \\ &\not> (\sum_{i=1}^{n-1} |\Delta f(i+1)| + |f(n)|) \sup_{1 \leqq i \leqq n} |G(i)| \not> (n \cdot 2^{-n+1} + 2^{-n})||g|| \not> 2^{-k}. \end{aligned}$$

We shall prove that the full linear functional Φ has the supremum $||g||$. For, it is obvious that Φ is bounded by $||g||$ for all vectors f belonging to the closed unit sphere of bv. It is even the supremum since

$$\Phi(e) = \sum_{i=1}^{\infty} g(i),$$

and

$$\Phi({}^{n}e) = \sum_{i=1}^{n} g(i).$$

Let f belong to the closed unit sphere of bv and g be an element of w^1, then $[f, g]$ is convergent, and determines a full and hence normed linear functional on bv with $||\Phi|| = ||g||$. We now state our result as

THEOREM 4. $(bv)^* = (bv)' = w^1$.

It has been proved in [11] theo. XI, that the dual space of w^1 is bv, i.e., $(w^1)^* = bv$.

3. Dual pairs in which neither space is normable

3.1. The first space with which we commence in this section is the space b_0^p consisting of all those vectors of b^p which are also in c_0. b_0^p arises as the dual space of the non-linear space l_L^p treated in [12].

We now describe the cataloguing of the closed unit sphere of b_0^p. The closed unit sphere of b_0^p (which consists of those vectors f of b_0^p for which

$$\sum_{i=1}^{\infty} |f(i)|^p$$

is bounded by 1) permits a cataloguing of the second kind with respect to the norm of c_0. In the first place, since convergence with respect to the norm of c_0 implies convergence with respect to ω, it follows that the limit of any sequence $f_1, f_2, f_3, \cdots$ of vectors of the closed unit sphere of b_0^p which is fundamental with respect to the norm of c_0 lies in c_0 and also lies in the closed unit sphere of b^p, i.e., it lies in the closed unit sphere of b_0^p. Secondly, the species G_1 of all vectors in the closed unit sphere of b_0^p with finitely many non-zero rational complex components is dense in the closed unit sphere of b_0^p. Given vectors

$$f' = \sum_{i=1}^{n'} f'(i)e_i, \quad f'' = \sum_{i=1}^{n''} f''(i)e_i, \cdots$$

and

$$f^{(m)} = \sum_{i=1}^{n(m)} f^{(m)}(k)e_k$$

all belonging to the closed unit sphere of b_0^p, the distance of e_{n+1} (where $n = \max\ (n', n'', \cdots, n^{(m)})$ from each $f^{(j)}$ $(1 \leqq j \leqq m)$ (with respect to the norm of c_0) is $> \frac{1}{4}$. Hence the closed unit sphere of b_0^p permits a cataloguing of the second kind.

3.2. We now proceed to determine the dual space of b_0^p for $1 < p < \infty$. Let Φ be a full linear functional on b_0^p, and let $g(n) = \Phi(e_n)$. Further let $a_1, a_2, a_3, \cdots$ be any sequence of non-negative terms converging to zero. For each fixed k, for each n, we can choose exactly one assertion from those recognized as valid of the two assertions for a sequence g:

$$\text{(i) } |g(n)| < 2^{-k-n} \text{ and (ii) } |g(n)| > 2^{-k-n-1}.$$

Take $y_g^{(k)}(n) = 0$ if (i) has been chosen and

$$y_g^{(k)}(n) = \min(|g(n)|^{q-1}, a_n) \operatorname{sgn} \overline{g(n)}$$

if (ii) has been chosen. Obviously

$$|y_g^{(k)}(n) - y_g^{(k+k')}(n)| < 2^{-k} \text{ for all } k' \geqq 0 : \lim_{k\to\infty} y_g^{(k)}(n)$$

therefore exists and we denote it by $y_g(n)$. We have

$$g(n)y_g(n) = \lim_{k\to\infty} g(n)y_g^{(k)}(n) = \min(|g(n)|^q, a_n|g(n)|).$$

For the space b_0^p we take $\eta_k = y_g(k)$ to obtain the dressed spread $\Lambda(b_0^p)$. Considering the dressed spread $\Lambda(b_0^p)$ as a catalogued space, we conclude by the continuity theorem that to each k, an $N(k)$ can be assigned such that $|\Phi(0) - \Phi(f)| < 2^{-k}$ for any f in $\Lambda(b_0^p)$ passing through the dressed node consisting of N zeros. Applying this to the vector

$$(0, 0, \cdots, 0, y_g(n+1), \cdots, y_g(n+m), 0, 0, 0, \cdots)$$

(n zero in the beginning) for $n \geqq N(k)$ we have:

$$\sum_{i=n+1}^{n+m} g(i)y_g(i) = |\sum_{i=n+1}^{n+m} \min(|g(i)|^q, a_i|g(i)|)| < 2^{-k}.$$

From this follows that

$$\sum_{i=1}^{\infty} \min(|g(i)|^q, a_i|g(i)|)$$

si convergent, for any sequence $a_1, a_2, a_3, \cdots$ of non-negative terms tending to zero.

We now introduce the linear space ${}_1b^p$ which is the species of all those vectors g of b^p for which given any sequence $a = (a_1, a_2, a_3, \cdots)$ of non-negative terms converging to zero the series

$$||g||_a = \sum_{i=1}^{\infty} \min(|g(i)|^p, a_i|g(i)|)$$

converges. With this notation we have proved that to each full linear functional Φ on b_0^p, a vector g of ${}_1b^q$ can be assigned, such that $\Phi(f) = [f, g]$.

The continuity of Φ follows from the fact that the closed unit sphere of b_0^p is catalogued. We insert here an alternative proof of this continuity, which does not utilize this cataloguing. Given k, we first of all indicate an $N = N(k)$ such that

$$\sum_{i=n}^{n+m} \min(|g(i)|^q, |f(i)||g(i)|) < 2^{-k} \text{ and } n \cdot 2^{-n} \cdot C \ngtr 2^{-k}$$

for $n \geqq N(k)$ and $m \geqq 0$, C^q is any bound of

$$\sum_{i=1}^{\infty} |g(i)|^q.$$

If now f is in the closed unit sphere of b_0^p with $\|f\|_{c_0} < 2^{-n}$, then taking any $n \geqq N(k)$, we have

$$\begin{aligned} |\Phi(f)| &= |\sum_{i=1}^{\infty} f(i)g(i)| \ngtr \sum_{i=1}^{n} |f(i)g(i)| + \sum_{i=n+1}^{\infty} |f(i)g(i)| \\ &\ngtr (\sum_{i=1}^{n} |f(i)|^p)^{1/p} (\sum_{i=1}^{n} |g(i)|^q)^{1/q} + \sum_{i=n+1}^{\infty} |f(i)g(i)| \\ &\ngtr n \cdot 2^{-n} \cdot C + \sum_{i=n+1}^{\infty} |f(i)g(i)| \ngtr 2^{-k} + \sum_{i=n+1}^{\infty} |f(i)g(i)|. \end{aligned}$$

Now,

$$\sum_{i=n}^{n+m} \min(|g(i)|^q, |f(i)g(i)|) < 2^{-k} \text{ for every } m \geqq 0.$$

Fixing m, we can choose for each i exactly one assertion from those recognized as valid among the two assertions:

(A) $||f(i)g(i)| - |g(i)|^q| > 2^{-k-1}/m$ and (B) $||f(i)g(i)| - |g(i)|^q| < 2^{-k}/m$.

For each i for which (A) has been chosen, we can judge whether

(A_1) $\min(|f(i)g(i)|, |g(i)|^q) = |f(i)g(i)|$, or ($A_2$) $\min(|f(i)g(i)|, |g(i)|^q) = |g(i)|^q$,

and, in an obvious notation

$$\begin{aligned} \Sigma_{(A)} \min(|f(i)g(i)|, |g(i)|^q) &= \Sigma_{(A_1)} |f(i)g(i)| + \Sigma_{(A_2)} |g(i)|^q \\ &\ngtr \sum_{i=n}^{n+m} \min(|f(i)g(i)|, |g(i)|^q) < 2^{-k}, \end{aligned}$$

so that, a fortiori,

$$\Sigma_{(A_1)} |f(i)||g(i)| < 2^{-k}, \text{ and } \Sigma_{(A_2)} |g(i)|^q < 2^{-k}.$$

For those indices i for which (B) has been chosen,

$$\Sigma_{(B)}(|f(i)||g(i)| - \min(|f(i)||g(i)|, |g(i)|^q)) < m \cdot 2^{-k}/m = 2^{-k},$$

i.e.

$$\begin{aligned} \Sigma_{(B)} |f(i)||g(i)| &< \Sigma_{(B)} \min(|f(i)||g(i)|, |g(i)|^q) \\ &\quad + 2^{-k} < 2^{-k} + 2^{-k} = 2^{-k+1}. \end{aligned}$$

Thus we have:

$$\begin{aligned} \sum_{i=n}^{n+m} |f(i)||g(i)| &= \Sigma_{(A_1)} |f(i)||g(i)| + \Sigma_{(A_2)} |f(i)||g(i)| + \Sigma_{(B)} |f(i)||g(i)| \\ &< 2^{-k} + (\Sigma_{(A_2)} |f(i)|^p)^{1/p} (\Sigma_{(A_2)} |g(i)|^q)^{1/q} + 2^{-k+1} \\ &< 2^{-k} + 1.2^{-k/q} + 2^{-k+1} < 3.2^{-k} + 2^{-k/q}, \end{aligned}$$

which is valid for every $m \geqq 0$, so that

$$\sum_{i=n}^{\infty} |f(i)||g(i)| \not> 3.2^{-k}+2^{-k/q}.$$

Hence

$$|\Phi(f)| \not> 2^{-k}+3.2^{-k}+2^{-k/q} \not> 2^{-k+2}+2^{-k/q},$$

whenever

$$||f||_{c_0} < 2^{-N(k)},$$

which establishes the continuity of Φ. The proof of the fact that $|||\Phi||| = |||g|||$ follows exactly as in the proof of $(l^p)^* = b^q$.

Let f belong to the closed unit sphere of b_0^p and g be an element of ${}_1b^q$, then $[f, g]$ is convergent, as proved immediately above, and determines a full linear functional Φ on b_0^p. Thus we have proved that corresponding to each full linear functional Φ of b_p^0 a vector g of ${}_1b^q$ can be assigned such that $\Phi(f) = [f, g]$, and conversely, to each f belonging to b_0^p and g belonging to ${}_1b^q$, $[f, g]$ is convergent and defines a full linear functional on b_0^p with $|||\Phi||| \equiv |||g|||$. Thus we have proved

THEOREM 5.

$$(b_0^p)^* = {}_1b^p \left(1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1\right).$$

Let Φ be a full normed linear functional on b_0^p. We shall prove that every normed linear functional Φ of b_0^p is a normed linear functional on l^p. Firstly, the supremum $||\Phi||$ of Φ on the closed unit sphere of b_0^p is a bound on the closed unit sphere of $l^p \subseteq b_0^p$. Secondly, we shall show that Φ possesses a supremum on the closed unit sphere of l^p. Since $||\Phi||$ is the supremum of Φ on the closed unit sphere of b_0^p, corresponding to given k, there exists a unit vector f of b_0^p such $|\Phi(f)| > ||\Phi||-2^{-k-1}$. As ${}^nf$ converges to f in b_0^p, we can indicate an n such that $|\Phi({}^nf)-\Phi(f)| < 2^{-k-1}$. Hence we have

$$|\Phi(f)| > |\Phi(f)|-2^{-k-1} > ||\Phi||-2^{-k-1}-2^{-k-1} = ||\Phi||-2^{-k},$$

which proves that $||\Phi||$ is the supremum of Φ on the closed unit sphere of l^p and the vector g corresponding to Φ lies in l^q.

Conversely, if g lies in l^q, then it determines a normed linear functional Φ with $||\Phi|| = ||g||$ on $b^p \supseteq b_0^p$, so that Φ is a normed linear functional on b_0^p. We state our result as

THEOREM 6. $(b_0^p)' = l^q$.

3.3. We now proceed to determine the dual space of ${}_1b^p$ $(1 < p < \infty)$. It is obvious that ${}_1b^p$ contains l^p. We shall prove that it also contains b^1. Let f be any vector of b^1. We wish to prove that f also lies in ${}_1b^p$, i.e.,

$$\sum_{i=1}^{\infty} \min(|f(i)|^p, a_i|f(i)|)$$

converges for any sequence $a_1, a_2, a_3, \cdots$ of non-negative term converging to zero. Now corresponding to given k, there exists an $N = N(k)$, such that $a_n < 2^{-k}$ for every $n \geqq N(k)$, and then

$$\sum_{i=N}^{N+m} \min(|f(i)|^p, a_i|f(i)|) \not> \sum_{i=N}^{N+m} a_i|f(i)| < 2^{-k} \sum_{i=N}^{N+m} |f(i)| < 2^{-k} \cdot C$$

(where C is any bound of

$$\sum_{i=1}^{\infty} |f(i)|),$$

which establishes the required convergence, so that f also lies in ${}_1b^p$.

Let Φ be a full linear functional on ${}_1b^p$, which is therefore full on l^p as well as on b^1. Putting $\Phi(e_k) = g(k)$, g lies in $(l^p)^*$ as well as in $(b^1)^*$, and so the dual space of ${}_1b^p$ is contained in the

$$(l^p)^* \cap (b^1)^*, \text{ i.e., in } b_0^q.$$

Conversely, given any vector g of b_0^q, $[f, g] = [g, f]$ converges for any f in ${}_1b^p$, as proved above, which establishes the following

THEOREM 7. $({}_1b^p)^* = b_0^q$.

A proof similar to that of $(b_0^p)' = l^q$ can be given here to prove that $({}_1b^p)' = l^q$.

We now indicate a system of neighbourhoods of the null vector which generates a locally convex topology of ${}_1b^p$, as follows: Corresponding to a given sequence $a = (a_1, a_2, a_3, \cdots)$ converging to zero and a natural number k, $U_{a,k}$ is the species of all those vectors f belonging to ${}_1b^p$ for which $||f||_a < 2^{-k}$. As for every sequence $a = (a_1, a_2, a_3, \cdots)$ converging to zero, and every k, a certain $N = N(a, k)$ can be indicated such that

$$\sum_{i=N}^{N+m} \min(|f(i)|^p, a_i|f(i)|) < 2^{-k}$$

for $m \geqq 0$, it follows that $f - {}^Nf$ lies in $U_{a,k}$; ${}^nf \to f$ with respect to the above topology. From this follows the denseness of the linear manifold of all those vectors in the closed unit sphere of ${}_1b^p$ consisting of finitely many rational complex numbers followed by zeros, which allows us to conclude the separability of the said closed unit sphere.

We now prove that the closed unit sphere of ${}_1b^p$ is complete with respect to this topology. Let $f^1, f^2, f^3, \cdots$ be a fundamental sequence of vectors of the closed unit sphere of ${}_1b^p$, so that given any a and k, an

$N = N(a, k)$ can be indicated such that $||f^n - f^{n+m}||_a < 2^{-k}$ for $n \geqq N(a, k)$ and $m \geqq 0$. However, by taking a to be the sequence $(0, 0, \cdots, 0, 2, 0, 0, \cdots)$ in which 2 occurs at the ith place, we conclude that $|f^n(i) - f^{n+m}(i)| < 2^{-k}$ for $n \geqq N(a, k)$, $m \geqq 0$. It follows from this, that for each fixed $i, f^1(i), f^2(i), f^3(i), \cdots$ is a convergent sequence of complex numbers: let its limit be $f(i)$. We shall show that f belongs to the closed unit sphere of ${}_1b^p$. First, convergence in the topology of ${}_1b^p$ implies convergence with respect to the distance delimitation ω defined for the space b^p. Therefore f lies in the closed unit sphere of b^p. Secondly,

$$||f^r - f^s||_a < 2^{-k} \text{ for any } r, s \geqq N(a, k), \text{ i.e., } \sum_{i=1}^{\infty} \min(|f^r(i) - f^s(i)|^p,$$

$a_i|f^r(i) - f^s(i)|) < 2^{-k}$ implies a fortiori that

$$\sum_{i=1}^{m} \min(|f^r(i) - f^s(i)|^p, a_i|f^r(i) - f^s(i)|) < 2^{-k}$$

for every m; hence in the limit as $s \to \infty$, we have

$$\sum_{i=1}^{m} \min(|f^r(i) - f(i)|^p, a_i|f^r(i) - f(i)|) \not> 2^{-k}$$

for every m. Therefore

$$\sum_{i=1}^{\infty} \min(|f^r(i) - f(i)|^p, a_i|f^r(i) - f(i)|) \not> 2^{-k}, \text{ i.e.,}$$

$||f^r - f||_a < 2^{-k}$, so that $f^r - f$ is a vector in ${}_1b^p$ and hence f lies in ${}_1b^p$ which proves the completeness of the closed unit sphere of ${}_1b^p$.

We now prove the continuity of Φ with respect to the topology of ${}_1b^p$. If f is any vector of ${}_1b^p$, then

$$\sum_{i=1}^{\infty} \min(|f(i)|^p, a_i|f(i)|)$$

converges for any $a = (a_1, a_2, \cdots)$ converging to zero. Now if we take f in the closed unit sphere of ${}_1b^p$ with

$$||f||_a < 2^{-k}, \text{ i.e., } \sum_{i=1}^{\infty} \min(|f(i)|^p, a_i|f(i)|) < 2^{-k},$$

then

$$|\Phi(f)| = |\sum_{i=1}^{\infty} f(i)g(i)| \not> \sum_{i=1}^{\infty} |f(i)g(i)|.$$

Now, taking $a_i = |g(i)|$, we obtain

$$\sum_{i=1}^{n} \min(|f(i)|^p, |f(i)g(i)|) < 2^{-k}$$

for every n. Now we can prove in the same way as in the proof of theorem 5 above that $|\Phi(f)| < 3.2^{-k} + C \cdot 2^{-k/p}$ where C^q is any bound of

$$\sum_{i=1}^{\infty} |g(i)|^q,$$

which establishes the continuity of Φ.

3.4. The following spaces analogous to b_0^p can now be introduced.

For $1 < p < \infty$, $b_r^p (r > p)$ is the space of all those vectors of b^p which also lie in l^r. The quasi-norm is the same as in the space b^p. Let Φ be a full linear functional on b_r^p, and let $g(n) = \Phi(e_n)$. Further, let $a_1, a_2, a_3, \cdots$ by any sequence of non-negative terms such that $\Sigma\, a_i$ converges. For each fixed k, for each n, we can choose exactly one assertion from those recognized as valid of the two assertions for a sequence g:

$$\text{(i) } |g(n)| < 2^{-k-n} \text{ and (ii) } |g(n)| > 2^{-k-1-n}.$$

Take $y_g^{(k)}(n) = 0$ if (i) has been chosen and

$$y_g^{(k)}(n) = \min\left(|g(n)|^{q-1}, a_n^{1/r}\right) \operatorname{sgn} \overline{g(n)}$$

if (ii) has been chosen. Obviously

$$|y_g^{(k)}(n) - y_g^{(k+k')}| < 2^{-k} \text{ for all } k' \geqq 0 : \lim_{k \to \infty} y_g^k(n)$$

therefore exists and we denote it by $y_g(n)$, and we have

$$g(n) y_g(n) = \lim_{k \to \infty} g(n) y_g^k(n) = \min\left(|g(n)|^q, a_n^{1/r} |g(n)|\right).$$

For the space b_r^p, we take $\eta_k = y_g(k)$ to obtain the dressed spread $\Lambda(b_r^p)$.

We now conclude that to each k, an $N(k)$ can be assigned such that $|\Phi(0) - \Phi(f)| < 2^{-k}$ for any f in $\Lambda(b_r^p)$ passing through the dressed node consisting of N zeros. Applying this to the vector

$$(0, 0, 0, \cdots, y_g(n+1), y_g(n+2), \cdots, y_g(n+m), 0, \cdots)$$

(with n zeros in the beginning) for $n \geqq N(k)$ we have

$$\left|\sum_{i=n+1}^{n+m} g(i) y_g(i)\right| = \sum_{i=n+1}^{n+m} \min\left(|g(i)|^q, a_i^{1/r} |g(i)|\right) < 2^{-k} \text{ for } n \geqq N(k).$$

From this follows that

$$\sum_{i=1}^{\infty} \min\left(|g(i)|^q, a_i |g(i)|\right)$$

is convergent, for any sequence $a_1, a_2, a_3, \cdots$ of non-negative terms with convergent $\sum_{i=1}^{\infty} a_i$.

We now introduce the linear space

$$ {}_sb^q \left(\frac{1}{p}+\frac{1}{q}=1, \frac{1}{r}+\frac{1}{s}=1, r>p, q>s\right) $$

which is the species of all those vectors f of b^q for which given any sequence $a_1, a_2, a_3, \cdots$ of non-negative numbers such that $\Sigma\, a_i$ converges, the series

$$ \|f\|_a = \sum_{i=1}^{\infty} \min\,(|f(i)|^q, a_i^{1/r}|f(i)|) $$

is convergent. With this notation we have proved that to each full linear functional Φ on b_r^p, a vector g of ${}_sb^q$ can be assigned, such that $\Phi(f) = [f, g]$.

The continuity of Φ follows from the fact that the closed unit sphere of b_r^p is catalogued. For, the closed unit sphere of b_r^p (which consists of all those vectors f of b_r^p for which

$$ \sum_{i=1}^{\infty} |f(i)|^p $$

is bounded by 1) permits a cataloguing of the second kind with respect to the norm of l^r which follows in the same way as for the space b_0^p. We can now complete this proof exactly as in the case of theorem 5 to obtain

THEOREM 8.

$$ (b_r^p)^* = {}_sb^q \left(\frac{1}{p}+\frac{1}{q}=1, \frac{1}{r}+\frac{1}{s}=1, r>p, s<q\right). $$

The proof of the fact that $(b_r^p)' = l^q$ follows in the same way as in theorem 6 above.

3.5. We now consider the space

$$ {}_rb^p \left(1<r<p, \frac{1}{p}+\frac{1}{q}=1, \frac{1}{r}+\frac{1}{s}=1\right). $$

It is obvious that ${}_rb^p$ contains l^p. We shall prove that it also contains b^r. Let f be any vector belonging to b^r. For any sequence $a_1, a_2, a_3, \cdots$ of non-negative terms for which $\sum_{i=1}^{\infty} a_i$ is convergent, the series

$$ \|f\|_a = \sum_{i=1}^{\infty} \min\,(|f(i)|^p, a_i^{1/s}|f(i)|) $$

is convergent in view of

$$\sum_{i=n}^{n+m} \min(|f(i)|^p, a_i^{1/s}|f(i)|) \ngtr \sum_{i=n}^{n+m} a_i^{1/s}|f(i)| \ngtr (\sum_{i=n}^{n+m} a_i)^{1/s}(\sum_{i=n}^{n+m} |f(i)|^r)^{1/r}$$

in which

$$\sum_{i=n}^{n+m} |f(i)|^r$$

is bounded by any bound of

$$\sum_{i=1}^{\infty} |f(i)|^r.$$

From this follows that f lies in ${}_rb^p$.

Let Φ be a full linear functional on ${}_rb^p$, which is therefore full on l^p as well as on b^r. Putting $\Phi(e_k) = g(k)$; g lies in $(l^p)^*$ as well as in $(b^r)^*$, and so the dual space of ${}_rb^p$ is contained in the intersection of $(l^p)^* = b^q$ and $(b^r)^* = l^s$, i.e., in b_s^q. Conversely, given any vector g of b_s^q, $[f, g] = [g, f]$ converges for any f in ${}_rb^p$, as proved above, which establishes completely

THEOREM 9. $({}_rb^p)^* = b_s^q$.

The proof of the fact that $({}_rb^p)' = l^q$ follows similarly as given in theorem 7.

3.6. b_∞^p is the non-linear space of all those vectors f of l^∞ which also lie in b^p. The quasi-norm of b^p is used also in this space.

By mapping a vector f of $\Lambda(b^p)$ into a vector f' of b_∞^p given by $f'(1) = 1$, $f'(n) = f(n-1)$ for $n \geqq 2$, we obtain a species $\Lambda'(b^p)$ contained in b_∞^p, on which the considerations, given in the proof of theorem 1 (sec. 2.2A), are applicable without change and we conclude that

$$(b_\infty^p)^* = (b_\infty^p)' = l^q.$$

If we were to extend the definitions given above to introduce a space ${}_1b^\infty$, this would have to be the species of those vectors of b^∞ for which given any sequence $a_1, a_2, a_3, \cdots$ of non-negative terms converging to zero, the supremum of min $(|g(i)|, a_i|g(i)|)$ exists; but this species coincides with the whole of b^∞ and has not to be treated separately.

3.7. We now introduce the space b_0^1 which is the species of all those vectors of b^1 which also lie in c_0. The quasi-norm of any vector f in this is the same as for the space b^1. Let Φ be a full linear functional on b_0^1 and let $g(n) = \Phi(e_n)$. Further, let $a_1, a_2, a_3, \cdots$ be any sequence of non-negative terms converging to zero. For each fixed j, for each n, we can choose exactly one assertion from those recognized as valid of the two assertions for a sequence g:

$$\text{(i) } |g(n)| < 2^{-j-n} \text{ and (ii) } |g(n)| > 2^{-j-n-1}.$$

Take $y^j_g(n) = 0$ if (i) has been chosen and

$$y^j_g(m') = [a_{m'}/\sum_{i=n}^{n+m'} a_i] \operatorname{sgn} \overline{g(m')}$$

if (ii) has been chosen for a fixed $m \geqq 0$ and $n \leqq m' \leqq n+m$. We dress the direction of the spread Λ by taking $\eta_k = y^j_g(k)$ to obtain the dress spread $\Lambda(b^1_0)$. We now conclude by the continuity theorem that to each k an $N(k)$ can be assigned such that $|\Phi(0)-\Phi(f)| < 2^{-k}$ for any f in $\Lambda(b^1_0)$ passing through the dressed node consisting of N zeros. Applying this to the vector

$$(0, 0, \cdots, 0, y^j_g(n), y^j_g(n+1), \cdots, y^j_g(n+m), 0, 0, 0, \cdots)$$

(with n-1 zeros in the beginning) for $n \geqq N(k)$, we have

$$\left(\frac{a_n}{\Sigma_{i=n}^{n+m} a_i}|g(n)| + \frac{a_{n+1}}{\Sigma_{i=n}^{n+m} a_i}|g(n+1)| + \cdots + \frac{a_{n+m}}{\Sigma_{i=n}^{n+m} a_i}|g(n+m)|\right) - 2^{-j-n}$$

$$- \cdots - 2^{-j-(n+m)} \not> \sum_{i=n}^{n+m} a_i|g(i)|/\sum_{i=n}^{n+m} a_i^{-2^{-j-n+1}} < 2^{-k}, \text{ i.e.,}$$

$$\frac{\Sigma_{i=n}^{n+m} a_i|g(i)|}{\Sigma_{i=n}^{n+m} a_i} < 2^{-k+2^{-j-n+1}}, \text{ i.e., } \sum_{i=n}^{n+m} a_i|g(i)| = 0(\sum_{i=n}^{n+m} a_i).$$

We now introduce the linear space ${}_1b_0$ which is the species of all those vectors f of b^∞ for which given any sequence $a_1, a_2, \cdots$ of strictly positive numbers converging to zero, we have

$$\sum_{i=n}^{n+m} a_i|f(i)| = 0(\sum_{i=n}^{n+m} a_i).$$

The quasi-norm of b^∞ is used in ${}_1b_0$. With this notation we have proved that to each full linear functional Φ on b^1_0 a vector g of ${}_1b_0$ can be assigned such that $\Phi(f) = [f, g]$.

The continuity of Φ follows from the fact that the closed unit sphere of b^1_0 is catalogued. An alternative proof can also be given as in the previous discussion. The proof of the fact that $|||\Phi||| = |||g|||$ follows exactly as in the proof of $(l^1)^* = b^\infty$.

Conversely, let f belong to the closed unit sphere of b^1_0 and g be an element of ${}_1b_0$, then $[f, g]$ is obviously convergent, and determines a full linear functional Φ on b^1_0 which proves

THEOREM 10. $(b^1_0)^* = {}_1b_0$.

3.8. As ${}_1b_0$ contains both c_0 and b^1, its dual space will be contained in

$(c_0)^* \cap (b^1)^* = b^1 \cap c_0 = b_0^1$: it can be shown that it is exactly equal to b_0^1, exactly as in similar cases treated above.

3.9. It can be easily proved from the above discussion that $(b_0^1)' = {}_1l_0$ where ${}_1l_0$ is the species of all complex sequences f of l^∞ which also lie in ${}_1b_0$. The norm of l^∞ is used in ${}_1l_0$. Now we show that ${}_1l_0$ is a non-linear space. We dress the direction of L by replacing each node $(c_1, c_2, \cdots, c_n)$ of order n by $(1, c_2-1, c_3-1, \cdots, c_n-1)$ to obtain the dressed fan direction $L({}_1b_0)$. To every sequence c of 1's and 2's extracted from the fan L, we construct a sequence as follows:

$$\begin{aligned} f_c(1) &= 1; \text{ and for } n > 1, \\ f_c(n) &= 0 \text{ if the } n\text{-th constituent of } c \text{ is } 1; \\ &= 1 \text{ if the } n\text{-th constituent of } c \text{ is } 2. \end{aligned}$$

$L({}_1b_0)$ is contained in b^1; and so, a fortiori, in ${}_1b_0$, since b^1 is contained, in ${}_1b_0$. Thus $L(b_0)$ is contained in l^∞ as well as in ${}_1b_0$, i.e., in ${}_1l_0$. We also know that $e_1 = (1, 0, 0, \cdots)$ lies in ${}_1l_0$. Now by the usual technique we can prove that it is contradictory that $f_c - e_1$ lies in ${}_1l_0$ for every c in L, from which it follows that ${}_1l_0$ is a non-linear space.

By applying a full linear functional Φ on ${}_1l_0$ to $L({}_1b_0)$, we conclude that g given by $g(k) = \Phi(e_k)$ lies in c_0. As ${}_1l_0$ also contains c_0 (as c_0 is contained in l^∞ as well as in ${}_1b_0$), it follows that $({}_1l_0)^*$ is contained in b_0^1 and is hence even equal to it.

Exactly as in the proof of $(b_0^p)' = l^q$, we can show that every normed linear functional on ${}_1b_0$ is also a normed linear functional of c_0, and the proof can be completed to show that $({}_1b_0)' = l^1$. Similarly, it can be shown that $({}_1l_0)' = l^1$.

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BIBLIOGRAPHY

L. E. J. Brouwer
[1] Points and spaces. Can. J. Math. 6, 1–17 (1954).

A. Heyting
[2] Intuitionism, an introduction. Amsterdam 1966.

A. Heyting
[3] Note on the Riesz-Fischer theorem. Proc. Kon. Ned. Ak. v. Wet. ser. A, 54 35–40 (1951).

A. HEYTING
[4] Espaces de Hilbert et intuitionnisme, Colloque internationaux du C.N.R.S. Paris 1953, 59–63.

A. HEYTING
[5] Intuïtionistische theorie der Hilbert-ruimten. (Unpublished lecture notes in Netherlandish. Amsterdam, 1949–50).

ASHVINIKUMAR
[6] Hilbert spaces in intuitionism-Hilbertaj spacoj en intuiciismo. Thesis. Amsterdam, 1966.

ASHVINIKUMAR
[7] Über Katalogisierte Räume. Comp. Math. 21, 431–456 (1970).

ASHVINIKUMAR
[8] On the intuitionist theory of Stieltjes integration and its applications. Proc. Kon. Ned. Ak. v. Wet. ser. A, 73, 62–76, (1970).

ASHVINIKUMAR
[9] On Brouwer-Stieltjes integration. Proc. Kon. Ned. Ak. v. Wet. ser. A, 73, 161–171 (1970).

ASHVINIKUMAR
[10] A note on quasi-numbers. (To appear.)

ASHVINIKUMAR and S. L. SHUKLA
[11] Intuitionist determination of dual spaces of certain catalogued linear spaces. I., II. Proc. Kon. Ned. Ak. v. Wet. ser. A, 74, no. 3 (1971).

S. L. SHUKLA
[12] On intuitionist analogues of classically inseparable spaces. (To appear.)

S. L. SHUKLA
[13] On some linear spaces which coincide classically but are different intuitionistically. (To appear.)

S. L. SHUKLA
[14] Intuitionist treatment of sequence spaces satisfying a condition of convergence, or absolute convergence or their analogues and related spaces. (To appear.)

E. BISHOP
[15] Foundations of constructive analysis, New York. 1967.

J. D. HILL
[16] On the space (γ) of convergent series, Tôhoku math. J. 45, 332–337 (1939).

G. H. HARDY
[17] Divergent series, Oxford, 1949.

G. KÖTHE
[18] Topologische lineare Räume I., Berlin Göttingen-Heidelberg 1960.

L. V. KANTOROVICH and G. P. AKILOV
[19] Functional analysis in normed spaces. Oxford-London New York etc. 1964. (Translated from the Russian 'Funktoional'nyi analiz v normirovannykh prostranstvakh, Moscow 1959).

F. RIESZ and B. SZ. NAGY
[20] Leçons d'analyse fonctionnelle. Budapest 1953.

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